# INSTITUTO TECNOLÓGICO DE AERONÁUTICA

## MP-288 - Exercises on Multicopter Inertia with KKT Conditions

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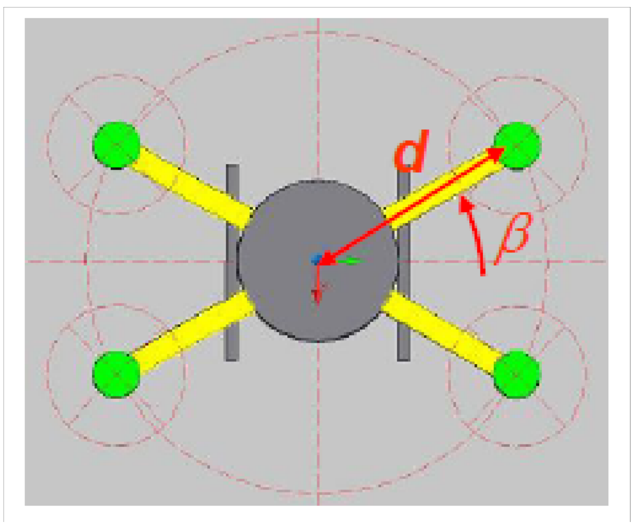
1) Solve the optimization problem for the inertial characteristics of a multicopter proposed in the series of slides "ITA MP-288 Multicopter Inertia.pdf".

This problem is presented in the slide 16. Important data are shown in slide 17.

Use KKT optimality conditions to find optimum points for the problem.




$$\min_{\substack{\beta, d \\ \underline{d} \le d \le \bar{d} \\ 0 \le \beta \le \frac{\pi}{2}}} I_{yy}$$

s.t.:

$$m_q \le \bar{m}_q$$

$$d \sin\beta \ge \frac{d_r}{2} + f$$

$$d \cos\beta \ge \frac{d_r}{2} + f$$

$$I_{yy}(\beta, d) = 4\left[\left(d^2 m_m + d^3 \frac{\rho_a A_a}{3}\right)\sin^2\beta + z_G^2 (m_m + d\rho_a A_a)\right]$$

$$I_{xx}(\beta, d) = 4\left[\left(d^2 m_m + d^3 \frac{\rho_a A_a}{3}\right)\cos^2\beta + z_G^2 (m_m + d\rho_a A_a)\right]$$

$$m_q(d) = m_c + 4(m_m + d\rho_a A_a)$$

Assuming as data $150mm \le d \le 250mm$, $m_m = 59g$, $A_a = 20*7mm^2$, $\rho_a = 2.7g/cm^3$ (aluminium), $z_G = 30mm$, $m_c = 600g$, $d_r = 120mm$, $f = 15mm$.

Maximum quadcopter mass $1150g$.

Definimos as constantes:

$m_m := 59\ gr$

$A_a := 20 \cdot 7\ mm^2$

$\rho_a := 2.7 \cdot 10^{-3}\ \frac{gr}{mm^3}$

$Z_G := 30\ mm$

$m_c := 600\ gr$

$d_r := 120\ mm$

$f_{gap} := 15\ mm$

$m'_q := 1150\ gr$

$150mm \le d \le 250mm$

Definimos a função objetiva:

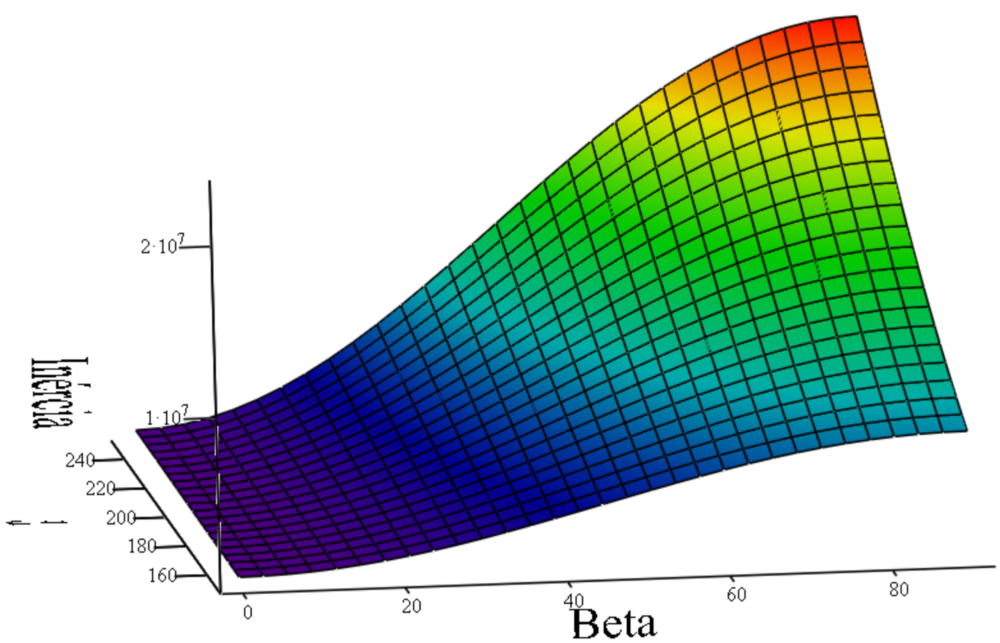
$$I_{yy}(\beta, d) := 4 \cdot \left[ \left( d^2 \cdot m_m + d^3 \cdot \frac{\rho_a \cdot A_a}{3} \right) \cdot \sin(\beta \cdot deg)^2 + Z_G{}^2 \cdot \left( m_m + d \cdot \rho_a \cdot A_a \right) \right] \quad gr \cdot mm^2$$

Funções auxiliares :

$$m_q(d) := m_c + 4 \cdot \left( m_m + d \cdot \rho_a \cdot A_a \right)$$

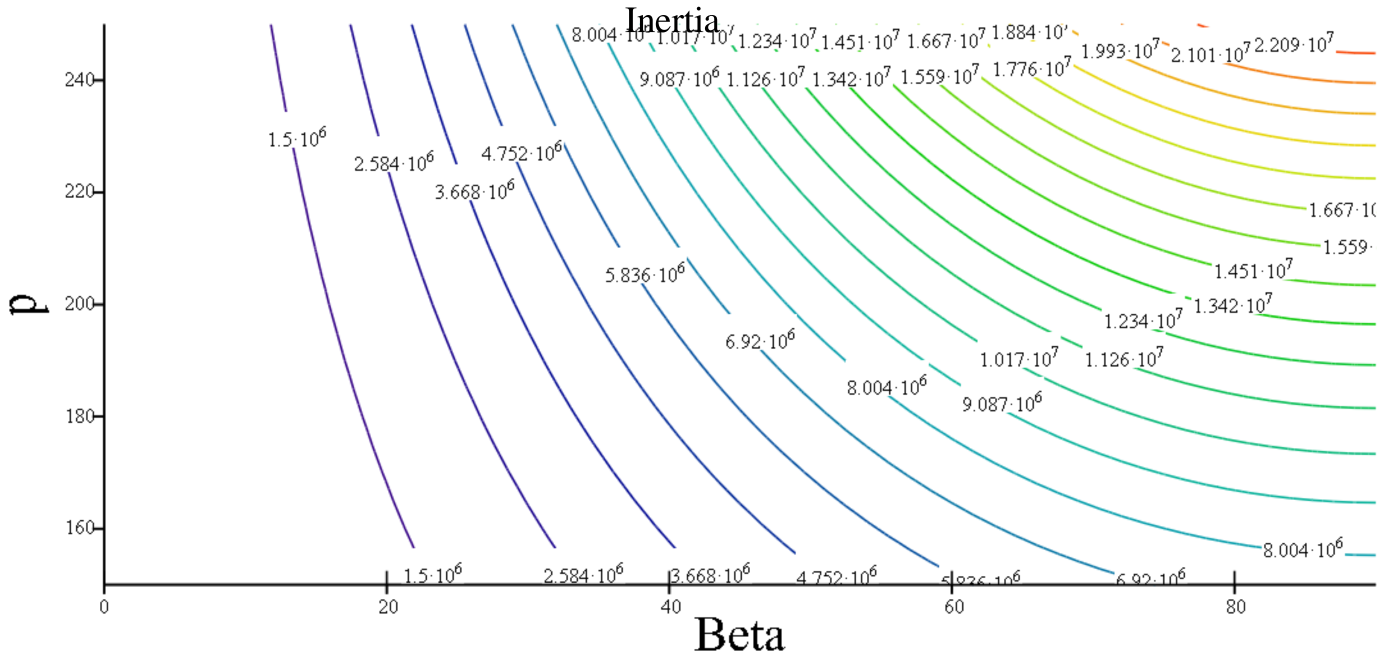
$$I_{xx}(\beta, d) := 4 \cdot \left[ \left( d^2 \cdot m_m + d^3 \cdot \frac{\rho_a \cdot A_a}{3} \right) \cdot \cos(\beta \cdot deg)^2 + Z_G{}^2 \cdot \left( m_m + d \cdot \rho_a \cdot A_a \right) \right] \quad gr \cdot mm^2$$

Primeiramente vamos plotar a função objetiva :




$I_{yy}$

Plotamos as função objetiva:




$I_{yy}$

Definimos funções de restrição :

$g_m(d) := m_q(d) - m'_q$ onde : $g_m(d) \le 0$

$g_{rl}(\beta, d) := -d \cdot \sin(\beta \cdot deg) + \frac{d_r}{2} + f_{gap}$ onde : $g_{rl}(\beta, d) \le 0$

$g_{ru}(\beta, d) := -d \cdot \cos(\beta \cdot deg) + \frac{d_r}{2} + f_{gap}$ onde : $g_{ru}(\beta, d) \le 0$

$g_d(d) := -d + 150$ onde : $g_d(d) \le 0$

Plotando as funções de restrição na função objetiva temos :

$\beta_{min} := 0$ $\quad d_{min} := 150$

$\beta_{max} := 90$ $\quad d_{max} := 250$

$d_{gm}(\beta) := g_m(d) + 0.01\cdot\beta = 0 \; = \; 207.7 - 0\cdot\beta$

$Line_{gm} := Trace_{3DLine}\left(\beta_{min}, \beta_{max}, d_{gm}, I_{yy}\right)$

$Plane_{gm} := Trace_{VerticalPlane}\left(\beta_{min}, \beta_{max}, d_{gm}\right)$

$\Delta d_{gm}(\beta) := g_m(d) + 0.01\cdot\beta = 10 \; = \; 214.3 - 0\cdot\beta$

$Line_{\Delta gm} := Trace_{3DLine}\left(\beta_{min}, \beta_{max}, \Delta d_{gm}, I_{yy}\right)$

$Plane_{\Delta gm} := Trace_{VerticalPlane}\left(\beta_{min}, \beta_{max}, \Delta d_{gm}\right)$

$d_{grl}(\beta) := g_{rl}(\beta, d) = 0 \; = \; \frac{75}{\sin(\beta\cdot deg)}$

$Line_{grl} := Trace_{3DLine}\left(18, \beta_{max}, d_{grl}, I_{yy}\right)$

$Plane_{grl} := Trace_{VerticalPlane}\left(18, \beta_{max}, d_{grl}\right)$

$\Delta d_{grl}(\beta) := g_{rl}(\beta, d) = 5 \; = \; \frac{70}{\sin(\beta\cdot deg)}$

$Line_{\Delta grl} := Trace_{3DLine}\left(17, \beta_{max}, \Delta d_{grl}, I_{yy}\right)$

$Plane_{\Delta grl} := Trace_{VerticalPlane}\left(17, \beta_{max}, \Delta d_{grl}\right)$

$d_{gru}(\beta) := g_{ru}(\beta, d) = 0 \; = \; \frac{75}{\cos(\beta\cdot deg)}$

$Line_{gru} := Trace_{3DLine}\left(\beta_{min}, 72, d_{gru}, I_{yy}\right)$

$Plane_{gru} := Trace_{VerticalPlane}\left(\beta_{min}, 72, d_{gru}\right)$

$\Delta d_{gru}(\beta) := g_{ru}(\beta, d) = 5 \; = \; \frac{70}{\cos(\beta\cdot deg)}$

$Line_{\Delta gru} := Trace_{3DLine}\left(\beta_{min}, 73, \Delta d_{gru}, I_{yy}\right)$

$Plane_{\Delta gru} := Trace_{VerticalPlane}\left(\beta_{min}, 73, \Delta d_{gru}\right)$

$d_{gd}(\beta) := g_d(d) + 0.001\cdot\beta = 0 \; = \; 0\cdot\beta + 150$

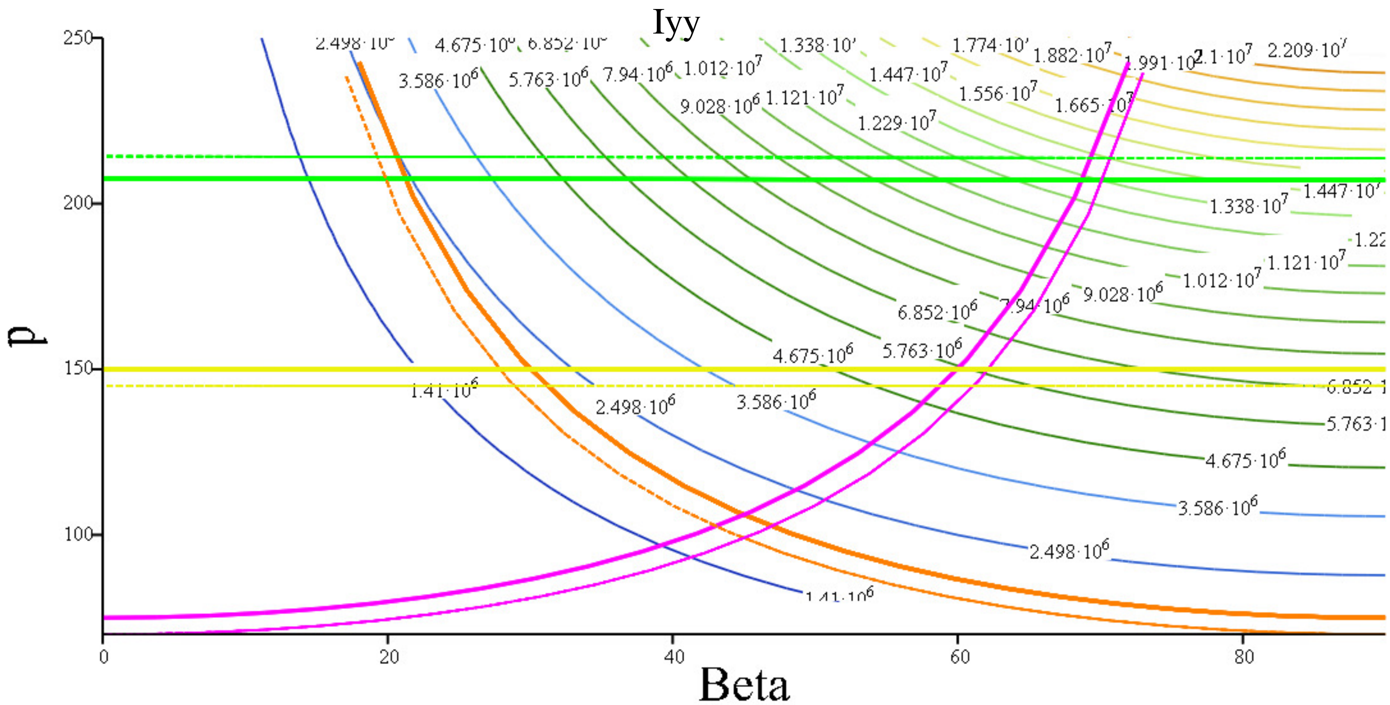
$Line_{gd} := Trace_{3DLine}\left(\beta_{min}, \beta_{max}, d_{gd}, I_{yy}\right)$

$Plane_{gd} := Trace_{VerticalPlane}\left(\beta_{min}, \beta_{max}, d_{gd}\right)$

$\Delta d_{gd}(\beta) := g_d(d) + 0.001\cdot\beta = 5 \; = \; 0\cdot\beta + 145$

$Line_{\Delta gd} := Trace_{3DLine}\left(\beta_{min}, \beta_{max}, \Delta d_{gd}, I_{yy}\right)$

$Plane_{\Delta gd} := Trace_{VerticalPlane}\left(\beta_{min}, \beta_{max}, \Delta d_{gd}\right)$




Legenda:

gm grl gru $g_d$

A definição do Lagrange pode ser vista a seguir. A função de restrição para $d_{min}$ não será adicionada ao Langrangeano, esta condição será imposta diretamente na verificação de cada combinação.

$$L(\beta, d, v_1, v_2, v_3) := I_{yy}(\beta, d) + v_1 \cdot g_m(d) + v_2 \cdot g_{rl}(\beta, d) + v_3 \cdot g_{ru}(\beta, d)$$

Assim :

$$G(\beta, d, v_1, v_2, v_3) := \nabla_{\beta, d, v_1, v_2, v_3} L(\beta, d, v_1, v_2, v_3)$$

$$G(\beta, d, v_1, v_2, v_3) = \begin{bmatrix} d \cdot deg \cdot \left(0.5 \cdot \sin(2 \cdot \beta \cdot deg) \cdot d^2 + 236 \cdot \sin(2 \cdot \beta \cdot deg) \cdot d - 1 \cdot v_2 \cdot \cos(\beta \cdot deg) + v_3 \cdot \sin(\beta \cdot deg)\right) \\ 1.5 \cdot d^2 \cdot \sin(\beta \cdot deg)^2 + 472 \cdot d \cdot \sin(\beta \cdot deg)^2 - 1 \cdot v_2 \cdot \sin(\beta \cdot deg) + 2 \cdot v_3 \cdot \sin\left(\frac{\beta \cdot deg}{2}\right)^2 + 1.5 \cdot v_1 - 1 \cdot v_3 + 1.4 \times 10^3 \\ 1.5 \cdot d - 314 \\ 75 - d \cdot \sin(\beta \cdot deg) \\ 75 - d \cdot \cos(\beta \cdot deg) \end{bmatrix}$$

Analisando cada opção, temos :

$$\frac{\partial}{\partial\beta}L(\beta, d, v_1, v_2, v_3) = \frac{\partial}{\partial\beta}I_{yy}(\beta, d) + v_1\cdot\frac{\partial}{\partial\beta}g_m(d) + v_2\cdot\frac{\partial}{\partial\beta}g_{rl}(\beta, d) + v_3\cdot\frac{\partial}{\partial\beta}g_{ru}(\beta, d)$$

$$\left|\begin{array}{l} \frac{\partial}{\partial\beta}L \geq 0 \quad \text{if} \quad \beta = \beta_{min} \\ \frac{\partial}{\partial\beta}L = 0 \quad \text{if} \quad \beta_{min} < \beta < \beta_{max} \\ \frac{\partial}{\partial\beta}L \leq 0 \quad \text{if} \quad \beta = \beta_{max} \end{array}\right.$$

$$\frac{\partial}{\partial d}L(\beta, d, v_1, v_2, v_3) = \frac{\partial}{\partial d}I_{yy}(\beta, d) + v_1\cdot\frac{\partial}{\partial d}g_m(d) + v_2\cdot\frac{\partial}{\partial d}g_{rl}(\beta, d) + v_3\cdot\frac{\partial}{\partial d}g_{ru}(\beta, d)$$

$$\left|\begin{array}{l} \frac{\partial}{\partial d}L \geq 0 \quad \text{if} \quad d = d_{min} \\ \frac{\partial}{\partial d}L = 0 \quad \text{if} \quad d_{min} < d < d_{max} \\ \frac{\partial}{\partial d}L \leq 0 \quad \text{if} \quad d = d_{max} \end{array}\right.$$

$$\frac{\partial}{\partial v_1}L(\beta, d, v_1, v_2, v_3) = g_m(d)$$

$$\left|\begin{array}{l} \frac{\partial}{\partial v_1}L < 0 \quad \text{if} \quad v_1 = 0 \\ \frac{\partial}{\partial v_1}L = v_1\cdot g_m = 0 \quad \text{if} \quad v_1 > 0 \end{array}\right.$$

$$\frac{\partial}{\partial v_2}L(\beta, d, v_1, v_2, v_3) = g_{rl}(\beta, d)$$

$$\left|\begin{array}{l} \frac{\partial}{\partial v_2}L < 0 \quad \text{if} \quad v_2 = 0 \\ \frac{\partial}{\partial v_2}L = v_2\cdot g_{rl} = 0 \quad \text{if} \quad v_2 > 0 \end{array}\right.$$

$$\frac{\partial}{\partial v_3}L(\beta, d, v_1, v_2, v_3) = g_{ru}(\beta, d)$$

$$\left|\begin{array}{l} \frac{\partial}{\partial v_3}L < 0 \quad \text{if} \quad v_3 = 0 \\ \frac{\partial}{\partial v_3}L = v_3\cdot g_{ru} = 0 \quad \text{if} \quad v_3 > 0 \end{array}\right.$$

Temos 8 combinações diferentes vistas a seguir :

$(v_1 = 0, v_2 = 0, v_3 = 0)$

$(g_m = 0, v_2 = 0, v_3 = 0)$

$(v_1 = 0, g_{rl} = 0, v_3 = 0)$

$(v_1 = 0, v_2 = 0, g_{ru} = 0)$

$(g_m = 0, g_{rl} = 0, v_3 = 0)$

$(g_m = 0, v_2 = 0, g_{ru} = 0)$

$(v_1 = 0, g_{rl} = 0, g_{ru} = 0)$

$(g_m = 0, g_{rl} = 0, g_{ru} = 0)$

Verificamos no gráfico que o ponto mínimo esta sobre a função de restrição $g_{rl}$, assim as possíveis combinações são :

$(v_1 = 0, g_{rl} = 0, v_3 = 0)$ para d qualquer

$(v_1 = 0, g_{rl} = 0, v_3 = 0)$ para $d = d_{min}$

$(g_m = 0, g_{rl} = 0, v_3 = 0)$

Para $(v_1 = 0, g_{rl} = 0, v_3 = 0, d = d_{min})$ , temos :

Primeiramente calculamos β :

$$\beta^* := g_{rl}(\beta, d) = 0 \;\left|\begin{array}{l}\text{substitute}, d = d_{min}\\ \text{solve}, \beta\end{array}\right. = \begin{pmatrix}\dfrac{\pi}{6\cdot deg}\\ \dfrac{5\cdot\pi}{6\cdot deg}\end{pmatrix}$$

Verificando os multiplicadores de Lagrange para as duas opções de β :

$$v^*_1 := g_m(d_{min}) = -87.2$$

$$v^*_2 := g_{rl}(\beta^*, d_{min}) = \begin{pmatrix}0\\0\end{pmatrix}$$

$$v^*_3 := g_{ru}(\beta^*, d_{min}) = \begin{pmatrix}-54.9\\204.9\end{pmatrix}$$

Temos que uma das opções para β não respeita a restrição $g_{ru}$ , assim essa opção será excluída.

$$\beta^* := \beta^*_1 = 30 \quad deg$$

$$v^*_2 := v^*_{2_1}$$

$$v^*_3 := v^*_{3_1}$$

Conferindo o gradiente do Lagrangeano :

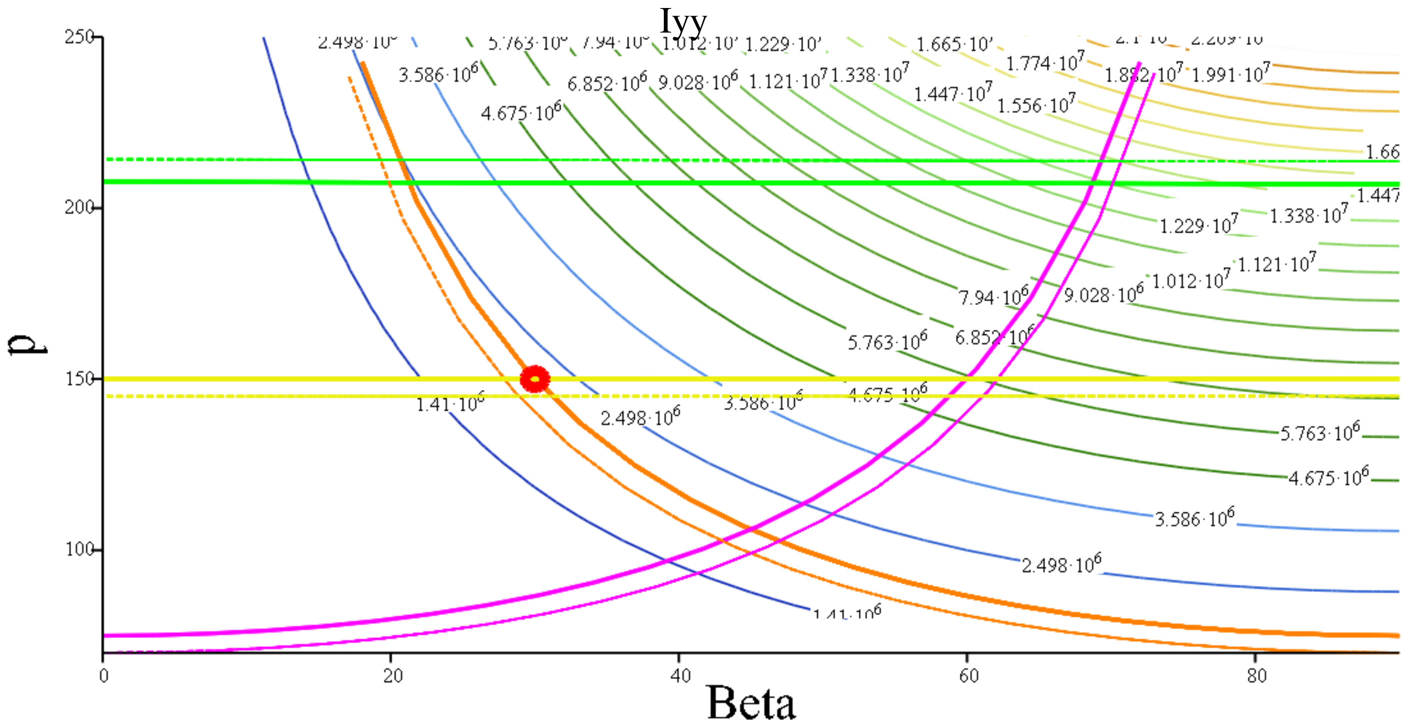
$$G\left(\beta^*, d_{min}, v^*_1, v^*_2, v^*_3\right) = \begin{pmatrix} 14022000\cdot deg\cdot \cos(30\cdot deg)\cdot \sin(30\cdot deg) - 8235.6\cdot deg\cdot \sin(30\cdot deg) - 0\cdot deg\cdot \cos(30\cdot deg) \\ 104820\cdot \sin(30\cdot deg)^2 - 0\cdot \sin(30\cdot deg) + 54.9\cdot \cos(30\cdot deg) + 1229 \\ -87.2 \\ 75 - 150\cdot \sin(30\cdot deg) \\ 75 - 150\cdot \cos(30\cdot deg) \end{pmatrix}$$

$$\begin{pmatrix} 14022000\cdot deg\cdot \cos(30\cdot deg)\cdot \sin(30\cdot deg) - 8235.6\cdot deg\cdot \sin(30\cdot deg) - 0\cdot deg\cdot \cos(30\cdot deg) \\ 104820\cdot \sin(30\cdot deg)^2 - 0\cdot \sin(30\cdot deg) + 54.9\cdot \cos(30\cdot deg) + 1229 \\ -87.2 \\ 75 - 150\cdot \sin(30\cdot deg) \\ 75 - 150\cdot \cos(30\cdot deg) \end{pmatrix} = \begin{pmatrix} 105899.4 \\ 27481.5 \\ -87.2 \\ 0 \\ -54.9 \end{pmatrix}$$

Vemos que o gradiente não é nulo.

Plotando o ponto extremo :

$$P1_{Line} := Trace_{VerticalLine}\left(\beta^{*}, d_{min}, 0.7, 2.5, 20\right)$$




Legenda:

gm grl gru $g_d$