Lecture 4 Numerical Method #1

Solving Systems of Linear Algebraic Equations

Gauss Elimination Method

Part 1. Theory — We will discuss it during the class.

Part 2. Programming

The Gauss elimination method consists of Forward Elimination and Back Substitution.

2-1. Implementation of *Gauss Elimination* in MathCAD

Let's consider a system of 3-linear equations

$$A_{11}x_1 + A_{12}x_2 + A_{13}x_3 = b_1$$
$$A_{21}x_1 + A_{22}x_2 + A_{23}x_3 = b_2$$
$$A_{31}x_1 + A_{32}x_2 + A_{33}x_3 = b_3$$

Then, cast the coefficients and the RHS (right hand side) terms into a matrix form to perform *Forward Elimination*

$$\left[\begin{array}{ccc|c} A_{11} & A_{12} & A_{13} & b_1 \\ A_{21} & A_{22} & A_{23} & b_2 \\ A_{31} & A_{32} & A_{33} & b_3 \end{array}\right]$$

$$\Downarrow$$

$$\underbrace{\left[\begin{array}{ccc|c} A_{11} & A_{12} & A_{13} & b_1 \\ 0 & A'_{22} & A'_{23} & b'_2 \\ 0 & 0 & A''_{33} & b''_3 \end{array}\right]}_{\text{After forward elmination is completed, we have an upper triangular system.}}$$

$$\Downarrow$$

$$\underbrace{\begin{array}{l} x_3 = b''_3 / A''_{33} \\ x_2 = (b'_2 - A'_{23} x_3) / A'_{22} \\ x_1 = (b_1 - A_{12} x_2 - A_{13} x_3) / A_{11} \end{array}}_{\text{Back-Substitution}}$$

A. Pseudocode for Forward Elimination

Iteration #1

1. Determine the pivot term (A_{11}): move down the matrix from one pivot row to the next as the iterations go on

2. Elimination Process:

   2-A Start from the second row and moves to the last row in order to determine the pivot term for each row

   (1) Compute the factor for Row#2

   $$\text{factor} \leftarrow \frac{A_{21}}{A_{11}}$$

   2-B Elimination process moves one column to the next (until the last column: cols(A))

   (2) Eliminate the terms in Row#2

   $$A_{21} \leftarrow \underbrace{A_{21} - \text{factor} * A_{11}}_{\text{Elimination of the 1st column of the 2nd row}}$$

   Elimination process repeats until the last column

$$A_{22} \leftarrow A_{22} - \text{factor}*A_{12}$$
$$A_{23} \leftarrow A_{23} - \text{factor}*A_{13}$$

(3) Modify the RHS

$$b_2 \leftarrow b_2 - \text{factor}*b_1$$

Go back to Step 2: move on to the next row and repeat Step 2 until the iteration reaches to the last row.

(1) Compute the factor for Row#3

$$\text{factor} \leftarrow \frac{A_{31}}{A_{11}}$$

(2) Eliminate terms in Row#3

$$A_{31} \leftarrow \underbrace{A_{31} - \text{factor}*A_{11}}_{\substack{\text{Elimination of} \\ \text{the 1st column} \\ \text{of the 3rd row}}}$$

$$A_{32} \leftarrow A_{32} - \text{factor}*A_{12}$$
$$A_{33} \leftarrow A_{33} - \text{factor}*A_{13}$$

(3) Modify the RHS

$$b_3 \leftarrow b_3 - \text{factor}*b_1$$

Part I : Forward Elimination

Iteration #1

$$\text{Fiteration01}(A,b) := \left|\begin{array}{l} \text{pivot} \leftarrow A_{1,1} \\ \text{for } i \in 2..\text{rows}(A) \\ \quad \left|\begin{array}{l} \text{factor} \leftarrow \dfrac{A_{i,1}}{\text{pivot}} \\ \text{for } k \in 1..\text{cols}(A) \\ \quad A_{i,k} \leftarrow A_{i,k} - \text{factor}\cdot A_{1,k} \\ b_i \leftarrow b_i - \text{factor}\cdot b_1 \end{array}\right. \\ \text{out} \leftarrow (A \;\; b) \end{array}\right.$$

$$A := \begin{pmatrix} 2 & 3 & -1 \\ 0 & 1 & -3 \\ 4 & 5 & -2 \end{pmatrix} \qquad b := \begin{pmatrix} 7 \\ -5 \\ 10 \end{pmatrix}$$

$\text{result01} := \text{Fiteration01}(A,b)$ $\qquad$ $\text{result01} = (\{3,3\} \;\; \{3,1\})$

$A01 := \text{result01}_{1,1}$ $\qquad$ $b01 := \text{result01}_{1,2}$

$$A01 = \begin{pmatrix} 2 & 3 & -1 \\ 0 & 1 & -3 \\ 0 & -1 & 0 \end{pmatrix} \qquad b01 = \begin{pmatrix} 7 \\ -5 \\ -4 \end{pmatrix}$$

At this point, Forward Elimination for Column #1 has been completed.

Iteration #2

1. Determine the pivot term, i.e., A_{22}:

Remember $[A]$ *and* $\{b\}$ *have already been modified once (after the first iteration) so that, for example,* A_{22} *and* b_2 *in the following pseudocode really mean* $A01_{22}$ *and* $b01_2$ *(refer to the Mathcad code on the previous page), respectively. However, for simplicity in developing an algorithm, I will use the notations of* A_{22} *and* b_2*, etc., instead of* $A01_{22}$ *and* $b01_2$*, etc. Please be aware of that.*

   This step continues (move down the matrix from one pivot row to the next) as the iterations go on

2. Elimination Process: starting from the second row and moves to the last row
   a. Compute the factor for Row#3

      $$\text{factor} \leftarrow \frac{A_{32}}{\text{pivot}} \quad (\text{or } \frac{A_{32}}{A_{22}})$$

   b. Eliminate the terms in Row#3

      $$A_{32} \leftarrow \underbrace{A_{32} - \text{factor}*A_{22}}_{\substack{\text{Elimination of} \\ \text{the 2nd column} \\ \text{in the 3rd row}}}$$

      then elimination process moves one column to the next (until the last column: cols(A), but in this specific case, cols(A) = 3)

      $$A_{33} \leftarrow A_{33} - \text{factor}*A_{23}$$

   c. Modify the RHS

      $$b_3 \leftarrow b_3 - \text{factor}*b_2$$

iteration#2

$$\text{Fiteration02}(A, b) := \left|\begin{array}{l} \text{pivot} \leftarrow A_{2,2} \\ \text{for } i \in \text{rows}(A) \\ \quad \left|\begin{array}{l} \text{factor} \leftarrow \dfrac{A_{i,2}}{\text{pivot}} \\ \text{for } k \in 1..\text{cols}(A) \\ \quad A_{i,k} \leftarrow A_{i,k} - \text{factor} \cdot A_{2,k} \\ b_i \leftarrow b_i - \text{factor} \cdot b_2 \end{array}\right. \\ \text{out} \leftarrow (A \quad b) \end{array}\right.$$

$\text{result02} := \text{Fiteration02}(A01, b01)$

$\text{result02} = (\{3,3\} \quad \{3,1\})$

$$\text{result02}_{1,1} = \begin{pmatrix} 2 & 3 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & -3 \end{pmatrix} \qquad \text{result02}_{1,2} = \begin{pmatrix} 7 \\ -5 \\ -9 \end{pmatrix}$$

$A02 := \text{result02}_{1,1}$

$b02 := \text{result02}_{1,2}$

$$A02 = \begin{pmatrix} 2 & 3 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & -3 \end{pmatrix} \qquad b02 = \begin{pmatrix} 7 \\ -5 \\ -9 \end{pmatrix}$$

Identify a pattern and then generalize the pseudocode for iterations #1 and #2.
What will be a complete algorithm for Forward Elimination?

2.2 Implementation of *Back-Substitution* in Mathcad.

Forward Elimination has been completed for $[\mathrm{A}]$ *and* $\{b\}$. *Thus,* A_{22} *and* b_2 *in the following pseudocode really mean* $\mathrm{A02}_{33}$ *and* $b02_3$ *(refer to the Mathcad code above), respectively. However, for simplicity in developing an algorithm, I will use the notations of* A_{22} *and* b_2*, etc., instead of* $\mathrm{A01}_{22}$ *and* $b01_2$*, etc.*

Iteration #1

1. Solve for the unknown in the last row (Row#3)

$$x_3 \leftarrow \frac{b_3}{\mathrm{A}_{33}}$$

: moves up the matrix from the last row to the next (one row up) as each iteration goes on

2. Back-Substitute x_3 into Row#2

$i \leftarrow 2$

: We use "i" to count iterations of back-substitution going on from one row (starting from (n-1)th row) to the next (n-2)th until the first row.

$sum \leftarrow 0.0$

(Why do you need this? please read a several lines in the following)

2-1. The dot product between $[\mathrm{A}]$ and $\{b\}$

(Recall the general equation of Back-Substitution:

$\sum_{j=i+1}^{n} \mathrm{A}_{ij} * x_j$ from $x_i = \dfrac{b_i - \sum_{j=i+1}^{n} \mathrm{A}_{ij} * x_j}{\mathrm{A}_{ii}}$

for $j = i+1, i+2 .. \mathrm{cols(A)}$

: column j of $[\mathrm{A}]$ is multiplied by row j of x

$$sum \leftarrow sum + \mathrm{A}_{2j} * x_j \quad (\text{or } sum \leftarrow sum + \mathrm{A}_{ij} * x_j)$$

(We use a variable "sum" to accumulate the summation of a dot product between $\mathrm{A}_{ij} * x_j$. Thus, I have to define this new variable before this line. That's a reason why I assign a value of zero to variable "sum" in step #2, previously. However, there is a more important reason. I will talk about this during the class. If NOT, please remind me of this)

2-2 Assign "*sum*" to x_2 (or x_i)

$x_2 \leftarrow \frac{b_2 - sum}{A_{22}}$ (or $x_i \leftarrow \frac{b_i - sum}{A_{ii}}$)

: where "*sum*" containing $A_{23} * x_3$ is passed from the result of the FOR-loop. A Mathcad code for iteration #1 is presented in the following:

Part II : Back-Substitution

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Iteration #1

These are the matrices resulted from
Forward Elimination

$$A := \begin{pmatrix} 2 & 3 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & -3 \end{pmatrix} \qquad b := \begin{pmatrix} 7 \\ -5 \\ -9 \end{pmatrix}$$

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Biteration01(A, b) := | x_3 ← b_3 / A_3,3
                      | i ← 2
                      | sum ← 0.0
                      | for  j ∈ i + 1, i + 2 .. cols(A)
                      |    sum ← sum + A_i,j · x_j
                      | x_i ← (b_2 − sum) / A_i,i
                      | x
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unknown_x01 := Biteration01(A, b)

$$\text{unknown\_x01} = \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix}$$

At this point, we have found two unknowns x_3 , x_2 since completion of iteration#1.

Iteration #2

1. Solve for the unknown x_1 in the first row (Since we are dealing with a 3X3 matrix here) (in general, move up the matrix from the previous row to the next as the iterations go on)

In other words, Back-Substitute x_3 , x_2 into Row#1

$i \leftarrow 1$: Row#1

$sum \leftarrow 0.0$

2. The dot product between [A] and $\{b\}$

   (recall the general equation of Back-Substitution:

$$\sum_{j=i+1}^{n} A_{ij} * x_j \text{ from } x_i = \frac{b_i - \sum_{j=i+1}^{n} A_{ij} * x_j}{A_{ii}})$$

for $j = i+1, i+2$ .. cols(A)
: column j of [A] is multiplied by row j of x

$$sum \leftarrow sum + A_{2j} * x_j \text{ (or } sum \leftarrow sum + A_{ij} * x_j)$$

(We use a variable "sum" to accumulate the summation of all the dot products between $A_{ij} * x_j$)

a. Assign "*sum*" ($= A_{12} * x_2 + A_{13} * x_3$) to x_2 (or x_i)

$$x_1 \leftarrow \frac{b_1 - sum}{A_{11}} \quad (\text{or } x_i \leftarrow \frac{b_i - sum}{A_{ii}})$$

where "*sum*" containing $A_{12} * x_2 + A_{13} * x_3$ is passed from the result of the "for"-loop.

Iteration #2

x_3 and x_2 were computed in iteration #1. However, to develop the Biteration02 as a continuation from Biteration01, I re-compute these two variables herein

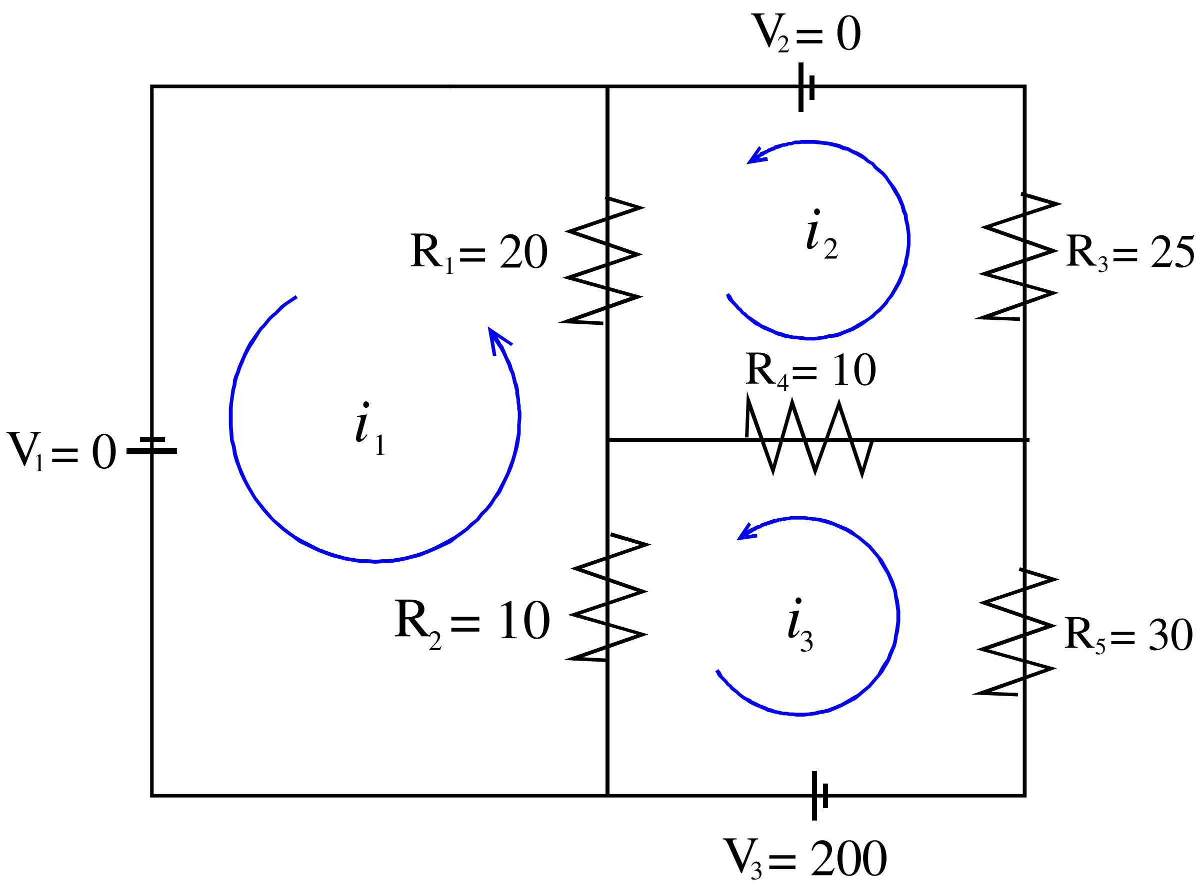
$$\text{Biteration02}(A,b) := \left| \begin{array}{l} x_3 \leftarrow \dfrac{b_3}{A_{3,3}} \\ x_2 \leftarrow \dfrac{b_2 - A_{2,3} \cdot x_3}{A_{2,2}} \\ i \leftarrow 1 \\ sum \leftarrow 0.0 \\ \text{for } j \in i+1, i+2 .. \text{cols}(A) \\ \quad sum \leftarrow sum + A_{i,j} \cdot x_j \\ x_1 \leftarrow \dfrac{b_1 - sum}{A_{i,i}} \\ x \end{array} \right.$$

$$\text{unknown\_x02} := \text{Biteration02}(A,b)$$

$$\text{unknown\_x02} = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}$$

Identify a pattern and then generalize the pseudocode for iterations #1 and #2.
What will be a complete algorithm for Back-Substitution?

Example of using the Gauss Elimination Method to solve a system of linear algebraic equations: How the Gauss Elimination method is applied to solve an electrical engineering problem.



A simple electrical network contains a number of resistances and three source of electromotive force (3 batteries) as shown above.

We define each unknown current to be positive if it flows in the counterclockwise direction; if a computed current i is negative, the flow is clockwise.

The analysis tools come from elementary physics:

1. The sum of the voltage drops around a closed loop is zero
2. The voltage drop across a resistor is the product of the current and the resistance.

The analysis of the voltages around the three loops gives three equations, which we solve by ***Gauss Elimination***. → We solve for electric current in each loop !

FLOW AROUND LEFT LOOP: $20(i_1 - i_2) + 10(i_1 - i_3) = 0$

FLOW AROUND UPPER RIGHT LOOP: $25i_2 + 10(i_2 - i_3) + 20(i_2 - i_1) = 0$

FLOW AROUNF LOWER RIGHT LOOP: $30i_3 + 10(i_3 - i_2) + 10(i_3 - i_1) = 200$

Thus,

$$\begin{aligned} &20(i_1 - i_2) + 10(i_1 - i_3) = 0 && \rightarrow 30i_1 - 20i_2 - 10i_3 = 0 \\ &25i_2 + 10(i_2 - i_3) + 20(i_2 - i_1) = 0 && \rightarrow -20i_1 + 55i_2 - 10i_3 = 0 \\ &30i_3 + 10(i_3 - i_2) + 10(i_3 - i_1) = 200 && \rightarrow -10i_1 - 10i_2 + 50i_3 = 200 \end{aligned}$$

In a matrix form, the system of three linear equations can be written as:

$$\begin{aligned} +30i_1 - 20i_2 - 10i_3 &= 0 \\ -20i_1 + 55i_2 - 10i_3 &= 0 \\ -10i_1 - 10i_2 + 50i_3 &= 200 \end{aligned}$$

$$\begin{bmatrix} 30 & -20 & -10 \\ -20 & 55 & -10 \\ -10 & -10 & 50 \end{bmatrix} \begin{Bmatrix} i_1 \\ i_2 \\ i_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 200 \end{Bmatrix}$$

First iteration, pivot = 30, factor for row #2 = -2/3; II-factor*I, and factor for row#3 = -1/3; III-factor*I

$$\underbrace{\left[\begin{array}{ccc|c} 30 & -20 & -10 & 0 \\ 0 & (125/3) & (-50/3) & 0 \\ 0 & (-50/3) & (140/3) & 200 \end{array}\right]}_{\text{After 1st iteration is completed}}$$

Second iteration; III-factor*II where pivot = (125/3), factor for row #3 = -2/5

$$\underbrace{\left[\begin{array}{ccc|c} 30 & -20 & -10 & 0 \\ 0 & (125/3) & (-50/3) & 0 \\ 0 & 0 & 40 & 200 \end{array}\right]}_{\text{Since 2nd iteration was completed, the system has been transformed into an upper triangular system}}$$

Back-Substitution

$$40 * i_3 = 200 \quad \Rightarrow \quad i_3 = 5$$
$$(125/3) * i_2 - (50/3) * i_3 = 0 \quad \Rightarrow \quad i_2 = 2$$
$$30 * i_1 - 20 * i_2 - 10 * i_3 = 0 \quad \Rightarrow \quad i_1 = 3$$

Therefore,

$$\begin{Bmatrix} i_1 \\ i_2 \\ i_3 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 2 \\ 5 \end{Bmatrix}$$