Urban Maths

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Virtual Unreality!

Driving through an unfamiliar city centre recently, with its poorly signposted routes and confusing one-way systems, I became increasingly frustrated and completely lost. Sometimes I went with the flow and took the direction being followed by the majority of the traffic; at other times I deliberately avoided such directions. I began to feel like a particle randomly diffusing through an incomprehensible network of roads! When I eventually reached my destination and regained my equilibrium I realised the process I'd followed through the streets reminded me of a technique for solving linear networks that I've only come across in the relatively recent past.

Imagine an electrical circuit comprising a number of interconnected electrical resistors. Focus on a particular interconnection node and the nodes and resistors to which it is directly connected. For example, let's look at a segment of the circuit where node, m, say, might be surrounded by nodes, q, r and s, to which it is connected via resistors R_1, R_2 and R_3, as in Figure 1.

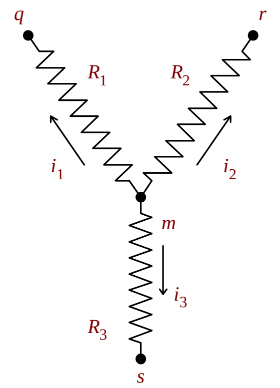

Figure 1: Segment of electrical network

We start conventionally by analysing this part of the circuit using Kirchhoff's current law and Ohm's law. Kirchhoff's current law says that the sum of the electrical currents out of any interconnection node must be zero (this is basically a statement of the conservation of electric charge). So, with currents i_1, i_2 and i_3 from Figure 1 we have:

$$i_1 + i_2 + i_3 = 0. \tag{1}$$

Ohm's law relates the voltage difference across a resistor to the current flowing through it and the value of the resistance. Using V_x to represent the voltage at node x, we have for the three branches of Figure 1:

$$\begin{aligned} V_m - V_q &= i_1 R_1, \\ V_m - V_r &= i_2 R_2, \\ V_m - V_s &= i_3 R_3. \end{aligned} \tag{2}$$

Divide each of the equations in (2) by its value of resistance, sum the resulting equations and make use of equation (1) to obtain, after a little rearrangement:

$$V_m = \frac{\frac{1}{R_1}V_q + \frac{1}{R_2}V_r + \frac{1}{R_3}V_s}{\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)}. \tag{3}$$

So far we've done nothing out of the ordinary, and if we were to continue in the normal manner we would write similar equations for all the nodes in the network, assemble them in a matrix equation and proceed to find the unknown voltages using a standard matrix solution method. However, we are going to rewrite equation (3) and interpret the result in a somewhat unusual way. First, we express it in the form:

$$V_m = p_1 V_q + p_2 V_r + p_3 V_s, \tag{4}$$

where we have

$$p_x = \frac{\frac{1}{R_x}}{\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)}, \tag{5}$$

for each $x = 1$, 2 and 3.

Equation (4) says that V_m is a weighted sum of the voltages of the nodes to which it is connected. Because of the way they are defined, the weighting factors, p_x, sum to unity. That is: $p_1 + p_2 + p_3 = 1$. In this respect they behave like probabilities, for which the sum over a complete set must equal unity. This leads us to our unusual interpretation of equation (4).

Imagine that we have a virtual particle at node, m, which moves to one or other of nodes q, r and s. The choice of which node we move the particle to depends on a uniform random number in the range 0 to 1. If the value of this number is less than p_1 the particle moves to node q; if the number is between p_1 and $p_1 + p_2$ the particle moves to node r; and if the number is greater than $p_1 + p_2$ the particle moves to node s. Record the value of voltage at the new node that the particle is now on (assuming for the moment that we know what it is). Think of this as a single trial. If we repeat this for a large enough number of trials we should have recorded voltage V_q in a fraction p_1 of the trials, V_r in a fraction p_2 of the trials and V_s in a fraction p_3 of the trials. Hence, it follows from equation (4) that the average voltage recorded will be, at least approximately, V_m. If we know the voltages of the surrounding nodes we will then know that of node m. In general, we won't know the voltages immediately adjacent to all the nodes within a network, so we repeat the process, randomly moving the virtual particle, where the moves are weighted by the inverse resistances of the paths, as indicated in equations (4) and (5), until the particle reaches a boundary node whose voltage is known. The average boundary voltage arising from many such particles starting from node m then gives an estimate of the voltage at node m.

Note that the virtual particles involved are not to be associated in any way with real, physical electrical particles, such as electrons; they are entirely unreal!

Let's look at a network comprising a cube of resistors, each of 1 ohm resistance, with a 1 volt potential difference applied across a pair of diagonally opposite corners. Figure 2 shows the arrangement, where node 7 is grounded at 0 volts and node 8 is fixed at 1 volt. What are the voltages at the other nodes?

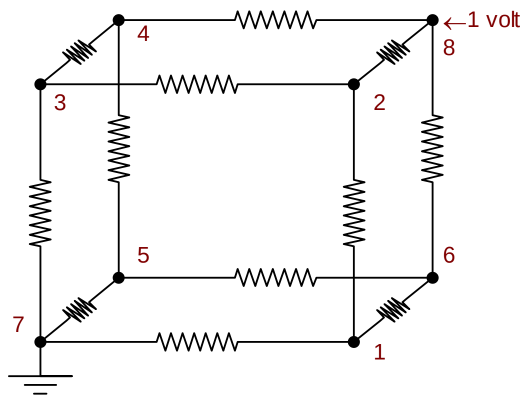

Figure 2: Cube of unit resistances

If we imagine a virtual particle at node 1, say, we can see that although a move to node 7 will result in a known voltage, moves to nodes 2 or 6 will not. If our particle moves to node 2, say, it must then choose another node to move to (which could include moving back to node 1). It must keep repeating this process until it lands on one of the boundary nodes, 7 or 8. Having started enough trial particles from node 1 to allow us to estimate its voltage by averaging all the recorded boundary voltages reached, we then need to repeat the whole process starting from each of the other nodes for which we want to find the voltage.

Actually, the symmetry of the cube allows us to focus on only two nodes, say 1 and 2. The voltages at nodes 3 and 5 are clearly identical to that at node 1, while the voltages at nodes 4 and 6 are identical to that at node 2.

A further simplification follows from the fact that all the resistances have the same value. This makes the probability that our virtual particle will move to any one of its three neighbouring nodes as $\frac{1}{3}$ for each (this follows from equation (5) by making all the resistances the same).

Figure 3 shows the cumulative estimates of the voltages of nodes 1 (hence, 3 and 5 also) and 2 (hence, 4 and 6 also) determined as a function of number of trial particles (I wrote a short Matlab program to generate this). After 1,000 trials this run estimated that $V_1 = 0.405$ volts and $V_2 = 0.607$ volts. The dotted lines mark the true values of 0.4 volts (nodes 1, 3 and 5) and 0.6 volts (nodes 2, 4 and 6).

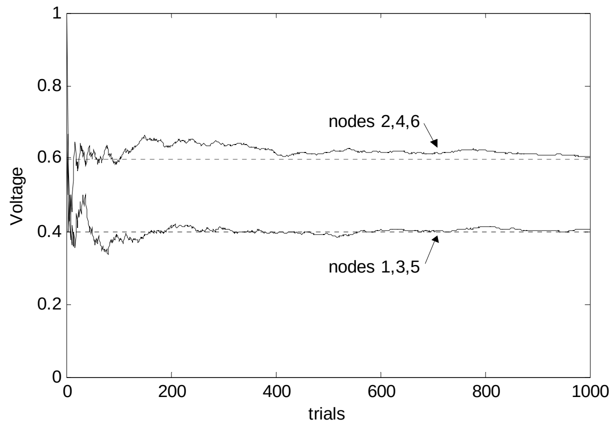

Figure 3: Unit-resistance cube voltages

This approach of having virtual particles perform random walks around a network is not confined to electric circuits. Any network for which we can ascribe fixed probabilities to the paths from node to node, and for which we know values of the appropriate analogue of electrical voltage at the boundary nodes, can be treated this way. In heat transfer networks, for example, we would use temperature as the analogue of voltage, with heat transferred by conduction (assuming that the temperature variations are not so great that the thermal conductivities become temperature dependent).

Unfortunately, I can't see a way to make it work for me while driving through unfamiliar cities! ❑