5.1 Vertical stiffness of pile groups

The vertical settlement of a particular pile in a group is given by:

$$w_k = w_1\left(\sum_{j=1}^{n}(V_j\,\alpha_{v_{kj}})\right) \qquad 5.2$$

where: w_1 is the settlement of the pile under unit load $= 1/K_{V\,(\text{isolated pile})}$,
V_j is the vertical force carried by the jth pile,
and $\alpha_{V_{kj}}$ is the interaction factor between piles k and j, $= 1$ when $k = j$.

Randolf and Wroth (1979) provide a simple means for estimating interaction factors between rigid piles. They developed expressions for the vertical stiffness of a single pile and a 2x1 pile group. From these the interaction factor is obtained from:

$$\alpha_v = \frac{1 - \dfrac{s}{(D/\pi + s)} + \pi(1-\nu)\rho\mathscr{L}\left(\dfrac{1}{\Upsilon} - \dfrac{1}{\Gamma}\right)}{1 + \pi(1-\nu)\rho\mathscr{L}/\Upsilon} \qquad 5.3$$

where: s is the pile spacing (centre to centre),
ρ is the ratio $E_{SL/2}/E_{SL}$ (a measure of the inhomogeneity of the soil profile along the pile shaft = 1 for a constant modulus profile, = 0.5 for a linear distribution of modulus with depth, E_{SL} is the soil modulus at the pile tip and $E_{SL/2}$ is the modulus at a depth of L/2),
Υ is $\ln(2r_m/D)$ with $r_m = 2.5(1-\nu)\rho L$,
Γ is $\ln(2r_m^2/Ds)$.

As an example of the use of this equation we will develop an expression for the vertical stiffness of a 2x2 pile group, the details of which are shown in Fig. 5.2. If unit load is applied to all the piles the settlement of each pile is:

$$w = w_1(1 + \alpha_{v12} + \alpha_{v13} + \alpha_{v14})$$

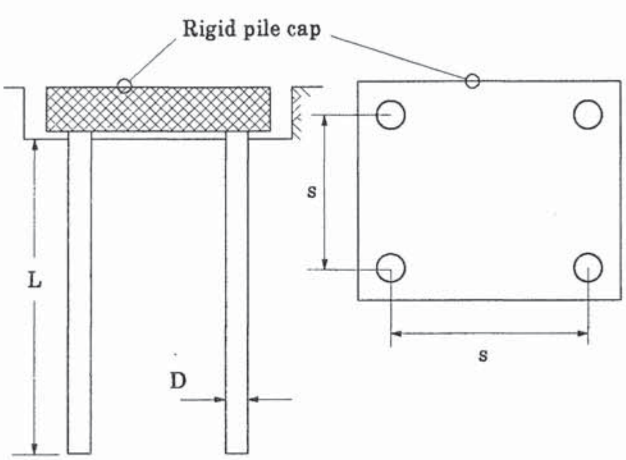


Fig. 5.2 *Definition diagram for a free-standing fixed-head 2x2 pile group.*

because of symmetry $\alpha_{v12} = \alpha_{v14}$, so that:

$$w = w_1(1 + 2\alpha_{v12} + \alpha_{v13})$$

Because we have a 2x2 pile group the load carried by all piles is the same, so the vertical stiffness of the group becomes:

$$K_{VG\ 2x2\ group} = \frac{4K_{V\ isolated\ pile}}{(1 + 2\alpha_{v12} + \alpha_{v13})}$$

This result is easily generalised to cater for a group with n piles:

$$K_{VG\ n\ pile\ group} = K_{V\ isolated\ pile}\left(\sum_{i=1}^{n}\left(\frac{1}{\sum_{j=1}^{n}\alpha_{v_{ij}}}\right)\right) \qquad 5.4$$

Example 5.1 Consider a 2x2 pile group having piles with the same dimensions and properties as those in example 4.1 and which are also embedded in the same soil profile. Estimate the vertical stiffness of the group when (i) the spacing is 5 pile diameters and (ii) when it is 10 pile diameters.

From the earlier example K = 1000 and, from equation 4.1,

$K_V = 1.9 \times 25 \times 0.75 \times 26.7^{0.67} \times 1000^{-0.03}$
$= 267.4$ kN/mm
$r_m = 2.5 \times (1 - 0.5) \times 20 = 25$ m
$\Upsilon = \ln(2 \times 25/0.75) = 4.2$

(i) 5 pile diameters

Spacing between piles 1 & 2 and 1 & 4 is $5 \times 0.75 = 3.75$ m
This gives $\Gamma_{12} = \ln(2 \times 25^2/0.75 \times 3.75) = 6.10$.

Substitution into equation 5.3 gives $\alpha_{v12} = 0.288$

Spacing between piles 1 & 3 is $5 \times \sqrt{2} \times 0.75 = 5.30$ m.
This gives $\Gamma_{13} = \ln(2 \times 25^2/0.75 \times 5.30) = 5.75$.

Substitution into equation 5.3 gives $\alpha_{v13} = 0.249$

$K_{VG} = 4 \times 267.4/(1 + 2 \times 0.288 + 0.249)$
$= 586$ kN/mm.

Considering the "efficiency" of the group as the ratio of the group vertical stiffness to the sum of the individual stiffnesses of the isolated piles, we find that the group has a vertical stiffness efficiency of 55%.

(ii) 10 pile diameters

Spacing between piles 1 & 2 and 1 & 4 is $10 \times 0.75 = 7.5$ m
This gives $\Gamma_{12} = \ln(2 \times 25^2/0.75 \times 3.75) = 5.40$.

Substitution into equation 5.3 gives $\alpha_{v12} = 0.205$

Spacing between piles 1 & 3 is $10 \times \sqrt{2} \times 0.75 = 10.6$ m.
This gives $\Gamma_{13} = \ln(2 \times 25^2/0.75 \times 5.30) = 5.06$.

Substitution into equation 5.3 gives $\alpha_{v13} = 0.156$

From equation 5.4:

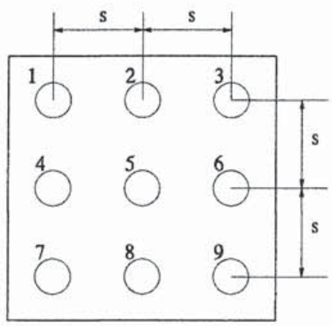
$$K_{VG} = 4\times 267.4/(1 + 2\times 0.205 + 0.156)$$
$$= 683 \text{ kN/mm}.$$

With this group stiffness we find that the group has a vertical stiffness efficiency of 64%.

Usually the settlement of all the piles in a group will be the same, as the piles are connected to a rigid cap. This means that the vertical load will not be distributed equally amongst the piles of a group having many piles. Example 5.1 for a 2x2 group does not illustrate this.

Example 5.2 Consider a 3x3 pile group with a rigid pile cap carrying a vertical load of 1000 kN. The piles are the same as those for example 5.1. Find the vertical settlement of the group and the load carried by each pile for a pile spacing of 5 diameters.

The details of the example are given in the following diagram:



Piles 1, 3, 7, & 9 carry the same vertical load, V_a

Piles 2, 4, 6, & 8 carry the same vertical load, V_b

Piles 5 carries V_c

The preliminary details are as for example 5.1.

There is symmetry in the group so there are only three different pile forces, the force carried by the corner piles, that by the mid-side piles, and that by the centre pile. There are 5 different interactions to consider: $1 \Rightarrow 2$, $1 \Rightarrow 3$, $1 \Rightarrow 5$, $1 \Rightarrow 6$, and $1 \Rightarrow 9$. The steps in the calculation of the interaction factors with equation 5.3 are:

Inter-action	Spacing (m)	Γ	α_v
$1 \Rightarrow 2$	3.75	6.097	0.288
$1 \Rightarrow 3$	7.50	5.404	0.205
$1 \Rightarrow 5$	5.30	5.750	0.249
$1 \Rightarrow 6$	8.39	5.292	0.190
$1 \Rightarrow 9$	10.61	5.057	0.156

As there are three different forces we need three equations to solve the system. Firstly from vertical equilibrium:

$$V_c + 4V_a + 4V_b = V$$

Since the pile cap is rigid the settlement of all the piles must be the same. From $w_a = w_c$:

$$(1 + 2\alpha_{v13} + \alpha_{v19})V_a + 2(\alpha_{v12} + \alpha_{v16})V_b + \alpha_{v15}V_c =$$
$$4\alpha_{v15}V_a + 4\alpha_{v12}V_b + V_c$$

From $w_b = w_c$:

$$2(\alpha_{v12} + \alpha_{v16})V_a + (1 + \alpha_{v13} + 2\alpha_{v16})V_b + \alpha_{v12}V_c =$$
$$4\alpha_{v15}V_a + 4\alpha_{v12}V_b + V_c$$

Making the required substitutions and solving the three equations gives:

$V_a = 130$, $V_b = 102$, and $V_c = 72$ kN.

We see that the centre pile carries the smallest force and the corner piles the largest. The settlement of the group (evaluated for the centre pile) is:

$$w = (4\alpha_{v15}V_a + 4\alpha_{v12}V_b + V_c)/267.4 = 1.2 \text{ mm}.$$

The vertical stiffness of the group is:

$$K_{VG} = 1000/1.2 = 833 \text{ kN/mm}.$$

The stiffness efficiency of the group is: $833\times 100/9\times 267.4 = 35\%$.

5.2 Rotational stiffness of a free-head free-standing pile group

The mechanism by which this stiffness is generated is shown in Fig. 5.3. As the piles of the group are of the free-head variety the piles themselves offer no rotational restraint. Thus the rotational stiffness of the group is generated by the vertical stiffness of the piles. If the moment acts about an axis parallel to one the sides of the group then two of the piles will receive an increment of downward force and two will

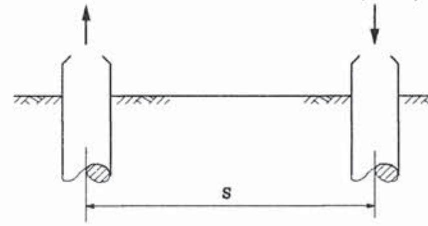


Fig. 5.3 *Rotational stiffness mechanism for a free-standing free-head pile group.*

receive an increment of upward force i.e. the loading is asymmetrical. The vertical force increments carried by the piles of a 2x2 group, as a consequence of the moment M, are $\pm M/2s$. The vertical displacement of the piles caused by the moment is now:

$$w = \pm V_1 w_1 (1 + \alpha_{v14} - \alpha_{v12} - \alpha_{v13})$$

because of symmetry $\alpha_{14} = \alpha_{12}$, so the pile displacement is:

$$w = \frac{\pm M w_1 (1 - \alpha_{v13})}{2s}$$

The rotation of the pile group is:

$$\theta = \frac{M w_1 (1 - \alpha_{v13})}{s^2}$$

Thus the rotational stiffness from the asymmetric axial loading of the piles becomes:

$$K_{\theta V \ 2x2 \ group} = \frac{K_{V \ isolated \ pile} s^2}{(1 - \alpha_{v13})} \qquad 5.5$$

Example 5.3 Consider a 2x2 free standing free head pile group with a moment applied about an axis parallel to the side. The piles and soil profile are the same as in examples 5.1 and 5.2, the pile spacing is 5 diameters. Evaluate the rotational stiffness of the group.

Substituting into equation 5.5 gives:

$$K_{\theta V} = 267.4 \times (5 \times 0.75)^2 / (1 - 0.249)$$
$$= 5007 \text{ kNm/mrad.}$$

Example 5.4 Consider the same pile group as in example 5.3 but now evaluate the rotational stiffness about a diagonal axis.

The applied moment is equilibrated by vertical forces generated in the piles off the diagonal about which the moment is applied. The piles on the diagonal axis carry no force generated by the the moment. If the force is denoted by ΔV then the settlement of the corner piles is:

$$w = w_1 \Delta V (1 - \alpha_{v13})$$

the rotation is:

$$\theta = 2w/\sqrt{2}s$$

the moment is:

$$M = \sqrt{2} s \Delta V$$

The rotational stiffness is:

$$K_{\theta V \ diag} = M/\theta = s^2 / w_1 (1 - \alpha_{v13})$$

$$= 5007 \text{ kNm/mrad.}$$

5.3 LATERAL STIFFNESS OF PILE GROUPS

Poulos and Davis (1980) introduce interaction factors for laterally loaded piles:

$$\alpha_u = \frac{\text{Displacement caused by unit action on an adjacent pile}}{\text{Displacement from unit action applied at pile head}} \qquad 5.6$$

$$\alpha_\theta = \frac{\text{Rotation caused by unit action on an adjacent pile}}{\text{Rotation from unit action applied at the pile head}} \qquad 5.7$$

There are a number of different types of lateral interaction factors:

- α_{uH} $\alpha_{\theta H}$: values of α_u and α_θ for a free head pile subject to horizontal load only,
- α_{uM} $\alpha_{\theta M}$: values of α_u and α_θ for a free head pile subject to moment loading only,
- α_{uF}: value of α_u for a fixed head pile group.

Values of α for various conditions are given by Poulos and Davis. For a two pile group 17 diagrams are needed to cover the range of possibilities. The use of the simple formulae given below is thus an attractive alternative.

The horizontal displacements for a group containing more than two piles are obtained from an equation similar to that for the vertical displacement of a vertically loaded pile group. For a pile group with n free head piles the displacement of pile k is given by:

$$u_k = u_1 \left(\sum_{j=1}^{n} (H_j \alpha_{uH_{kj}}) \right) \qquad 5.8$$

where: u_1 is the displacement of an isolated free head pile under unit horizontal force ($= f_{uH}$),
H_j is the horizontal force carried by pile j, α_{uHkj} is the value of α_{uH} for piles k and j corresponding to the spacing between them and the angle between the direction of loading and the line joining the centres of the piles, Fig. 5.4, $\alpha_{uHkj} = 1$ when $j = k$.

In addition there is an equilibrium equation for the group:

$$H = \sum_{j=1}^{n} H_j \qquad 5.9$$

In a free head pile group with n piles, each sustaining a

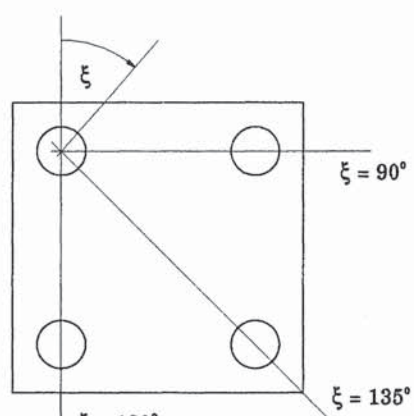


Fig. 5.4 *Angular relationship between various piles in a group.*

known force, there are n unknown pile displacements. In a fixed head pile group there are n+1 unknowns: the n pile forces and the displacement. Symmetry of a pile group can be used to reduce the number of unknowns. The displacement of a fixed head group is obtained with equation 5.8 in which the appropriate unit reference displacement for a single pile in the group.

Randolf (1981) gives the following equation for α_{uF}:

$$\alpha_{uF} = 0.3\left(\frac{D}{s}\right)\left[2(1+\nu)K\right]^{0.143}\left(1+\cos^2\xi\right) \quad 5.10a$$

where: ξ is defined in Fig. 5.4.

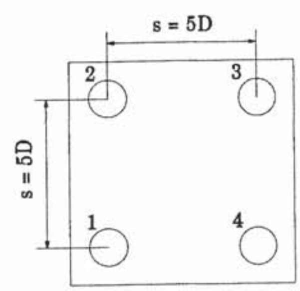
If the values for α_{uF} calculated with equation 5.10a exceed 0.5 Randolf substitutes the following:

$$\alpha_{uF} = 1 - \frac{1}{4\,\alpha_{uF\ \mathrm{from 5.10a}}} \quad 5.10b$$

The value of α_{uF} given by equation 5.10 is applicable to a soil profile having a constant modulus with depth. Randolf suggests that for a profile with a linear distribution of modulus with depth that α_{uF} is half that from the above equation.

Example 5.5 Consider a 2x2 fixed-head pile group with the same properties as that in example 5.1. For a pile spacing of 5 diameters and zero rotation of the pile cap determine the lateral stiffness of the groupK_{HFG}.

The layout of the pile group is as follows:



From the inverse of equation 3.24 the lateral stiffness of an isolated fixed head pile in a uniform soil profile is:

$K_{HF} = 25 \times 0.75/0.8 \times 1000^{-0.18} = 81.3$ MN/m (kN/mm).

The constant term for the interaction factors is:

$C = 0.3 \times 0.75 \times (2 \times (1 + 0.5) \times 1000)^{0.143} = 0.707$

The steps in the calculation of the interaction factors with equation 5.10 are given in the table at the top of the next column.

Using these interaction factors the lateral stiffness of the group is:

$K_{HFG} = 4 \times 81.3/(1 + 0.377 + 0.189 + 0.20) = 184.1$ kN/mm.

Inter-action	Spacing metres	ξ degrees	α_{uF}
1 ⇒ 2	3.75	0	0.377
1 ⇒ 3	5.30	45	0.189
1 ⇒ 4	3.75	90	0.200

Example 5.6 Extend example 5.5 to cover a 3x3 fixed head pile group which is subject to a horizontal force of 1000 kN.

There are now nine piles and each interacts with the other eight. The full matrix of ξ angles is:

$$\xi = \begin{pmatrix} 0 & 0 & 0 & 90 & 45 & 27 & 90 & 63 & 45 \\ 180 & 0 & 0 & 135 & 90 & 45 & 117 & 90 & 63 \\ 180 & 180 & 0 & 153 & 135 & 90 & 135 & 117 & 90 \\ 90 & 45 & 27 & 0 & 0 & 0 & 90 & 45 & 27 \\ 135 & 90 & 45 & 180 & 0 & 0 & 135 & 90 & 45 \\ 153 & 135 & 90 & 180 & 180 & 0 & 153 & 135 & 90 \\ 90 & 63 & 45 & 90 & 45 & 27 & 0 & 0 & 0 \\ 117 & 90 & 63 & 135 & 90 & 45 & 180 & 0 & 0 \\ 135 & 117 & 90 & 153 & 135 & 90 & 180 & 180 & 0 \end{pmatrix}$$

The first row of the ξ matrix deals with the interactions for pile 1 (using the numbering of the figure in example 5.2). The first entry in the row is the angle between pile 1 and itself, the second is between pile 1 and 2, the third between pile 1 and 3, the fourth entry in row one is the angle from pile 1 to pile 4, the seventh is the angle from pile 1 to pile 7, and the final value in the first row is the angle from pile 1 to pile 9. Each succeeding row deals with the interaction angles for one of the other piles, the second row for pile 2 and the third for pile 3 etc. The pattern of values in the matrix indicates the symmetry of the pile group.

The interaction factors calculated with equation 5.10 are:

$\alpha_{hF} =$

$$\begin{pmatrix} 1 & 0.377 & 0.189 & 0.189 & 0.200 & 0.152 & 0.094 & 0.101 & 0.100 \\ 0.377 & 1 & 0.377 & 0.200 & 0.189 & 0.200 & 0.101 & 0.094 & 0.101 \\ 0.189 & 0.377 & 1 & 0.152 & 0.200 & 0.189 & 0.100 & 0.101 & 0.094 \\ 0.189 & 0.200 & 0.152 & 1 & 0.377 & 0.189 & 0.189 & 0.200 & 0.152 \\ 0.200 & 0.189 & 0.200 & 0.377 & 1 & 0.377 & 0.200 & 0.189 & 0.200 \\ 0.152 & 0.200 & 0.189 & 0.189 & 0.377 & 1 & 0.152 & 0.200 & 0.189 \\ 0.094 & 0.101 & 0.100 & 0.189 & 0.200 & 0.152 & 1 & 0.377 & 0.189 \\ 0.101 & 0.094 & 0.101 & 0.200 & 0.189 & 0.200 & 0.377 & 1 & 0.377 \\ 0.100 & 0.101 & 0.094 & 0.152 & 0.200 & 0.189 & 0.189 & 0.377 & 1 \end{pmatrix}$$

These are interpreted in the same manner as the above matrix for interaction angles. The diagonal terms equal to 1.0 account for each pile interacting with itself. It is apparent that there is symmetry in the table which could be exploited in solving the equations.

The equations which have to be solved to obtain the unknown actions are an equilibrium equation and eight equations obtained by equating horizontal displacements, $u_1 = u_2$, $u_2 = u_3$ etc. This gives a system of equations of the form:

$$\mathbb{C}\mathbb{H} = \mathbb{L}$$

where: $\mathbb{C}$ is a 9x9 matrix of coefficients,
$\mathbb{H}$ is a vector of pile loads, with a transpose: $(H_1, H_2, H_3, H_4, H_5, H_6, H_7, H_8, H_9)$
and $\mathbb{L}$ is the applied load vector, with a transpose: $(0, 0, 0, 0, 0, 0, 0, 0, H)$.

The coefficient matrix is:

$$\mathbb{C} = \begin{pmatrix} 0.62 & -0.62 & -0.19 & -0.01 & 0.01 & -0.05 & -0.01 & 0.01 & -0.00 \\ 0.19 & 0.62 & -0.62 & 0.05 & -0.01 & 0.01 & 0.00 & -0.01 & 0.01 \\ 0.0 & 0.18 & 0.85 & -0.85 & -0.18 & 0.0 & -0.09 & -0.10 & -0.06 \\ -0.01 & 0.01 & -0.05 & 0.62 & -0.62 & -0.19 & -0.01 & 0.01 & -0.05 \\ 0.05 & -0.01 & 0.01 & 0.19 & 0.62 & -0.62 & 0.05 & -0.01 & 0.01 \\ 0.05 & 0.10 & 0.09 & 0.0 & 0.18 & 0.85 & -0.85 & -0.18 & 0.0 \\ -0.01 & 0.01 & -0.00 & -0.01 & 0.01 & -0.05 & 0.62 & -0.62 & -0.19 \\ 0.00 & -0.01 & 0.01 & 0.05 & -0.01 & 0.01 & 0.19 & 0.62 & -0.62 \\ 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 \end{pmatrix}$$

The first row of the coefficient matrix is obtained from equating u_1 and u_2, thus it is the difference between the row 1 and 2 of the α_{hF} matrix; row 2 of $\mathbb{C}$ is the difference between rows 2 and 3 of α_{hF} etc. The final row of $\mathbb{C}$ is the equilibrium equation.

The solution of this set of equations, with $\mathbb{H}$ expressed as a 3x3 matrix to show the distribution of load on the piles of the group, is:

$$H_p = \begin{pmatrix} 138.1 & 85.8 & 138.1 \\ 110.9 & 54.1 & 110.9 \\ 138.1 & 85.8 & 138.1 \end{pmatrix} \text{ kN}$$

Once the pile head forces have been obtained the head moments can be found using equation 3.25.

The lateral displacement of the group is obtained by substitution of the loads and the influence coefficients into equation 5.8. Performing the calculations for pile 1 gives:

$$\begin{aligned} u_G &= (1/81.3)\times(1.0\times138.1 + 0.377\times85.8 + 0.189\times138.1 \\ &\quad + 0.189\times110.9 + 0.200\times54.1 + 0.152\times110.9 \\ &\quad + 0.094\times138.1 + 0.101\times85.8 + 0.100\times138.1) \\ &= 3.45 \text{ mm.} \end{aligned}$$

The lateral stiffness of the pile group is:

$$K_{HFG} = 1000/3.45 = 290 \text{ kN/mm.}$$

The efficiency of the group is: $290\times100/9\times81.3 = 40\%$.

Randolf gives the following relations for calculating other interaction factors from α_{uF}: factors are obtained from:

$$\begin{aligned} \alpha_{uH} &= \frac{5\alpha_{uF}}{6} \\ \alpha_{uM} &\approx \alpha_{\theta H} \approx \alpha_{uH}^2 \\ \alpha_{\theta M} &\approx \alpha_{uH}^3 \end{aligned} \qquad 5.11$$

These equations show that rotational interaction between piles is less than that for horizontal displacement.

Example 5.7 Return to example 5.5 and evaluate the lateral stiffness of the group if the pile heads are free.

As we have the free head condition there will be no moment at the pile head. Equation 5.11 gives $\alpha_{uH} = 0.833\alpha_{uF}$.

From example 3.3 we have, for the piles in this example, $f_{uh} = 1.97\times10^{-2}$ (this is the lateral displacement of a free head pile under unit horizontal load). Thus K_h for an isolated pile is 50.8 kN/mm. The group lateral stiffness is now:

$$\begin{aligned} K_{HG} &= 4\times50.8/0.833(1 + 0.377 + 0.189 + 0.200) \\ &= 138 \text{ kN/mm.} \end{aligned}$$

The calculation of the group lateral stiffness in this example could be done with an equation similar to 5.4. Equation 5.4 gives the group vertical stiffness and consequently has α_v for the interaction coefficient. An equation of the same form with α_{uH} instead of α_v would give K_{HG} directly. Similarly an equation with α_{uF} would give K_{HFG}.

5.4 Stiffness of a fixed head free standing pile group

There is now a fixing moment at the pile head so the applied moment is spilt between the part carried by the moments at the pile heads and the part carried by the moment generated by axial force increments in the pile shafts. This axial force mechanism is essentially the same as the free head case illustrated in Fig. 5.5. The coupling between the pile head response to moment and shear complicates the calculation of the response of a fixed head group. For a fixed head pile the rotation of the head is the same as that of the pile cap. This compatibility condition gives a means of dividing the applied moment between the two mechanisms. Note that this broadens our definition of a fixed head group. Thus far a fixed head pile or pile group has been understood to have zero head rotation. Now we allow some rotation of the pile cap (as must occur when moment loading is applied) but still regard the group as having a fixed head if the rotation of the pile cap is the same as that of the pile head.

The material presented in this section is developed for 2x2 pile groups. Extension to groups with a larger number of piles is along the same lines as the work in examples 5.5 and 5.6 in going from 2x2 to 3x3 groups.

Since we have a 2x2 group the horizontal load and the portion of the moment not equilibrated by axial loading of the piles are shared equally between the 4 piles. Using equations 3.15, 3.5 and 3.10 to write the rotational compatibility conditions gives:

$$\begin{aligned} \frac{M - 4M_p}{K_{\theta V}} &= H_p f_{\theta H}(1 + \alpha_{\theta H_{12}} + \alpha_{\theta H_{13}} + \alpha_{\theta H_{14}}) \\ &\quad + M_p f_{\theta M}(1 + \alpha_{\theta M_{12}} + \alpha_{\theta M_{13}} + \alpha_{\theta H_{14}}) \end{aligned}$$

where: M_p is the moment generated at the pile head,
and H_p is the horizontal force carried by each pile head.

Once M_p is obtained the rotation and lateral displacement of the pile group can be evaluated.