CYCLOID SWEEPER

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A *cycloid* is the plane curve swept out by a point fixed on the rim of a wheel rolling in a vertical plane along a straight line. What we will do in this worksheet is to "roll" a circle along the x-axis (the line y = 0).

But what we will want is an inverted cycloid, that is, the wheel rolls upside down along the x-axis. Thus, the generating circle also rolls upside down, with its center moving at a uniform rate along the line y = -1. The inverted cycloid is the curve of fastest descent of a Newtonian particle falling to a point not directly beneath the starting point -- it is the celebrated "brachistochrone" curve.

We start by setting the radius of the generating circle to one unit.

$$a := 1$$

Set the range variable t_2 as needed to plot the inverted cycloid curve in green.

$$t_2 := 0, \frac{1}{100} .. 1.0$$

Set t_1 to the FRAME variable, divided by 100 for a smooth animation, and parametrize the cycloid coordinates x_1 and y_1 as functions of t_1.

$$t_1 := \frac{FRAME}{100}$$ (FRAME is the animation variable.)

$$x_1(t_1) := a\cdot(2\cdot\pi\cdot t_1 - \sin(2\cdot\pi\cdot t_1)) \quad y_1(t_1) := -a\cdot(1 - \cos(2\cdot\pi\cdot t_1))$$

Generating Circle for Inverted Cycloid

We can use what we have just defined, and the following equations, to animate the upside-down, rolling circle that generates an inverted cycloid.

$$t_1 := \mathrm{if}\left(FRAME \le 100, \frac{FRAME}{100}, 1 - \frac{FRAME - 100}{100}\right)$$

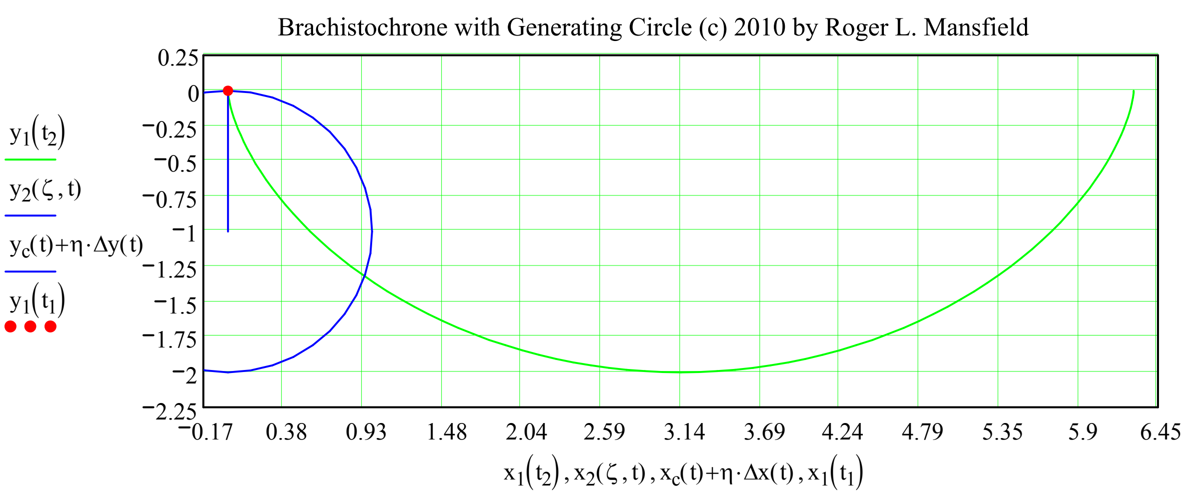
$\eta := 0..1$ $\qquad t := t_1$ $\qquad \zeta := 0, \frac{\pi}{20}..2\cdot\pi$

$x_c(t) := 2\cdot\pi\cdot t$ $\qquad y_c(t) := -1$ $\qquad x_2(\zeta, t) := a\cdot\cos(\zeta) + x_c(t)$

$\Delta x(t) := x_1(t) - x_c(t)$ $\qquad \Delta y(t) := y_1(t) - y_c(t)$ $\qquad y_2(\zeta, t) := a\cdot\sin(\zeta) + y_c(t)$

This definition of t_1 permits an animation wherein the circle rolls from left to right and then back again, so as to enable continuous looping in a video player (select "Loop" or "Repeat," depending upon the player).

Using the assignments just made, we can now draw and animate the generating circle for the cycloid curve. Select Animation from the Tools menu, then click on Record. Drag-select the plot, set FRAME to go from 0 to 200, and click on Animate.



REFERENCES

[1] Thomas, George B. Jr. and Finney, Ross L., *Calculus and Analytic Geometry, 8th Edition* (Addison-Wesley, 1992). Proof that the brachistochrone (inverted cycloid) is also a tautochrone is given on pp. 654-655.

[2] Mansfield, Roger L., "Tautochrone Balls: A Mathcad Animation," an article submitted for the June 2010 issue of the *PTC Express* Newsletter.