Reduction of Order in the Three-Body Problem

R. Mansfield - Mathcad Virtual Event, April 14, 2011

1. Let the positions and velocities of the three bodies be denoted by the following 3-vectors

$$\mathbf{r1} = \begin{bmatrix} x1 \\ y1 \\ z1 \end{bmatrix} \qquad \mathbf{r2} = \begin{bmatrix} x2 \\ y2 \\ z2 \end{bmatrix} \qquad \mathbf{r3} = \begin{bmatrix} x3 \\ y3 \\ z3 \end{bmatrix}$$

$$\mathbf{v1} = \begin{bmatrix} vx1 \\ vy1 \\ vz1 \end{bmatrix} \qquad \mathbf{v2} = \begin{bmatrix} vx2 \\ vy2 \\ vz2 \end{bmatrix} \qquad \mathbf{v3} = \begin{bmatrix} vx3 \\ vy3 \\ vz3 \end{bmatrix}$$

2. Newton's universal law of gravitation gives, for each body

$$m1 \cdot \frac{d^2}{dt^2}\mathbf{r1} = G \cdot m1 \cdot m2 \cdot \frac{(\mathbf{r2}-\mathbf{r1})}{(\|\mathbf{r2}-\mathbf{r1}\|)^3} + G \cdot m1 \cdot m3 \cdot \frac{(\mathbf{r3}-\mathbf{r1})}{(\|\mathbf{r3}-\mathbf{r1}\|)^3} \qquad \textbf{(1)}$$

$$m2 \cdot \frac{d^2}{dt^2}\mathbf{r2} = G \cdot m2 \cdot m1 \cdot \frac{(\mathbf{r1}-\mathbf{r2})}{(\|\mathbf{r1}-\mathbf{r2}\|)^3} + G \cdot m2 \cdot m3 \cdot \frac{(\mathbf{r3}-\mathbf{r2})}{(\|\mathbf{r3}-\mathbf{r2}\|)^3} \qquad \textbf{(2)}$$

$$m3 \cdot \frac{d^2}{dt^2}\mathbf{r3} = G \cdot m3 \cdot m1 \cdot \frac{(\mathbf{r1}-\mathbf{r3})}{(\|\mathbf{r1}-\mathbf{r3}\|)^3} + G \cdot m3 \cdot m2 \cdot \frac{(\mathbf{r2}-\mathbf{r3})}{(\|\mathbf{r2}-\mathbf{r3}\|)^3} \qquad \textbf{(3)}$$

3. Now velocity is related to position as follows

$$\mathbf{v1} = \frac{d}{dt}\mathbf{r1} \quad \textbf{(4)} \qquad \mathbf{v2} = \frac{d}{dt}\mathbf{r2} \quad \textbf{(5)} \qquad \mathbf{v3} = \frac{d}{dt}\mathbf{r3} \quad \textbf{(6)}$$

4. Eqs. (4), (5), and (6) are a system of three 3-vector, first-order, ordinary differential equations. We get three more 3-vector, first-order ODEs by writing the following three equations, and then substituting the right sides of Eqs. (1), (2), and (3)

$$m1 \cdot \frac{d}{dt}\mathbf{v1} = m1 \cdot \frac{d^2}{dt^2}\mathbf{r1} \qquad (7)$$

$$m2 \cdot \frac{d}{dt}\mathbf{v2} = m2 \cdot \frac{d^2}{dt^2}\mathbf{r2} \qquad (8)$$

$$m3 \cdot \frac{d}{dt}\mathbf{v3} = m3 \cdot \frac{d^2}{dt^2}\mathbf{r3} \qquad (9)$$

5. Now if we write the *state vector* for our three-body system as

$$X = \begin{bmatrix} x1 \\ y1 \\ z1 \\ vx1 \\ vy1 \\ vz1 \\ x2 \\ y2 \\ z2 \\ vx2 \\ vy2 \\ vz2 \\ x3 \\ y3 \\ z3 \\ vx3 \\ vy3 \\ vz3 \end{bmatrix}$$

then $\frac{d}{dt}X = X_{dot}(X)$

is a system of eighteen scalar first-order, ordinary differential equations. It is equivalent to the system of nine scalar second-order ODEs embodied in Eqs. (1), (2), and (3). The derivative function

$$X_{dot}(X)$$

is as specified in the Mathcad Prime 1.0 worksheet N=3_2D_Cusps.mcdx that I am about to show to you again.

If you compare Dr. Valery Ochkov's 3-Planets worksheet (as obtained via the link at the beginning of the N=3_2D_Cusps.mcdx worksheet), you will see that he uses Odesolve with Eqs. (1), (2), and (3) directly.

Why does Valery's approach seem to be so much simpler? Because Mathcad's Odesolve *itself* employs the technique of reduction of order -- you just cannot see that Mathcad is doing all this!

But in my own work, I often need to be cognizant of the internal workings of Mathcad, because any algorithm that I develop has to be coded eventually in FORTRAN, C++, or Java, etc., before it can be used in any operational space command and control system.

REFERENCE

Mansfield, Roger L., "Perturbed Orbital Motion in Astronomy and Astrodynamics," *Tutorial Paper,* Astronomical Data Service, January 22, 2005.