ANIMATING KEPLER'S LAWS

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In this worksheet we state Kepler's laws of orbital motion. We then use Mathcad to animate them. In order to do this, we have to quantify Kepler's laws mathematically. Mathcad makes this easy.

<u>Kepler's Laws</u>

I. Each planet moves in an elliptical orbit around the sun, with the sun at one focus of the ellipse.

II. The radius vector from sun to planet sweeps out equal areas in equal times.

III. The square of the orbital period of a planet is proportional to the cube of the semimajor axis of its elliptical orbit around the sun.

In my book, *Topics in Astrodynamics* [1], I state these laws for the motions of planets around the sun in Chapter 1, and prove them using Newton's laws in Chapter 2.

The laws are stated with respect to planets, but they hold for other natural celestial bodies, such as moons, comets, and asteroids, too. And of course they hold for "cruising" interplanetary spacecraft, i.e., spacecraft which are not firing their onboard rocket engines. (A spacecraft fires its rocket engine to change the trajectory; such a maneuver is called a "Trajectory Correction Maneuver", or TCM. Each TCM is modeled under the assumption that all of the thrusting occurred impulsively at a single point of the trajectory, the midpoint of the TCM thrusting interval. Kepler's laws hold for the trajectory segments that joint the TCM midpoints.)

Let us choose a hypothetical asteroid for our example of orbital motion around the sun. Kepler's First Law says that the orbit of the asteroid is elliptical, with the sun, at one focus. To illustrate this law all we need to do is to draw an ellipse via the polar equation for a conic, using Mathcad's XY plot capability. But if we do it in the most simple and straightforward manner possible, our animation will not obey Kepler's Second Law. Let's do it and see why.

Suppose that the semilatus rectum, p, of our asteroid's orbit is 3.0 astronomical units (A.U.) and that the orbital eccentricity, ecc, is 0.75. (Semilatus rectum, astronomical unit, and orbital eccentricity are all defined in my book.) The polar equation of the orbit is as follows, where r is the radius vector (actually the radius vector magnitude, or distance from the sun to the asteroid), and υ is the "true anomaly", or angle that the radius vector makes with the positive x-axis of the XY orbital reference plane.

$p := 3.0$ $\qquad\qquad$ $ecc := 0.75$

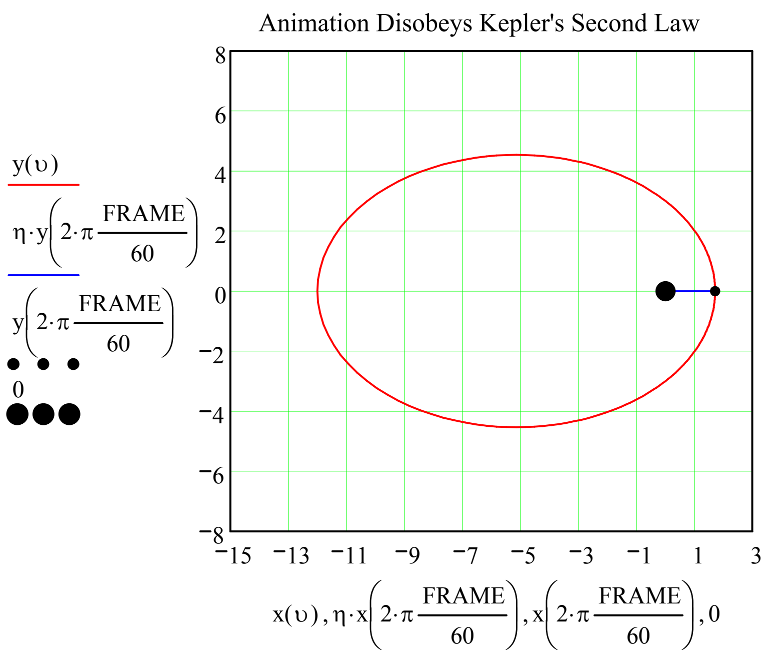
$$r(\upsilon) := \frac{p}{1 + ecc\cdot\cos(\upsilon)}$$

If we define values of υ as a range variable, as follows, then we can write and plot the cartesian coordinates of the elliptical orbit. (The variable FRAME, as used below, is a special Mathcad parameter that makes animation possible. For further information see the Mathcad manual.)

$$\upsilon := -\pi, -\pi + \frac{\pi}{100} .. \pi \qquad \eta := 0..1$$

$$x(\upsilon) := r(\upsilon)\cdot\cos(\upsilon) \qquad y(\upsilon) := r(\upsilon)\cdot\sin(\upsilon)$$

To animate the plot, choose Tools>Animation>Record, and a "Record Animation" dialog box will pop up. Move the dialog box to the upper right of the worksheet window to get it out of the way. Set the number of frames to go from 0 to 120 at 10 frames per second, then use the mouse to click and drag from the upper left, just above the XY plot, to the lower right, just below it. This specifies the animation display. Now click the "Animate" button. When animation creation is done, an animation window will pop up and you can test your animation file and then save it, if you wish. The radius vector should sweep out two orbits in the animation, starting at perihelion.



Note: the radius vector is pointing to perihelion. You can position it at other points around the orbit, without creating an animation, by setting FRAME to values such as 1, 2, ... 59. (Remember that you must specify, say, FRAME := 1 above the plot.)

The problem with this animation is that the radius vector actually appears to sweep out faster near aphelion (the point on the far left of the ellipse, farthest from the sun) than at perihelion (the point on the far right, nearest to the sun). This is the opposite of what Kepler's Second Law dictates.

To get the radius vector to sweep out equal areas in equal times we need to calculate the true anomaly, υ, as a function of time t since perihelion. To do this, we implement equations given on p. 56 of my book. Here "a" is the semimajor axis (half the length of the major axis of the ellipse) of the orbit and K is the Gaussian gravity constant for the sun, expressed in A.U.$^{3/2}$/day.

$$a := \frac{p}{1 - ecc^2} \qquad K := 0.01720209895$$

$$\upsilon_1(t) := \left| \begin{array}{l} n \leftarrow K \cdot a^{\frac{-3}{2}} \\ M \leftarrow n \cdot t \\ E \leftarrow M \\ E \leftarrow \mathrm{root}(E - ecc \cdot \sin(E) - M, E) \\ E + 2 \cdot \mathrm{atan}\left(\dfrac{ecc \cdot \sin(E)}{1 + \sqrt{1 - ecc^2} - ecc \cdot \cos(E)}\right) \end{array} \right.$$

We now have a true anomaly function, υ_1, that is a function of t. We set our animation up to sweep out radius vectors for 60 time points per orbit. To do this we calculate the orbital period via Kepler's Third Law.

$$P := \frac{2 \cdot \pi}{K} \cdot a^{\frac{3}{2}}$$

(P^2 is proportional to a^3, and the constant of proportionality is $4\pi^2/K^2$. This is another way of stating Kepler's Third Law.)

$$m := \frac{P}{60}$$

(The parameter m is a time step, exactly 1/60 of the orbital period.)

$$\frac{P}{365.25} = 17.957$$

(The orbital period is 17.957 Julian years consisting of 365.25 mean solar days.)

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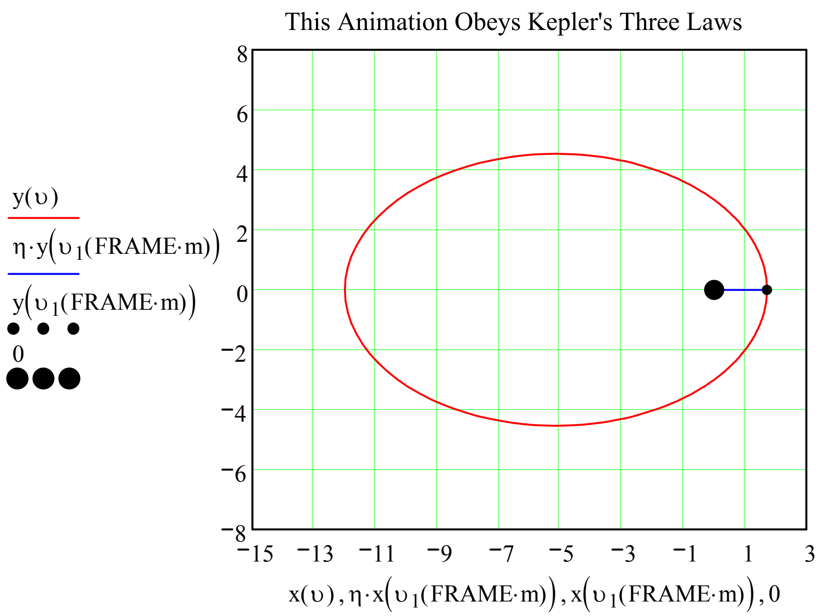

This second animation plot will indeed sweep out equal areas in equal times. Note how the radius vector "whips around" perihelion, but moves much more slowly near aphelion. This is the way an actual asteroid behaves as it orbits the sun.

Two other points to note:

1. The fact that the asteroid sweeps out a short radius vector swiftly, and a longer radius vector more slowly, is due to its orbit obeying Kepler's Second Law. But Kepler's Second Law itself is a consequence of the Law of Conservation of Angular Momentum.

2. When the asteroid is close to the sun its kinetic energy is great and its potential energy is small. When it is far from the sun the opposite is true. These facts are consequences of the Law of Conservation of Energy, i.e., the kinetic and potential energies of the asteroid orbit sum to a constant which is a constant of the orbital motion.

These conservation laws are quite important in orbital analysis and are proved in my book.

REFERENCE

[1] Mansfield, Roger L., *Topics in Astrodynamics*, Astronomical Data Service (Colorado Springs, Colorado, October 2003). See http://astrotopics.astroger.com.