Perturbed Orbital Motion in Astronomy and Astrodynamics

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Abstract. This tutorial paper derives the equations of perturbed orbital motion for application to heliocentric orbits (planets, asteroids, comets, space probes, etc.) and to planetocentric orbits (moons and artificial satellites).

1. Introduction

During approximately the first ten years of the space age, artificial Earth satellite orbital motion was a major new research area for dynamical astronomy. The *Astronomical Journal* featured seminal papers on the motions of artificial Earth satellites by, for example, Kozai [1] and Brouwer [2]. Papers by Goodyear [3] and Herrick [4] on universal variables and the uniform Kepler equation also come readily to mind. The latter two papers were no do doubt stimulated by interest in the new discipline of astrodynamics (prediction of the motions of manmade bodies in the gravity fields of the Sun, its planets, and the planets' moons).

But today, as we approach the 50th anniversary of the launch of the Russian satellite Sputnik 1 on October 4, 1957, we find that orbital mechanics research, as applied to artificial Earth satellites and interplanetary space probes, is largely done by aerospace engineering departments and by government laboratories and commercial space companies rather than by astronomy departments. Indeed, papers on orbital mechanics are more likely to be found in the *Journal of Celestial Mechanics and Dynamical Astronomy* and in the *Journal of the Astronautical Sciences* than in the *Astronomical Journal.*

The American student (or foreign student studying in America), who nowadays wants to pursue a career in orbital mechanics in the United States, will probably wind up in academia, in the space industry, in a governmental or quasi-governmental laboratory such as NASA's Jet Propulsion Laboratory or Johns Hopkins University's Applied Physics Laboratory, or in the military (most likely in the United State Air Force; see *Air Force* magazine [5] regarding the role of the U.S. Air Force in affairs of military space).

This student, articulate in the concepts of, and able in the techniques of the calculus, differential equations, and linear algebra, will find himself or herself working

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with the orbits of major planets, minor planets, robotic space probes, manned or unmanned Earth-orbital spacecraft, or all of these. So what I want to do here is to construct a mathematical foundation that the interested student, whether a physics major or an engineering major, will find to be of permanent reference value no matter what kind of orbit happens to be under investigation. And I hope that this paper will somehow lead to more astronomy departments giving greater emphasis to the teaching of orbital mechanics.

2. Equations of Absolute and Relative Motion

We start with a system of three bodies (three gravitating Newtonian particles) having masses, m_1, m_2,and m_3, and having position vectors $\mathbf{r}_1, \mathbf{r}_2$, and $\mathbf{r}_3$, as follows

$$\mathbf{r}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \mathbf{r}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}, \mathbf{r}_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}, \tag{1}$$

in some inertial reference frame. The equations of absolute Newtonian motion are

$$m_1\ddot{\mathbf{r}}_1 = Gm_1m_2\frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3} + Gm_1m_3\frac{\mathbf{r}_3 - \mathbf{r}_1}{|\mathbf{r}_3 - \mathbf{r}_1|^3} \tag{2}$$

$$m_2\ddot{\mathbf{r}}_2 = Gm_2m_1\frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} + Gm_2m_3\frac{\mathbf{r}_3 - \mathbf{r}_2}{|\mathbf{r}_3 - \mathbf{r}_2|^3} \tag{3}$$

$$m_3\ddot{\mathbf{r}}_3 = Gm_3m_1\frac{\mathbf{r}_1 - \mathbf{r}_3}{|\mathbf{r}_1 - \mathbf{r}_3|^3} + Gm_3m_2\frac{\mathbf{r}_2 - \mathbf{r}_3}{|\mathbf{r}_2 - \mathbf{r}_3|^3} \tag{4}$$

where $\ddot{\mathbf{r}}_i = \frac{d^2\mathbf{r}_i}{dt^2}$ and $r_i = |\mathbf{r}_i| = \sqrt{x_i^2 + y_i^2 + z_i^2}$ for $i = 1, 2$, and 3.

Let us put the origin at body 1 so that $\mathbf{r}_1 = \mathbf{0}$, let us assume that $\ddot{\mathbf{r}}_1 = \mathbf{0}$, and let us define the relative position vector, $\mathbf{r}$, as

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1. \tag{5}$$

Then Eq. 2 becomes

$$\mathbf{0} = Gm_1m_2\frac{\mathbf{r}}{r^3} + Gm_1m_3\frac{\mathbf{r}_3}{r_3^3}, \tag{6}$$

and upon dividing by m_2, Eq. 3 becomes

$$\ddot{\mathbf{r}} = -Gm_2\frac{\mathbf{r}}{r^3} + Gm_3\frac{\mathbf{r}_3 - \mathbf{r}}{|\mathbf{r}_3 - \mathbf{r}|^3}. \tag{7}$$

If we divide Eq. 6 by m_1 and subtract from Eq. 7, we get

$$\ddot{\mathbf{r}} = -G(m_1 + m_2)\frac{\mathbf{r}}{r^3} + Gm_3\left[\frac{\mathbf{r}_3 - \mathbf{r}}{|\mathbf{r}_3 - \mathbf{r}|^3} - \frac{\mathbf{r}_3}{r_3^3}\right], \tag{8}$$

or equivalently,

$$\ddot{\mathbf{r}} = -Gm_1(1 + \frac{m_2}{m_1})\frac{\mathbf{r}}{r^3} + Gm_1(\frac{m_3}{m_1})\left[\frac{\mathbf{r}_3 - \mathbf{r}}{|\mathbf{r}_3 - \mathbf{r}|^3} - \frac{\mathbf{r}_3}{r_3^3}\right]. \tag{9}$$

If we define the Gaussian gravity constant for body 1 as

$$k_1 = \sqrt{Gm_1}, \tag{10}$$

and further assume that all masses have been divided by m_1 ("normalized to m_1"), then we obtain the equations of orbital motion of body 2 relative to body 1:

$$\ddot{\mathbf{r}} = -k_1^2(1 + m_2)\frac{\mathbf{r}}{r^3} + k_1^2 m_3\left[\frac{\mathbf{r}_3 - \mathbf{r}}{|\mathbf{r}_3 - \mathbf{r}|^3} - \frac{\mathbf{r}_3}{r_3^3}\right]. \tag{11}$$

We can think of the first term on the right of the equals sign as being the acceleration acting on body 2 (the "secondary") that is due to body 1 (the "primary"). The second term is then the "perturbative acceleration acting on body 2, due to body 3". Body 3 is thus a "perturber".

3. Heliocentric Orbital Motion

For heliocentric orbital motion we can take body 1 as the Sun, body 2 as an asteroid, comet, Kuiper belt object, or interplanetary space probe, and bodies 3 through 12 as the nine perturbing major planets Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune, and Pluto. We note, then, that bodies 4 through 12 can be treated just like body 3: they simply add nine more terms to Eqs. 2 and 3. Thus Eq. 11 becomes

$$\ddot{\mathbf{r}} = -k_1^2(1 + m_2)\frac{\mathbf{r}}{r^3} + k_1^2\sum_{j=3}^{12} m_j\left[\frac{\mathbf{r}_j - \mathbf{r}}{|\mathbf{r}_j - \mathbf{r}|^3} - \frac{\mathbf{r}_j}{r_j^3}\right]. \tag{12}$$

If body 2 is an asteroid, comet, or interplanetary space probe, its mass, m_2, is negligible relative to that of the Sun. We are justified, then, in assuming that $m_2 = 0$. But body 2 could also be a major planet, e.g., Earth. In this case, we would have $m_2 \neq 0$, and bodies 3 through 11 would be Mercury through Pluto (with Earth omitted from the list, since it is body 2).

Finally, we should note that Pluto has been demoted to "Kuiper belt object" status by many planetary astronomers following the discoveries of other large, trans-Neptunian, planetlike bodies such as Varuna, Quaoar, and Sedna. If we subscribe to this demotion, then in applying Eq. 12 to the motion of an asteroid, comet, or interplanetary space probe, we would be justified in dropping Pluto as a perturbing major planet and summing from $j = 3$ to $j = 11$.

4. Geocentric Orbital Motion

For geocentric orbital motion we can take body 1 as Earth, body 2 as an artificial Earth satellite, body 3 as the Sun, body 4 as the Moon, and bodies 5 through 12 as the eight major planets Mercury through Pluto (with Earth omitted from the list, since it is body 1). Eq. 11 now becomes

$$\ddot{\mathbf{r}} = -k_1^2(1+m_2)\frac{\mathbf{r}}{r^3} + k_1^2 \sum_{j=3}^{12} m_j \left[\frac{\mathbf{r}_j - \mathbf{r}}{|\mathbf{r}_j - \mathbf{r}|^3} - \frac{\mathbf{r}_j}{r_j^3} \right]. \tag{13}$$

Here again we will want to assume that $m_2 = 0$. Now, if we take

$$k_1 = k_e, \text{ where } k_e \text{ is Earth's Gaussian gravity constant,}$$

$$\mu = 1 + \frac{m_2}{m_1}, \text{ and}$$

$$\mu_j = \frac{m_j}{m_1} \quad \text{for } 3 \le j \le 12,$$

then we have

$$\ddot{\mathbf{r}} = -k_e^2\mu\frac{\mathbf{r}}{r^3} + k_e^2 \sum_{j=3}^{12} \mu_j \left[\frac{\mathbf{r}_j - \mathbf{r}}{|\mathbf{r}_j - \mathbf{r}|^3} - \frac{\mathbf{r}_j}{r_j^3} \right], \tag{14}$$

and we have derived Eq. 10.30 in my book, *Topics in Astrodynamics* [6].

Eq. 13 would still apply, with appropriately assigned body indices, if body 2 were Earth's Moon. In this case we would have $m_2 \neq 0$. Also, for an accurate theory of lunar motion, we would need to take into account the fact that Earth's mass is neither uniformly nor even radially distributed. That is, we would need to incorporate a summation of the perturbative accelerations due to the adopted terms in Earth's gravitational potential. (Earth's gravity potential is discussed further in [6].) Finally, for the case of an artificial Earth satellite, there is also Earth's atmosphere to be dealt with. Indeed, the topic of the perturbative accelerations due to Earth's atmosphere is, in itself, worthy of a treatise. Accurate, long-term prediction of the positions of near-Earth artificial satellites, and even the positions of deep-space satellites with low perigees, is made quite difficult by the presence of atmospheric drag. This is why, for example, the U.S. Air Force must continually track artificial Earth satellites; not just to be able to determine when something new achieves orbit or when something old returns to Earth's surface, but also to be able to maintain the ongoing accuracy of the orbital elements in the space catalog (a catalog of the orbital elements of all space objects in Earth orbit).

5. Planetocentric Orbital Motion

The motions of artificial satellites around Earth are certainly of great interest to us who inhabit Earth. But successful missions of interplanetary exploration have put robotic probes into orbit around Venus, Mars, Jupiter, and Saturn, and a mission to orbit the planet Mercury is under way. Eq. 13 again applies, but with body 1 as the planet, body 2 as the orbiting space probe, body 3 as the Sun, bodies 4 through 13 as the other major planets, and bodies 14, 15, etc. as moons of the planet in question, insofar as their gravitational perturbations are to be accounted for.

References

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