MODELING BINARY STARS AND EXTRASOLAR PLANETS

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Chapter 2 of my book, *Topics in Astrodynamics*, [1], derives the equations of relative motion of two gravitating Newtonian particles: a primary, having mass m_1, and a secondary, having mass m_2. It then proceeds to the case where the mass m_2 of the secondary (e.g., an artificial Earth satellite) is much smaller than the mass m_1 of the primary (e.g., Earth).

If we seek to model the motions binary stars, or the motions of planets revolving around stars other than our Sun ("extrasolar planets", see [2]), then we can certainly use the equations of relative motion from Chapter 2 of my book. But we then find that the relative motion description is just "Part 1" of the problem. "Part 2" consists of describing and illustrating the absolute motions. We consider both parts of the problem below, for the case of a star-planet system, where m_1 = 1.0 mass unit (star) and m_2 = 0.1 mass unit (planet).

Part 1 - Plotting the Star-Planet Relative Orbit

Suppose that the semilatus rectum, p, of the relative orbit is 2.0 astronomical units (A.U.) and that the orbital eccentricity, ecc, is 0.5. (Semilatus rectum, astronomical unit, and orbital eccentricity are all defined in my book.) The polar equation of the orbit is as follows, where r is the radius vector (actually the radius vector magnitude, or distance from the star to the extrasolar planet), and υ is the "true anomaly", or angle that the radius vector makes with the positive x-axis of the XY orbital reference plane.
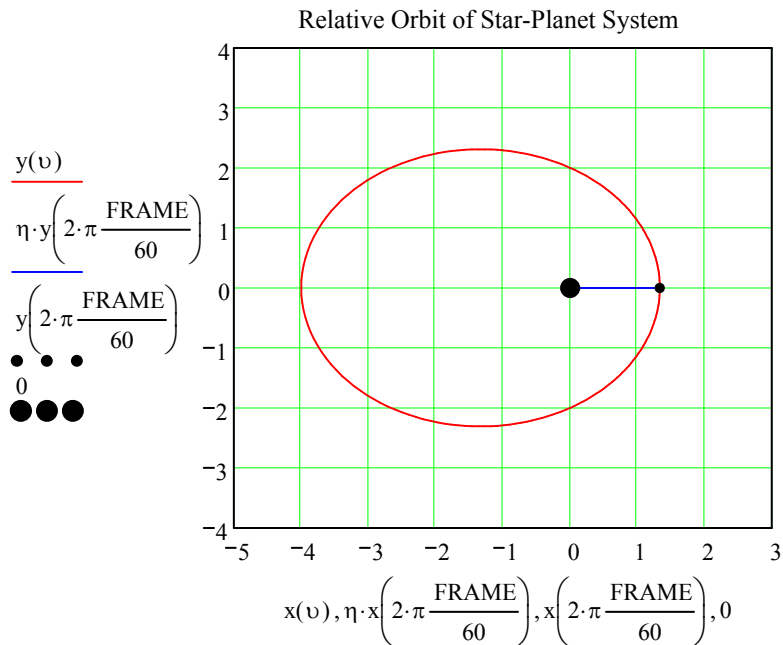
$$p := 2.0 \qquad\qquad ecc := 0.5$$

$$r(\upsilon) := \frac{p}{1 + ecc \cdot \cos(\upsilon)}$$

If we define values of υ as a range variable, as follows, then we can write and plot the cartesian coordinates of the relative elliptical orbit. This is the orbit of body 2 as seen from body 1, or dually, the orbit of body 1 as seen from body 2. (The variable FRAME, as used below, is a special Mathcad parameter that makes animation possible. For further information see the Mathcad manual.)

$$\upsilon := -\pi, -\pi + \frac{\pi}{100} .. \pi \qquad\qquad \eta := 0 .. 1$$

$$x(\upsilon) := r(\upsilon) \cdot \cos(\upsilon) \qquad\qquad y(\upsilon) := r(\upsilon) \cdot \sin(\upsilon)$$

Relative Orbit of Star-Planet System

$$y(\upsilon)$$

$$\eta \cdot y\left(2 \cdot \pi \frac{\text{FRAME}}{60}\right)$$

$$y\left(2 \cdot \pi \frac{\text{FRAME}}{60}\right)$$

$$x(\upsilon), \eta \cdot x\left(2 \cdot \pi \frac{\text{FRAME}}{60}\right), x\left(2 \cdot \pi \frac{\text{FRAME}}{60}\right), 0$$

Note: the radius vector is pointing to periastron. You can position it at other points around the orbit, without creating an animation, by setting FRAME to values such as 1, 2, ... 59. (Remember that you must specify, say, FRAME := 1 above the plot.)

To animate the plot, choose Tools>Animation>Record, and a "Record Animation" dialog box will pop up. Move the dialog box to the upper right of the worksheet window to get it out of the way. Set the number of frames to go from 0 to 60 at 10 frames per second, then use the mouse to click and drag from the upper left, just above the XY plot, to the lower right, just below it. This specifies the animation display. Now click the "Animate" button. When animation creation is done, an animation window will pop up and you can test your animation file and then save it, if you wish. The radius vector should sweep out one orbit in the animation, starting at periastron.

A problem with the animation is that the radius vector actually appears to sweep out faster near apastron (the point on the far left of the ellipse, farthest from the star) than at periastron (the point on the far right, nearest to the star). This is the opposite of what Kepler's Second Law dictates.

To get the radius vector to sweep out equal areas in equal times we need to calculate the true anomaly, υ, as a function of time t since periastron. To do this, we implement in Part 2 the equations given on p. 56 of my book. Here "a" is the semimajor axis (half the length of the major axis of the ellipse) of the orbit and k = 0.01720209895 is the Gaussian gravity constant for the Sun, expressed in A.U.$^{3/2}$/day. (We specify our own Sun's Gaussian constant in order to illustrate the problem using familiar distance units of A.U. and time units of mean solar days.)

Part 2 - Plotting the Star-Planet Absolute Orbits

In Part 1, the actual masses did not matter. It was implicitly assumed that body 1 had unit mass and body 2 had zero mass. We want now to plot the absolute orbits, so the masses do now matter. We set the masses as follows.

$$m_1 := 1.0 \qquad\qquad m_2 := 0.1$$

To make an animation that obeys Kepler's second law possible, we proceed as follows to generate points of the relative orbit.

$$a := \frac{p}{1 - ecc^2} \qquad\qquad K := 0.01720209895 \cdot \sqrt{m_1 + m_2}$$

$$\upsilon_1(t) := \left| \begin{array}{l} n \leftarrow K \cdot a^{-\frac{3}{2}} \\ M \leftarrow n \cdot t \\ E \leftarrow M \\ E \leftarrow root(E - ecc \cdot \sin(E) - M, E) \\ E + 2 \cdot atan\left(\dfrac{ecc \cdot \sin(E)}{1 + \sqrt{1 - ecc^2} - ecc \cdot \cos(E)} \right) \end{array} \right.$$

We now have a true anomaly function, υ_1, that is a function of t. We will set our animation up to sweep out radius vectors for 60 time points per orbit. To do this we calculate the orbital period via Kepler's Third Law.

$$P := \frac{2 \cdot \pi}{K} \cdot a^{\frac{3}{2}}$$ (P^2 is proportional to a^3, and the constant of proportionality is $4\pi^2/K^2$. This is another way of stating Kepler's Third Law.)

$$m := \frac{P}{60}$$ (The parameter m is a time step, exactly 1/60 of the orbital period.)

$$\frac{P}{365.25} = 4.152$$ (The orbital period is 4.152 Julian years consisting of 365.25 mean solar days.)

We plot two absolute orbits: the orbit of the secondary around the center of mass and the orbit of the primary around the center of mass. To do so, we need to calculate the cartesian coordinates of the primary's ellipse, x_1 and y_1, separately from the cartesian coordinates of the secondary's ellipse, x_2 and y_2, as follows.

$$f_1 := \frac{-m_2}{m_1 + m_2} \qquad x_1(\upsilon) := f_1 \cdot x(\upsilon) \qquad y_1(\upsilon) := f_1 \cdot y(\upsilon)$$

$$f_2 := \frac{m_1}{m_1 + m_2} \qquad x_2(\upsilon) := f_2\, x(\upsilon) \qquad y_2(\upsilon) := f_2 \cdot y(\upsilon)$$

We want to label the center of mass (i.e., the origin) with a plus sign, so we specify its coordinates for plotting (note that we did this in the Part 1 by plotting a black dot at (0,0) directly).

$$x_o := 0 \qquad\qquad y_o := 0$$

In order to be able to animate our plot, we specify the true anomaly angle in terms of the FRAME variable as follows.

$$\theta := \upsilon_1(\text{FRAME} \cdot m)$$

Absolute Orbits of Star-Planet System

$x_1(\upsilon), x_2(\upsilon), \eta \cdot x_1(\theta), \eta \cdot x_2(\theta), x_2(\theta), x_1(\theta), x_o, x(\upsilon)$
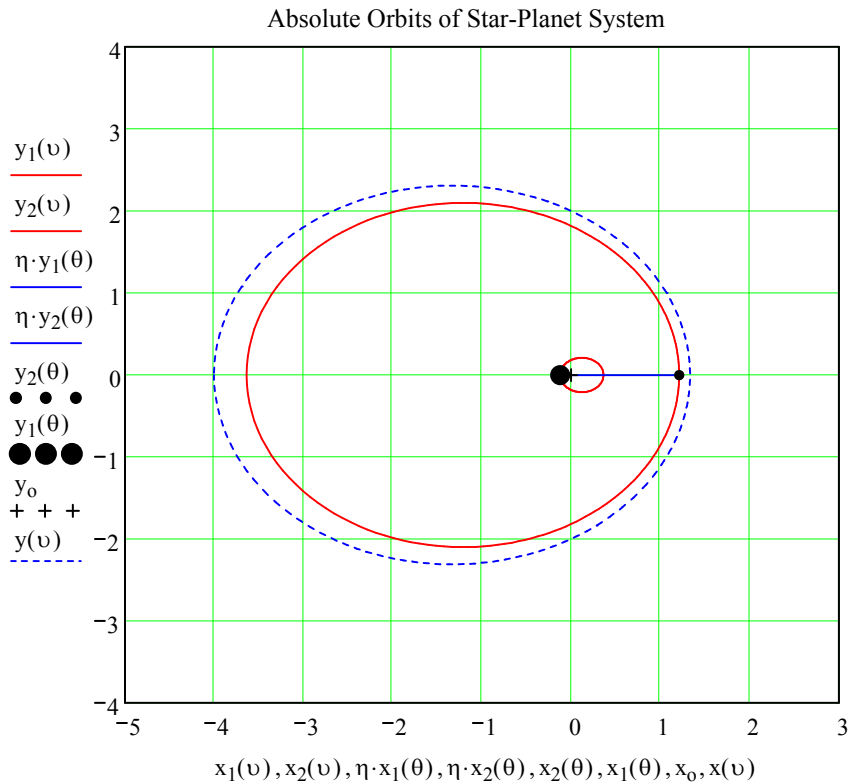
(The absolute orbits are the red ellipses. The blue ellipse, drawn with dashed lines, is the relative orbit plotted for comparison.)

To animate the plot, choose Tools>Animation>Record, and a "Record Animation" dialog box will pop up. Move the dialog box to the upper right of the worksheet window to get it out of the way. Set the number of frames to go from 0 to 60 at 10 frames per second, then use the mouse to click and drag from the upper left, just above the XY plot, to the lower right, just below it. This specifies the animation display. Now click the "Animate" button. When animation creation is done, an animation window will pop up and you can test your animation file, and then save it, if you wish. The radius vector should sweep out one orbit in the animation, starting at periastron.

This second animation plot will indeed sweep out equal areas in equal times. Note how the radius vector "whips around" periastron, but moves much more slowly near apastron. This is the way an actual extra-solar planet behaves as it orbits its own sun.

Other points to note:

1. The fact that the planet sweeps out a short radius vector swiftly, and a longer radius vector more slowly, is due to its orbit obeying Kepler's Second Law. But Kepler's Second Law itself is a consequence of the Law of Conservation of Angular Momentum.

2. When the planet is close to its sun its kinetic energy is great and its potential energy is small. When it is far from its sun the opposite is true. These facts are consequences of the Law of Conservation of Energy, i.e., the kinetic and potential energies of the planet orbit sum to a constant which is a constant of the orbital motion. These conservation laws are quite important in orbital analysis and are proved in my book.

3. It is possible in Mathcad to suppress all features of the absolute orbit plots except for the two bodies themselves. If you animate a plot of just the two bodies, then you will see what the motion would actually look like if you could observe it through a telescope, looking down on the orbit plane in the direction exactly opposite that of the orbital angular momentum vector. Try this!

4. The primary's absolute ellipse, the secondary's absolute ellipse, and the relative motion ellipse all have the same eccentricity. Further, the semimajor axis of the primary's ellipse is given by

$a_1 = a \cdot m_2/(m_1 + m_2)$ and the semimajor axis of the secondary's ellipse is given by

$a_2 = a \cdot m_1/(m_1 + m_2)$. Thus it holds that $a_1 + a_2 = a$ and $a_1/a_2 = m_2/m_1$.

5. Jupiter's mass is about 0.001 solar masses; mass m_2 in the example above was set to 0.1 in order to exaggerate the "wobble" of the primary. Large secondary masses are more commonly associated with binary stars than with star-planet systems. To see an example of the relative and absolute orbits of a more binary-star-like pair, set $m_2 := 0.5$ at the beginning of Part 2, then scroll down to the "Absolute Orbits of Star-Planet System" plot.

REFERENCES

[1] Mansfield, Roger L., *Topics in Astrodynamics*, Astronomical Data Service (Colorado Springs, Colorado, October 2003). See http://astrotopics.astroger.com.

[2] Appenzeller, Tim, "Someplace Like Earth," *National Geographic*, Washington, DC (December 2004), pp. 68-95.