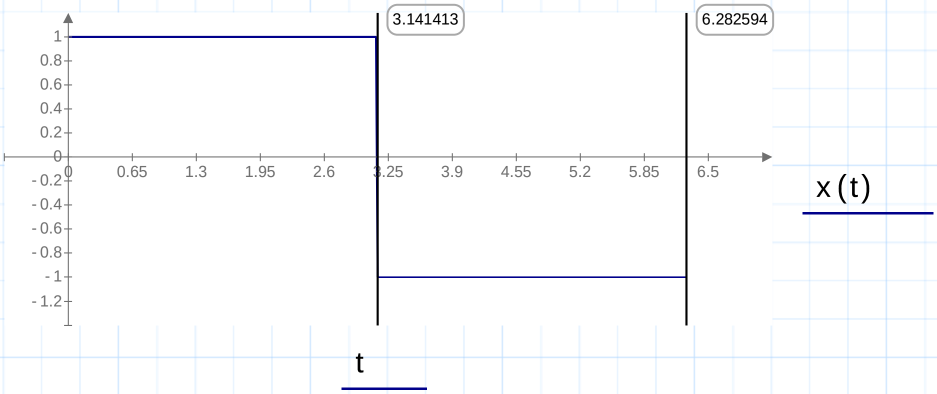
Example 2 - Trignometric Fourier Series

x(t) = 1 if 0<= t <= pi
= -1 if pi<= t <= 2pi
We can approx. x(t) by using the trignometric fourier series as follows:

$t := 0, \frac{\pi}{200} .. 2 \cdot \pi$ pi = 180 deg, 2pi = 360 deg

$$x(t) := \begin{array}{l} \text{if } (0 \le t \le \pi) \\ \quad x \leftarrow 1 \\ \text{if } (\pi \le t \le 2 \cdot \pi) \\ \quad x \leftarrow -1 \end{array}$$



Define the fourier coefficients:

$T := 2 \cdot \pi$ defining the fundamental period (one cycle of 2pi)

$\omega_0 := \frac{2 \cdot \pi}{T}$ defining the fundamental frequency w0

$N := 5$ defining the number of terms N

$n := 1 .. N$ set range for n

Set the integrals:

$$a_0 := \frac{2}{T} \cdot \int_0^T x(t)\, dt$$

$$a_n := \frac{2}{T} \cdot \int_0^T x(t) \cdot \cos(n \cdot \omega_0 \cdot t)\, dt$$

n is a matrix element in term 'an and bn' in the equations

$$b_n := \frac{2}{T} \cdot \int_0^T x(t) \cdot \sin(n \cdot \omega_0 \cdot t)\, dt$$

Defining x(t):

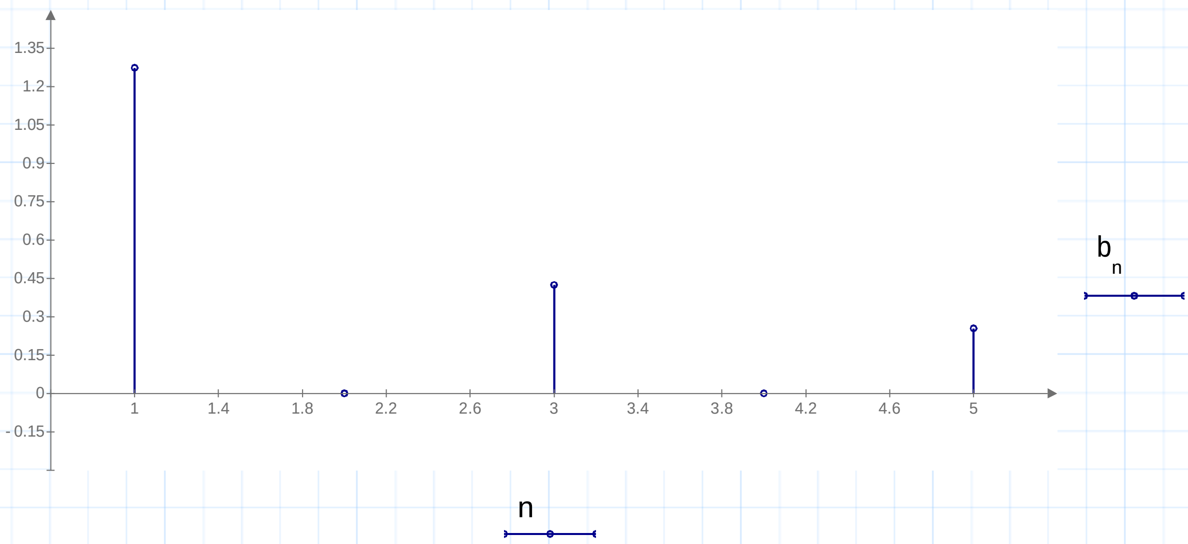
$$x(t) := \frac{a_0}{2} + \left(\sum_{n=1}^{N} \left(a_n \cdot \cos(n \cdot \omega_0 \cdot t) + b_n \cdot \sin(n \cdot \omega_0 \cdot t) \right) \right)$$

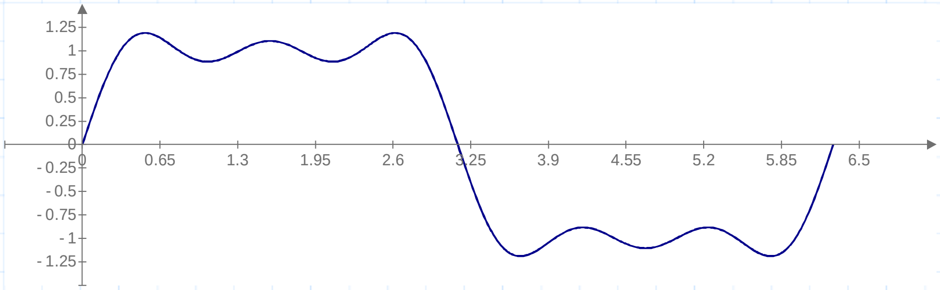
$$x(t) = \begin{bmatrix} 5.05305 \cdot 10^{-16} \\ 0.059971 \\ 0.11977 \\ 0.179225 \\ 0.238165 \\ 0.296424 \\ 0.353838 \\ 0.410246 \\ 0.465494 \\ 0.519434 \\ 0.571921 \\ 0.622822 \\ \vdots \end{bmatrix}$$

$$a_0 = 0$$

$$a_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad b_n = \begin{bmatrix} 1.27324 \\ 3.754602 \cdot 10^{-17} \\ 0.424413 \\ -1.656008 \cdot 10^{-17} \\ 0.254648 \end{bmatrix}$$

Plot the Fourier series coefficients:





Go back up to N=5 and change it to N=7
The plot of x(t) gives a better representation of the signal x(t).
It will have 4 peaks in the positive and negative sides instead of 3 when N=5.

Example 3-3 Symbolic Computation of Fourier Series
Same example as 3-2, but using the Mathcad Prime symbolic computation.

clear(x)

$t := 0, \frac{\pi}{200} .. 2 \cdot \pi$ defining the range for time t

$x(t) := \begin{cases} \text{if } (0 \le t \le 0) & x \leftarrow 1 \\ \text{if } (\pi \le t \le 2\pi) & x \leftarrow -1 \end{cases}$ defining the function to be approximated

$T := 2\pi$ defining the fundamental period

$\omega_0 := \frac{2\pi}{T}$

$N := 6$ defining the number of terms **clear**(n)

$n := 1 .. 6$ set a range for n

$\mathrm{TOL} := 10^{-2}$ reducing the tolerance

π 2π

$$a_0 := \frac{2}{T}\cdot\int_0 1\,dt + \frac{2}{T}\int_\pi -1\,dt$$

$a_0 \rightarrow 0$ calculate a0, arrow pointing to the right ---> calculate

next defining a(n) and calculating

$$a(n) := \frac{2}{T}\cdot\int_0^\pi \cos(n\cdot\omega_0\cdot t)\,dt + \frac{2}{T}\cdot\int_\pi^{2\cdot\pi}(-1)\cos(n\cdot\omega_0\cdot t)\,dt \rightarrow \frac{\sin(\pi\cdot n)}{\pi\cdot n} + \frac{\sin(\pi\cdot n) - \sin(2\cdot\pi\cdot n)}{\pi\cdot n}$$

next defining b(n) and calculating

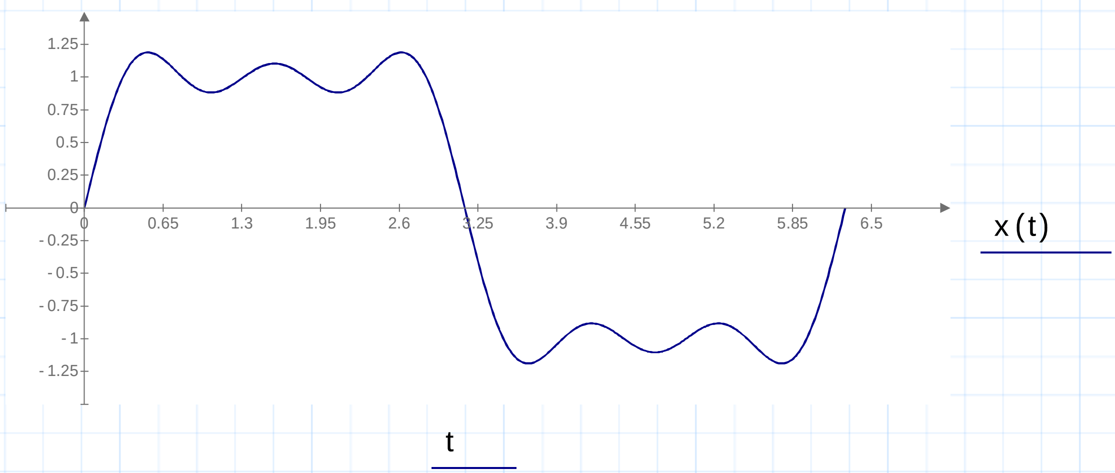
$$b(n) := \frac{2}{T}\cdot\int_0^\pi \sin(n\cdot\omega_0\cdot t)\,dt + \frac{2}{T}\cdot\int_\pi^{2\cdot\pi}(-1)\sin(n\cdot\omega_0\cdot t)\,dt \rightarrow \frac{2\cdot\sin\left(\frac{\pi\cdot n}{2}\right)^2}{\pi\cdot n} - \frac{\cos(\pi\cdot n) - \cos(2\cdot\pi\cdot n)}{\pi\cdot n}$$

defining the fourier series equation and calculating

$$x(t) := \frac{a_0}{2} + \left[\sum_{n=1}^{N}\left(a(n)\cdot\cos(n\cdot\omega_0\cdot t) + b(n)\cdot\sin(n\cdot\omega_0\cdot t)\right)\right] \rightarrow \left[\frac{4\cdot\sin(3\cdot t)}{3\cdot\pi} + \frac{4\cdot\sin(5\cdot t)}{5\cdot\pi} + \frac{4\cdot\sin(t)}{\pi}\right]$$

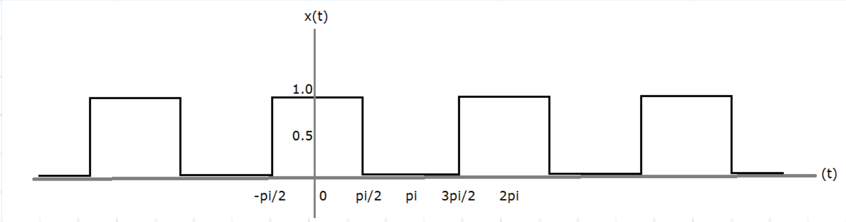
next for purpose of ploting manually set x(t) := to the evaluated answer above, the evaluated term above cannot be directly placed into x(t) for plotting into the plot's y-axis

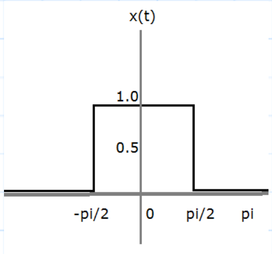
$$x(t) := \left(\frac{4\cdot\sin(3\cdot t)}{3\cdot\pi} + \frac{4\cdot\sin(5\cdot t)}{5\cdot\pi} + \frac{4\cdot\sin(t)}{\pi}\right)$$



Now go back and set N=7 for an improved representation of the original signal x(t), notice there is no change. This is because the problem here was solved using symbolic calculation rather than iterations with the summation function.

Example 3-5 Exponential fourier series in Prime 2.0





Refer to an engineering mathematics textbook for the pure mathematics derivation side for the exponential fourier series of this waveform

clear (x) **clear** (n)

$$x(t) := \begin{cases} \text{if } \left(\frac{-\pi}{2} \le t \le \frac{\pi}{2}\right) & x \leftarrow 1 \\ \text{else} & x \leftarrow 0 \end{cases}$$

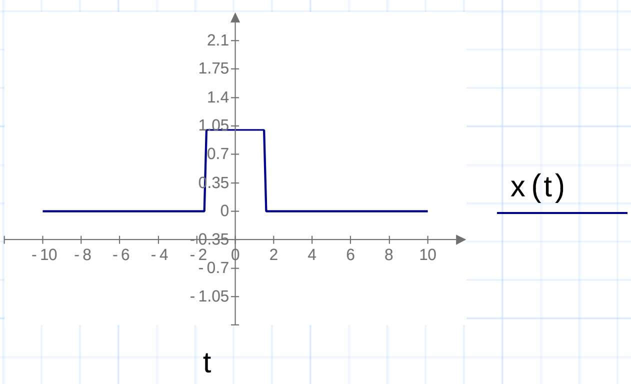
defining the function to be approximated

$t := -10, -9.9 .. 10$ defining the range for time t

$T := 2\pi$ defining the fundamental period

$$\omega_0 := \frac{2\pi}{T}$$

We plot the signal x(t) between -pi/2 and pi/2.
Rest of the values of x(t) are equal to zero.



Next we define Cn and change the origin of the array to start at a negative value since we plot the coefficients for negative values of n

$$\text{Origin} := -10$$

We have to define the range of n to be a suitable number since we cannot handle 'infinity'

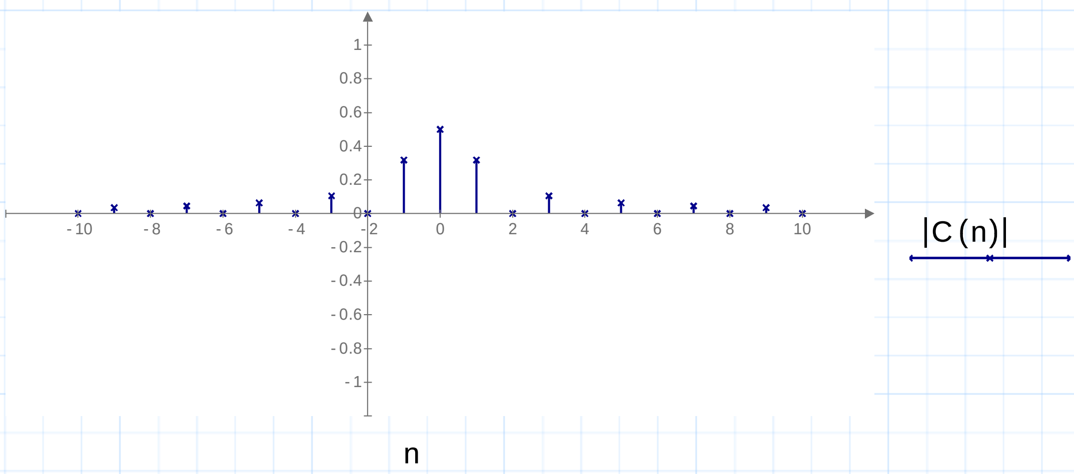
$$N := 10$$

$$n := -N .. N \qquad j := \sqrt{-1}$$

Finally we compute the Fourier coefficients Cn
Not matrix element n in Cn but C(x) more like a function:

$$C(n) := \frac{1}{T} \cdot \int_{\left(\frac{-\pi}{2}\right)}^{0} 1 \cdot e^{-j \cdot n \cdot \omega_0 \cdot t} \, dt + \frac{1}{T} \cdot \int_{0}^{\frac{\pi}{2}} 1 \cdot e^{-j \cdot n \cdot \omega_0 \cdot t} \, dt$$

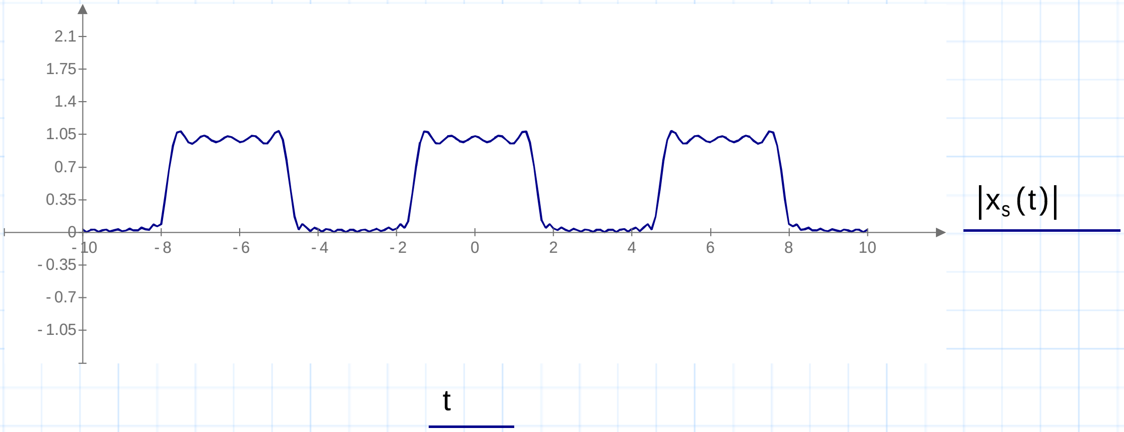
Take the magnitude of the coefficients Cn in plot, since the wave has positive values only:



Now we plot the fourier series representaton of the signal. To do this we define another signal xs(t), and equalize it to the exponential fourier series equation.

$$x_s(t) := \sum_{n=-N}^{N} C(n) \cdot \left(e^{j \cdot n \cdot \omega_0 \cdot t}\right)$$

Remember to take the magnitude of xs(t), because the original signal is positive values of x(t)



The Continous Time Fourier Series Property:

If x(t) is an odd function, where x(t) = -x(-t) OR -x(t) = x(-t) for all t
then a0 = 0 and an = 0.
If x(t) is even function, where x(t) = x(-t),
then bn = 0.
In other words, when we multiply an odd function with an even function, and intergrate it over one period, the area is 0.

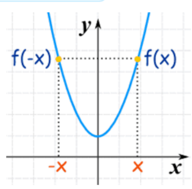
Next example demonstrates this. Notes on even and off function provided below.

Even Functions

A function is "even" when:

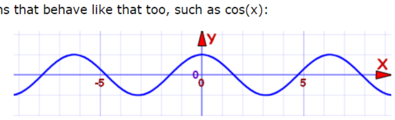
f(x) = f(−x) for all x

In other words there is symmetry about the y-axis (like a reflection):



This is the curve f(x) = x^2+1

They got called "even" functions because the functions x^2, x^4, x^6, x^8, etc behave like that, but there are other functions that behave like that too, such as cos(x):



Cosine function: f(x) = cos(x)
It is an even function

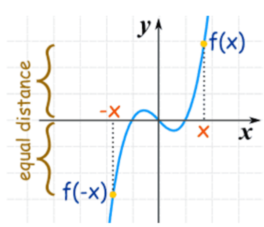
Odd Functions

A function is "odd" when:

−f(x) = f(−x) for all x

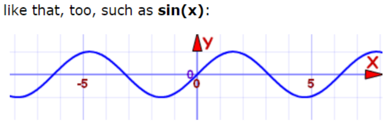
Note the minus in front of f: **−f(x)**.

And we get origin symmetry:



This is the curve f(x) = x^3−x

They got called "odd" because the functions x, x^3, x^5, x^7, etc behave like that, but there are other functions that behave like that, too, such as **sin(x)**:



Sine function: f(x) = sin(x)
It is an odd function

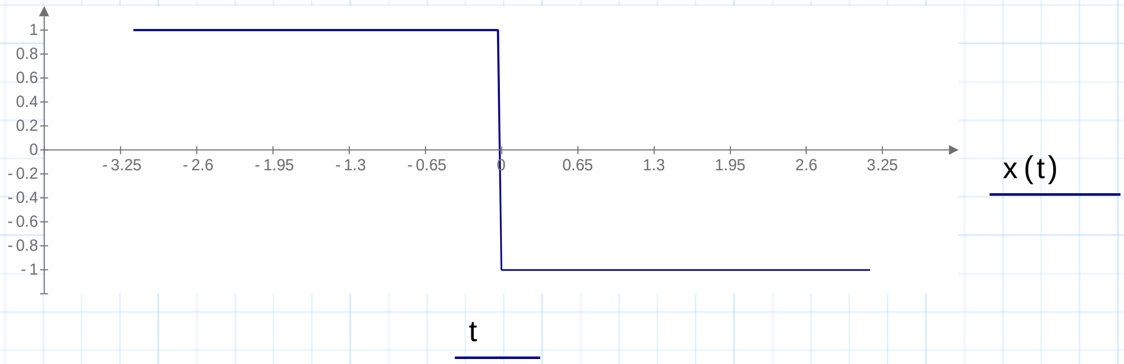
Example 3.6

$t := -\pi, -0.99 \cdot \pi .. \pi$

$x(t) := \text{if}(t < 0, 1, -1)$

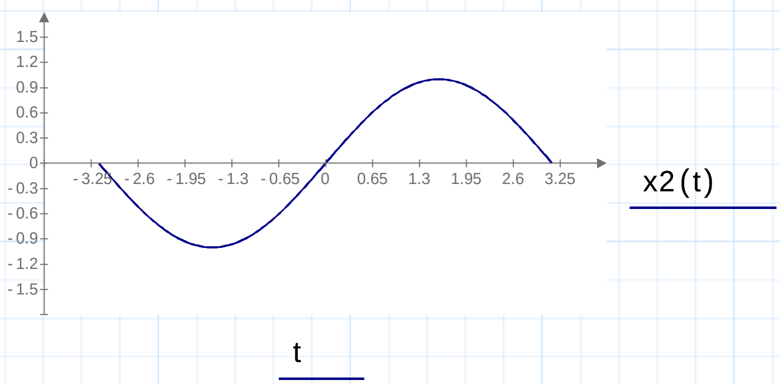
when t = - pi to 0, x(t) = -1
t = 0 , x(t) = 0
t = 0 to pi, x(t) = 1

plot below x(t) symmetry over the origin so its an odd function



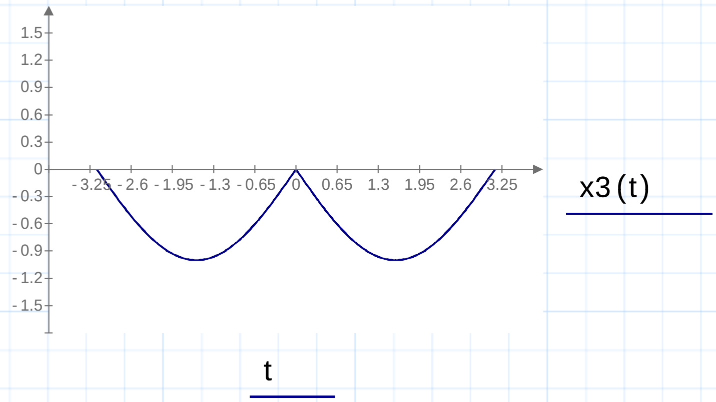
next define a sine function (odd) to multiply it to x(t) the original signal (odd)

$x2(t) := \sin(t)$



now multiplying original signal x(t) with the sine function x2(t)= sin(t)

$x3(t) := x(t) \cdot x2(t)$



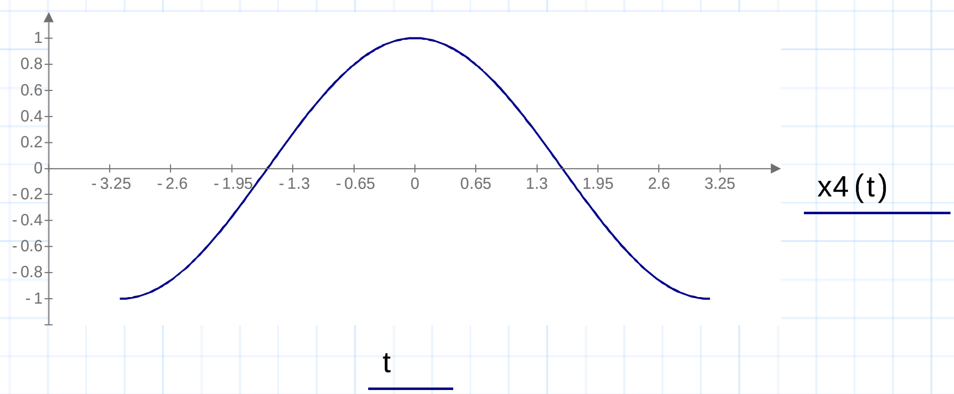
From the plot above the area of x3(t) with the 0 axis is Not equal to 0.
Odd function x(t) multiplied to odd function sin(x) results in NOT equal 0

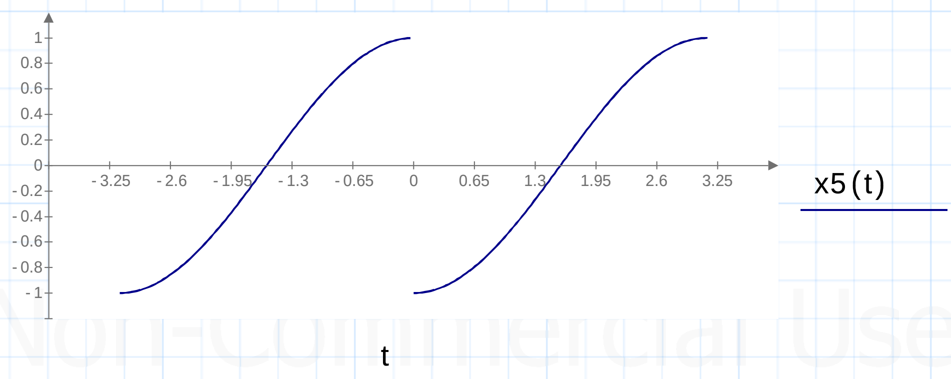
$$A(t) := \int_{-\pi}^{\pi} x3(t)\, dt = -4$$

Now multiply the same function x(t) which is even by a cosine function ie even function:

$x4(t) := \cos(t)$

$x5(t) := x(t) \cdot x4(t)$





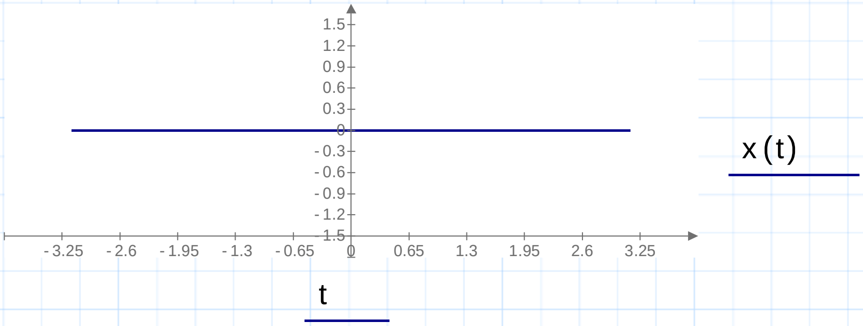
In the plot above at t = 0 and value of x5(t) = 1 and - 1, drops down vertically from 1 to -1.
This plot shows that its sum equals 0, the positive side of the plot cancels the negative side of the plot.
The Fourier Property - even x(t) multiply by even cos (t) = 0

$A(t) := \int_{-\pi}^{\pi} x5(t)\, dt = 0$ Here the Prime integral evaluated result equal 0

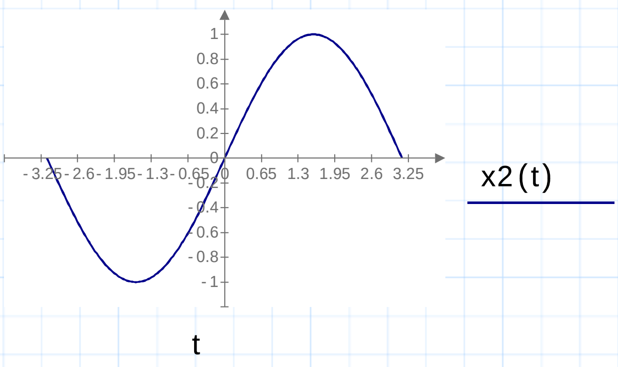
Example 3.7

$x(t) := 0$ Initialise x(t)

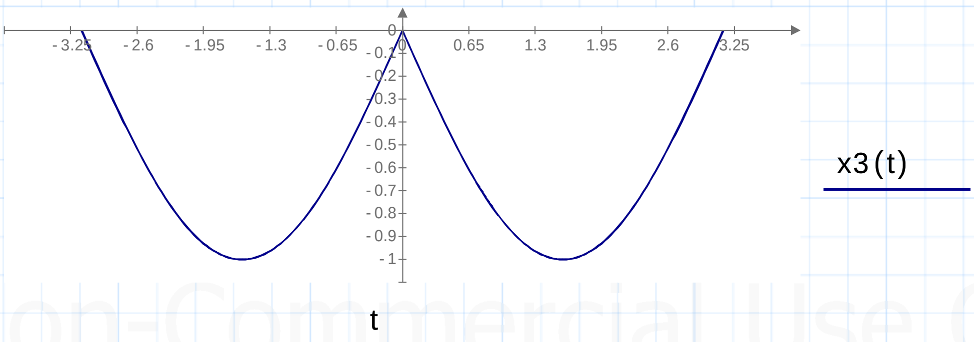
$x(t) := \text{if}(t < 0, -0.5\,t, 0.5\,t)$ x(t) is an even function - the original function to be evaluated



$x2(t) := \sin(t)$



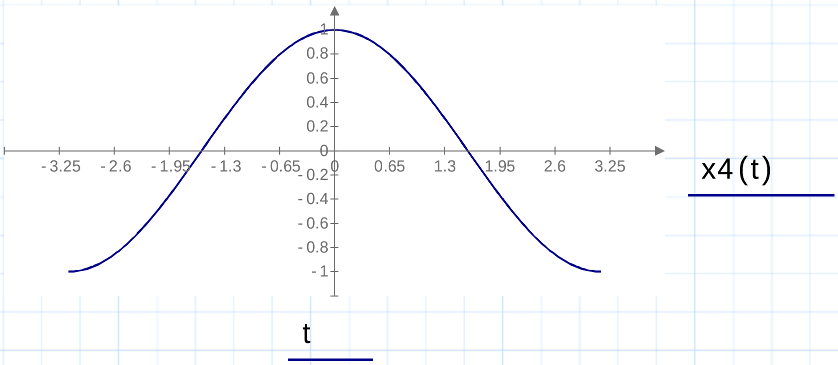
$x3(t) := x(t) \cdot x2(t)$



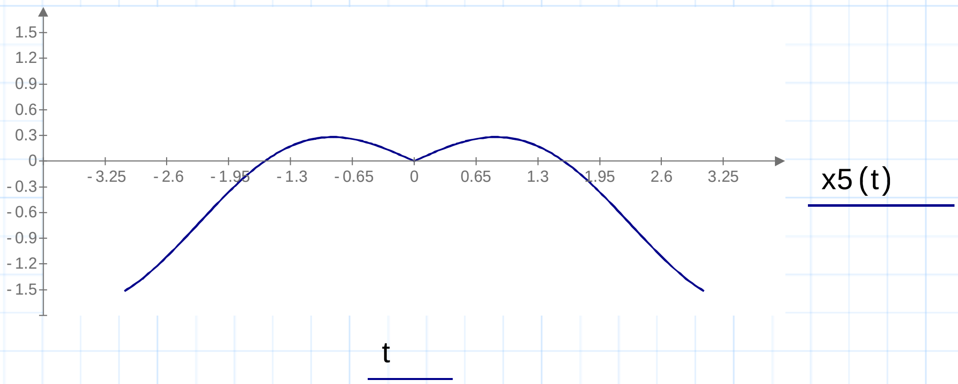
The even function after multiplying by an odd sine function equal 0

Next we multiply x(t) by a cosine function

$x4(t) := \cos(t)$



$x5(t) := 0$
$x5(t) := x(t)\ x4(t)$



The area for x5(t) is NOT equal 0.
Here even function x(t) multiplied by even cosine function results in NOT equal zero area.

The Continous Time Fourier Transform (Mixed explanation on this refer to your textbook, the objective is to get valuable results from Prime/Mathcad.

Notes from Chp 7 Linear Systems and Signals 2nd ed B.P. Lathi.

CHAPTER 7

Continuous-Time Signal Analysis: The Fourier Transform

We can analyze linear systems in many different ways by taking advantage of the property of linearity, where by the input is expressed as a sum of simpler components. The system response to any complex input can be found by summing the system's response to these simpler components of the input. In time-domain analysis, we separated the input into impulse components. In the frequency-domain analysis in Chapter 4, we separated the input into exponentials of the form e^{st} (the Laplace transform), where the complex frequency $s = \sigma + j\omega$. The Laplace transform, although very valuable for system analysis, proves somewhat awkward for signal analysis, where we prefer to represent signals in terms of exponentials $e^{j\omega t}$ instead of e^{st}. This is accomplished by the Fourier transform. In a sense, the Fourier transform may be considered to be a special case of the Laplace transform with $s = j\omega$. Although this view is true most of the time, it does not always hold because of the nature of convergence of the Laplace and Fourier integrals.

In Chapter 6, we succeeded in representing periodic signals as a sum of (everlasting) sinusoids or exponentials of the form $e^{j\omega t}$. The Fourier integral developed in this chapter extends this spectral representation to aperiodic signals.

7.1 Aperiodic Signal Representation by Fourier Integral

Applying a limiting process, we now show that an aperiodic signal can be expressed as a continuous sum (integral) of everlasting exponentials. To represent an aperiodic signal $x(t)$ such as the one depicted in Fig. 7.1a by everlasting exponentials, let us construct a new periodic signal $x_{T_0}(t)$ formed by repeating the signal $x(t)$ at intervals of T_0 seconds, as illustrated in Fig. 7.1b. The period T_0 is made long enough to avoid overlap between the repeating pulses. The periodic signal $x_{T_0}(t)$ can be represented by an exponential Fourier series. If we let $T_0 \to \infty$, the pulses in the periodic signal repeat after an infinite interval and, therefore

$$\lim_{T_0 \to \infty} x_{T_0}(t) = x(t)$$

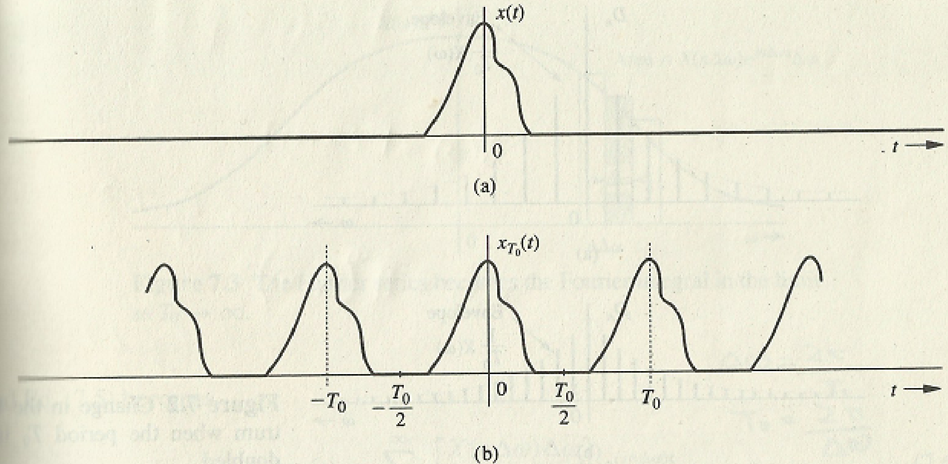


Figure 7.1 Construction of a periodic signal by periodic extension of $x(t)$.

Thus, the Fourier series representing $x_{T_0}(t)$ will also represent $x(t)$ in the limit $T_0 \to \infty$. The exponential Fourier series for $x_{T_0}(t)$ is given by

$$x_{T_0}(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \tag{7.1}$$

where

$$D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_{T_0}(t) e^{-jn\omega_0 t}\, dt \tag{7.2a}$$

and

$$\omega_0 = \frac{2\pi}{T_0} \tag{7.2b}$$

Observe that integrating $x_{T_0}(t)$ over $(-T_0/2, T_0/2)$ is the same as integrating $x(t)$ over $(-\infty, \infty)$. Therefore, Eq. (7.2a) can be expressed as

$$D_n = \frac{1}{T_0} \int_{-\infty}^{\infty} x(t) e^{-jn\omega_0 t}\, dt \tag{7.2c}$$

It is interesting to see how the nature of the spectrum changes as T_0 increases. To understand this fascinating behavior, let us define $X(\omega)$, a continuous function of ω, as

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt \tag{7.3}$$

$e^{-jn\omega_0 t}$
$e^{-j\omega t}$

A glance at Eqs. (7.2c) and (7.3) shows that

$$D_n = \frac{1}{T_0} X(n\omega_0) \tag{7.4}$$

$X(n\omega_0) = \int x(t) e^{-jn\omega_0 t}\, dt$

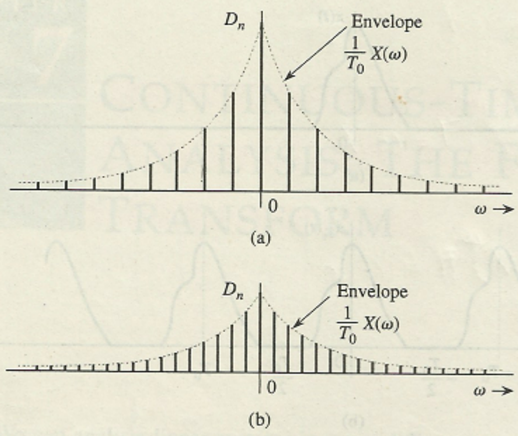


Figure 7.2 Change in the Fourier spectrum when the period T_0 in Fig. 7.1 is doubled.

This means that the Fourier coefficients D_n are $1/T_0$ times the samples of $X(\omega)$ uniformly spaced at intervals of ω_0, as depicted in Fig. 7.2a.[†] Therefore, $(1/T_0)X(\omega)$ is the envelope for the coefficients D_n. We now let $T_0 \rightarrow \infty$ by doubling T_0 repeatedly. Doubling T_0 halves the fundamental frequency ω_0 [Eq. (7.2b)], so that there are now twice as many components (samples) in the spectrum. However, by doubling T_0, the envelope $(1/T_0)X(\omega)$ is halved, as shown in Fig. 7.2b. If we continue this process of doubling T_0 repeatedly, the spectrum progressively becomes denser while its magnitude becomes smaller. Note, however, that the relative shape of the envelope remains the same [proportional to $X(\omega)$ in Eq. (7.3)]. In the limit as $T_0 \rightarrow \infty$, $\omega_0 \rightarrow 0$ and $D_n \rightarrow 0$. This result makes for a spectrum so dense that the spectral components are spaced at zero (infinitesimal) intervals. At the same time, the amplitude of each component is zero (infinitesimal). We have *nothing of everything, yet we have something!* This paradox sounds like *Alice in Wonderland,* but as we shall see, these are the classic characteristics of a very familiar phenomenon.[‡]

Substitution of Eq. (7.4) in Eq. (7.1) yields

$D_n = \frac{1}{T_0} X(n\omega_0)$ $x_{T_0}(t) = \sum_{-\infty}^{\infty} D_n e^{jn\omega_0 t}$ $\omega_0 = \frac{2\pi}{T_0}$, $T_0 \rightarrow \infty$, $\omega_0 \rightarrow 0$

$$x_{T_0}(t) = \sum_{n=-\infty}^{\infty} \frac{X(n\omega_0)}{T_0} e^{jn\omega_0 t} \tag{7.5}$$

As $T_0 \rightarrow \infty$, ω_0 becomes infinitesimal ($\omega_0 \rightarrow 0$). Hence, we shall replace ω_0 by a more appropriate notation, $\Delta\omega$. In terms of this new notation, Eq. (7.2b) becomes

$$\Delta\omega = \frac{2\pi}{T_0}$$

[†]For the sake of simplicity, we assume D_n, and therefore $X(\omega)$, in Fig. 7.2, to be real. The argument, however, is also valid for complex D_n [or $X(\omega)$].

[‡]If nothing else, the reader now has irrefutable proof of the proposition that 0% ownership of everything is better than 100% ownership of nothing.

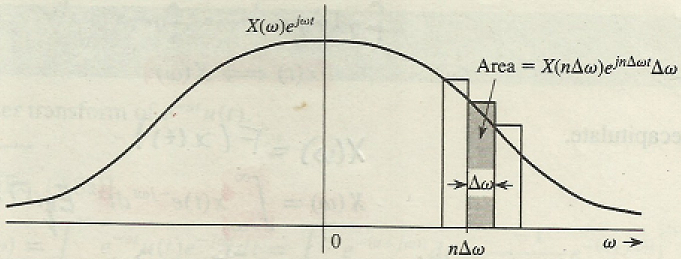


Figure 7.3 The Fourier series becomes the Fourier integral in the limit as $T_0 \to \infty$.

and Eq. (7.5) becomes

$$x_{T_0}(t) = \sum_{n=-\infty}^{\infty} \left[\frac{X(n\Delta\omega)\Delta\omega}{2\pi} \right] e^{(jn\Delta\omega)t} \qquad (7.6a)$$

$\Delta\omega = \frac{2\pi}{T_0}$, $T_0 = \frac{2\pi}{\Delta\omega}$, $\frac{1}{T_0} = \frac{\Delta\omega}{2\pi}$

Equation (7.6a) shows that $x_{T_0}(t)$ can be expressed as a sum of everlasting exponentials of frequencies $0, \pm\Delta\omega, \pm 2\Delta\omega, \pm 3\Delta\omega, \ldots$ (the Fourier series). The amount of the component of frequency $n\Delta\omega$ is $[X(n\Delta\omega)\Delta\omega]/2\pi$. In the limit as $T_0 \to \infty$, $\Delta\omega \to 0$ and $x_{T_0}(t) \to x(t)$. Therefore

$$x(t) = \lim_{T_0\to\infty} x_{T_0}(t)$$

$$= \lim_{\Delta\omega\to 0} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} X(n\Delta\omega)e^{(jn\Delta\omega)t}\Delta\omega \qquad (7.6b)$$

The sum on the right-hand side of Eq. (7.6b) can be viewed as the area under the function $X(\omega)e^{j\omega t}$, as illustrated in Fig. 7.3. Therefore

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\, d\omega \qquad (7.7)$$

The integral on the right-hand side is called the *Fourier integral*. We have now succeeded in representing an aperiodic signal $x(t)$ by a Fourier integral (rather than a Fourier series).† This integral is basically a Fourier series (in the limit) with fundamental frequency $\Delta\omega \to 0$, as seen from Eq. (7.6). The amount of the exponential $e^{jn\Delta\omega t}$ is $X(n\Delta\omega)\Delta\omega/2\pi$. Thus, the function $X(\omega)$ given by Eq. (7.3) acts as a spectral function. $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$

We call $X(\omega)$ the *direct* Fourier transform of $x(t)$, and $x(t)$ the *inverse* Fourier transform of $X(\omega)$. The same information is conveyed by the statement that $x(t)$ and $X(\omega)$ are a Fourier transform pair. Symbolically, this statement is expressed as

$$X(\omega) = \mathcal{F}[x(t)] \quad \text{and} \quad x(t) = \mathcal{F}^{-1}[X(\omega)]$$

†This derivation should not be considered to be a rigorous proof of Eq. (7.7). The situation is not as simple as we have made it appear.[1]

or

fourier transform pair

$$x(t) \Longleftrightarrow X(\omega)$$

To recapitulate,

$X(\omega) = F(x(t))$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \quad \text{Eq. 7.3} \tag{7.8a}$$

and

$x(t) = F^{-1}(X(\omega))$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega \quad \text{Eq. 7.7} \tag{7.8b}$$

It is helpful to keep in mind that the Fourier integral in Eq. (7.8b) is of the nature of a Fourier series with fundamental frequency $\Delta\omega$ approaching zero [Eq. (7.6b)]. Therefore, most of the discussion and properties of Fourier series apply to the Fourier transform as well. *The transform $X(\omega)$ is the frequency-domain specification of $x(t)$*.

We can plot the spectrum $X(\omega)$ as a function of ω. Since $X(\omega)$ is complex, we have both amplitude and angle (or phase) spectra

$$X(\omega) = |X(\omega)|e^{j\angle X(\omega)} \tag{7.9}$$

in which $|X(\omega)|$ is the amplitude and $\angle X(\omega)$ is the angle (or phase) of $X(\omega)$. According to Eq. (7.8a),

$-j\omega t \rightarrow j\omega t$

$$X(-\omega) = \int_{-\infty}^{\infty} x(t)e^{j\omega t}\,dt$$

Taking the conjugates of both sides yields

$$x^*(t) \Longleftrightarrow X^*(-\omega) \tag{7.10}$$

This property is known as the *conjugation property*. Now, if $x(t)$ is a real function of t, then $x(t) = x^*(t)$, and from the conjugation property, we find that

$$X(-\omega) = X^*(\omega) \tag{7.11a}$$

This is the *conjugate symmetry* property of the Fourier transform, applicable to real $x(t)$. Therefore, for real $x(t)$

$$|X(-\omega)| = |X(\omega)| \tag{7.11b}$$

$$\angle X(-\omega) = -\angle X(\omega) \tag{7.11c}$$

Thus, for real $x(t)$, the amplitude spectrum $|X(\omega)|$ is an even function, and the phase spectrum $\angle X(\omega)$ is an odd function of ω. These results were derived earlier for the Fourier spectrum of a periodic signal [Eq. (6.33)] and should come as no surprise.

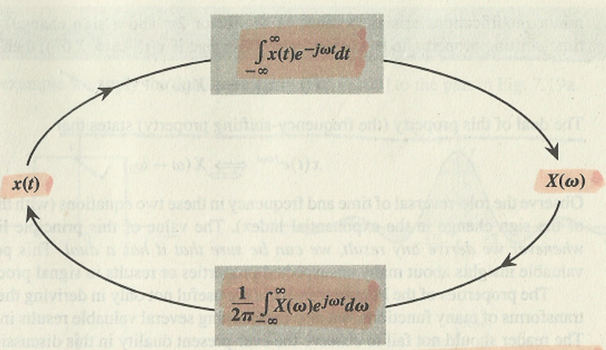


Figure 7.18 A near symmetry between the direct and the inverse Fourier transforms.

TABLE 7.1 Fourier Transforms

No.	$x(t)$	$X(\omega)$	
1	$e^{-at}u(t)$	$\frac{1}{a+j\omega}$	$a>0$
2	$e^{at}u(-t)$	$\frac{1}{a-j\omega}$	$a>0$
3	$e^{-a\lvert t\rvert}$	$\frac{2a}{a^2+\omega^2}$	$a>0$
4	$te^{-at}u(t)$	$\frac{1}{(a+j\omega)^2}$	$a>0$
5	$t^n e^{-at}u(t)$	$\frac{n!}{(a+j\omega)^{n+1}}$	$a>0$
6	$\delta(t)$	1	
7	1	$2\pi\delta(\omega)$	
8	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$	
9	$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$	
10	$\sin\omega_0 t$	$j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$	
11	$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$	
12	$\operatorname{sgn} t$	$\frac{2}{j\omega}$	
13	$\cos\omega_0 t\, u(t)$	$\frac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]+\frac{j\omega}{\omega_0^2-\omega^2}$	
14	$\sin\omega_0 t\, u(t)$	$\frac{\pi}{2j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]+\frac{\omega_0}{\omega_0^2-\omega^2}$	
15	$e^{-at}\sin\omega_0 t\, u(t)$	$\frac{\omega_0}{(a+j\omega)^2+\omega_0^2}$	$a>0$
16	$e^{-at}\cos\omega_0 t\, u(t)$	$\frac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$	$a>0$
17	$\operatorname{rect}\left(\frac{t}{\tau}\right)$	$\tau\operatorname{sinc}\left(\frac{\omega\tau}{2}\right)$	
18	$\frac{W}{\pi}\operatorname{sinc}(Wt)$	$\operatorname{rect}\left(\frac{\omega}{2W}\right)$	
19	$\Delta\left(\frac{t}{\tau}\right)$	$\frac{\tau}{2}\operatorname{sinc}^2\left(\frac{\omega\tau}{4}\right)$	
20	$\frac{W}{2\pi}\operatorname{sinc}^2\left(\frac{Wt}{2}\right)$	$\Delta\left(\frac{\omega}{2W}\right)$	
21	$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\omega_0\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_0)$	$\omega_0=\frac{2\pi}{T}$
22	$e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi}\,e^{-\sigma^2\omega^2/2}$	

Fourier transform is used to approximate aperiodic signals from time to frequency domains. For instance, a single pulse with duration T can be approximated by using the Fourier Transform.

The fourier transform of an aperiodic signal is defined as:

$$X(w) = \int_{-\infty}^{\infty} x(t)e^{-jwt}\,dt$$

A signal that is not periodic is called an aperiodic signal

From referenced textbook - example

For example we can compute the continuous time Fourier Transform of a rectangular pulse with duration T, using MathCAD as shown in Figure 3.19 and Equation (3.5).

$$x(t) = \begin{cases} A, & if\ |t| < \frac{T}{2} \\ 0, & otherwise \end{cases} \qquad \text{(Equ.3.5)}$$

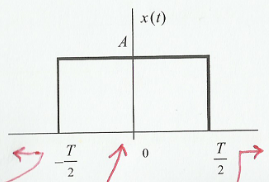


Figure 3.19: *A rectangular pulse to be approximated*

We can rewrite the Fourier integral equation for the above pulse as follows:

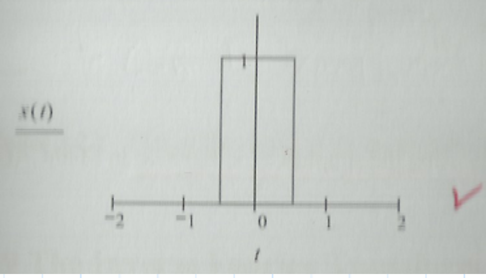
$$X(\omega) = \int_{-\infty}^{-\frac{T}{2}} 0e^{-j\omega t}dt + \int_{-\frac{T}{2}}^{\frac{T}{2}} Ae^{-j\omega t}dt + \int_{\frac{T}{2}}^{\infty} 0e^{-j\omega t}dt \qquad \text{(Equ.3.6)}$$

$A := 1$ Defining the amplitude

$T := 1$ Defining the time duration

$t := -5, -4.999..5$ Defining a range for t

$x(t) := if\left(|t| < \frac{T}{2}, A, 0\right)$ Defining the rectangular pulse



Let $j := \sqrt{-1}$ and $\omega := -5\cdot\pi, -4.9\cdot\pi..5\cdot\pi$

We can use the symbolic computation feature of MathCAD to get the result symbolically

$$X(\omega) := \int_{-\frac{T}{2}}^{\frac{T}{2}} A\cdot e^{-j\cdot\omega\cdot t}\,dt \rightarrow \frac{i}{\omega}\cdot exp\left(\frac{-1}{2}\cdot i\cdot\omega\right) - \frac{i}{\omega}\cdot exp\left(\frac{1}{2}\cdot i\cdot\omega\right)$$

The result above is the same as the one below

$$X(\omega) := \frac{j}{\omega}\cdot exp\left(\frac{-1}{2}\cdot j\cdot\omega\right) - \frac{j}{\omega}\cdot exp\left(\frac{1}{2}\cdot j\cdot\omega\right)$$

We can simplify the above expression by using the simplification function in MathCAD from the symbolic menu, which gives.

$$X(\omega) := 2\cdot\frac{sin\left(\frac{1}{2}\cdot\omega\right)}{\omega}$$

The answer can be expressed also as

$$X(\omega) = \frac{sin\left(\frac{1}{2}\cdot\omega\right)}{\left(\frac{\omega}{2}\right)}$$

Now, we can plot $X(\omega)$ (Figure 3.21) and its magnitude (Figure 3.22).

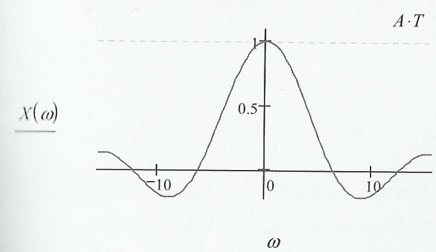


Figure 3.21: *Plot of the result of the Fourier Transform*

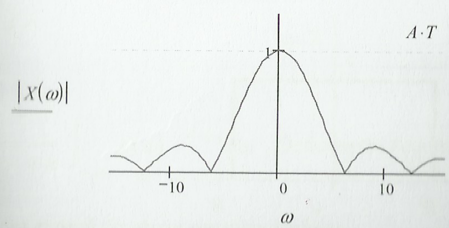


Figure 3.22: *Plot of the result of the magnitude*

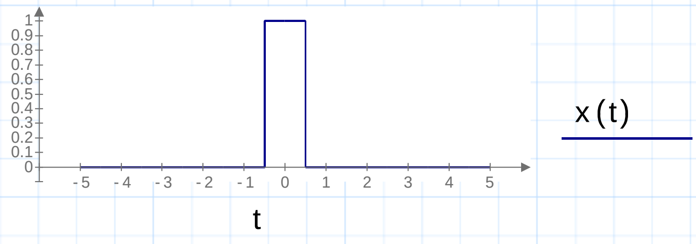
Example (of previous page worked).
An attempt to perform the explanation above in Prime/Mathcad

Define signal x(t) a rectangular pulse:

$\mathbf{clear}(x) \qquad \mathbf{clear}(n)$

$A := 1$ defining the amplitude
$T := 1$ defining the time duration
$t := -5, -4.999..5$ defining a range for t

$x(t) := \mathbf{if}\left(|t| < \left(\frac{T}{2}\right), A, 0\right)$ signal defined; magnitue of t less than T/2 at both ends sides of origin, then t = A, else 0



$j := \sqrt{-1} \qquad \omega := -5 \cdot \pi, -4.999 \cdot \pi .. 5 \cdot \pi$ remember w = 2 pi f

$$X(\omega) := \int_{-\frac{T}{2}}^{\frac{T}{2}} A \cdot e^{-j \cdot \omega \cdot t} \, dt$$

We can use the symbolic computation feature of Mathcad to get the result symbolically.

$$X(\omega) := \int_{-\frac{T}{2}}^{\frac{T}{2}} A \cdot e^{-j \cdot \omega \cdot t} \, dt \rightarrow \frac{2 \cdot \sin\left(\frac{\omega}{2}\right)}{\omega}$$ CORRECT!

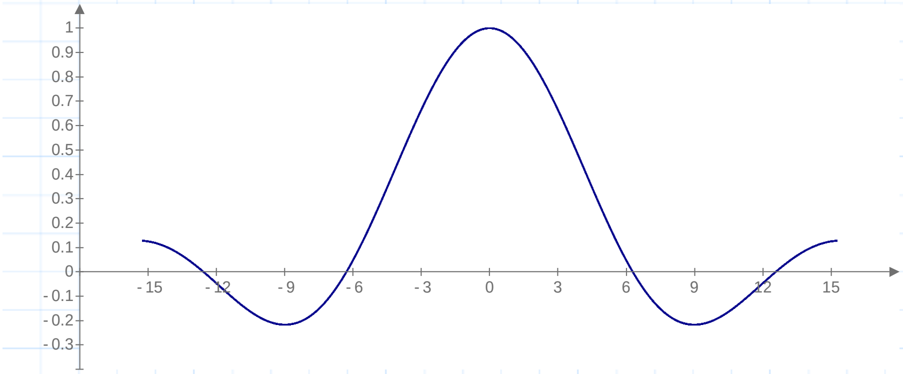
the intermediate mathematical steps:

X(w) = (j/w) exp^(-1/2)jw - (j/w) exp^(1/2)jw

this simplifies to

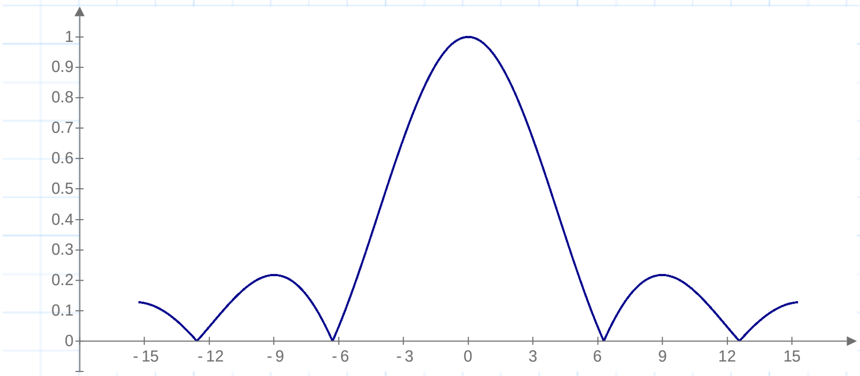
X(w) = sin(1/2)w / (w/2) ......the 2 in the numerator being at the denomintor w/2

Now the plot of X(w) and its magnitude plot



$X(\omega)$

ω



$|X(\omega)|$

ω

The first plot x(t) was in the time domain the x-axis is time, then the X(w) plot above is in the frequency domain the x-axis is frequency w=2 pi f, with f representing frequency in w.

Next 2 examples from Advanced Engineering Mathematics by H.K. Dass.

Example

Define signal x(t) a rectangular pulse:

clear (x)   **clear** (t)

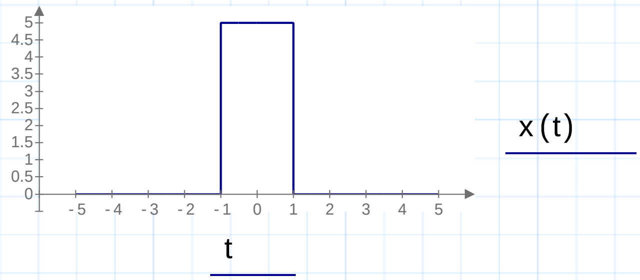
$a := 1$ defining amplitude value of 'a' for the function

$t := -5, -4.999 .. 5$ defining a range for t

clear (x)   **clear** (n)

$x(t) := \text{if}(|t| < a, 5, 0)$ defining the function to be approximated its equal to 1 when |t|<a, else 0,

Entered by: Karl S Bogha Dhaliwal - Grad Cert Power Systems Protection and Relaying Uni of Idaho. USA.
BSE - Arkansas State U 1990. BSc - USAO Oklahoma 1986.



Plot in time domain

$j := \sqrt{-1}$ $\omega := -3\cdot\pi, -2.999\cdot\pi .. 3\cdot\pi$ remember w = 2 pi f

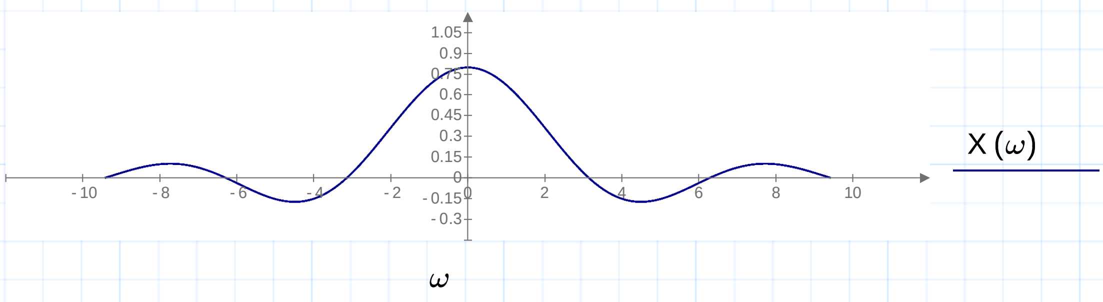
$$\left(\frac{1}{\sqrt{2\cdot\pi}}\right)\int_{-a}^{a} f(x)\cdot e^{-j\cdot\omega\cdot t}\,dt$$

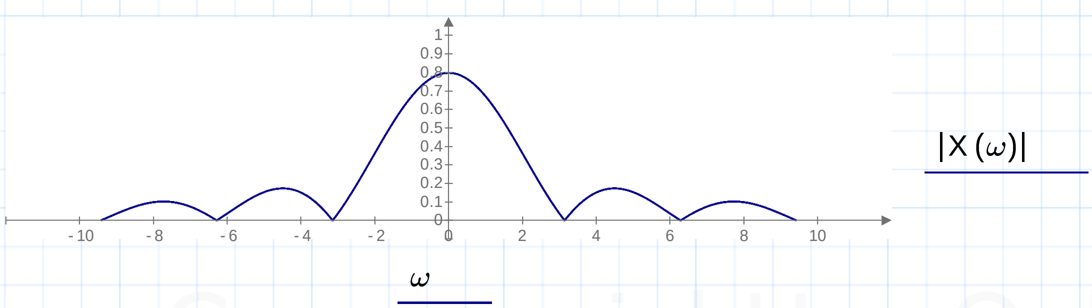
<--this is the equation we seek the Fourier transform
here f(x) is set to 'a'

$$X(\omega) := \left(\frac{1}{\sqrt{2\cdot\pi}}\right)\int_{-a}^{a} a\cdot e^{-j\cdot\omega\cdot t}\,dt$$ The Fourier Transform Formula

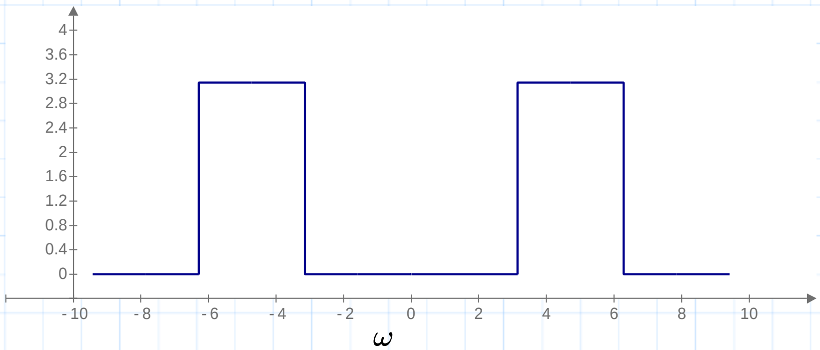
We can use the symbolic computation feature of Mathcad to get the result symbolically.

$$X(\omega) := \left(\frac{1}{\sqrt{2\cdot\pi}}\right)\int_{-a}^{a} a\cdot e^{-j\cdot\omega\cdot t}\,dt \rightarrow \frac{\sqrt{2}\cdot\sin(\omega)}{\sqrt{\pi}\cdot\omega}$$ Answer





Since X(w) is complex we have both magnitude and phase angle. Phase angle shown below.



Example

Define signal x(t):

clear(x)   clear(ω)

$a := 1$ defining amplitude value of 'a' for the function

$t := -5, -4.999 .. 5$ defining a range for t

clear(x)   clear(n)

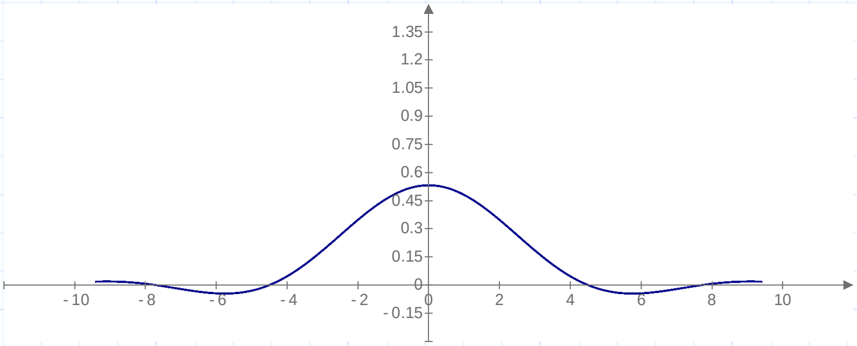
$x(t) := \text{if}\left(|t| \le a, \left(1 - t^2\right), 0\right)$

Plot in time domain

$j := \sqrt{-1}$ $\omega := -3\cdot\pi, -2.999\cdot\pi .. 3\cdot\pi$

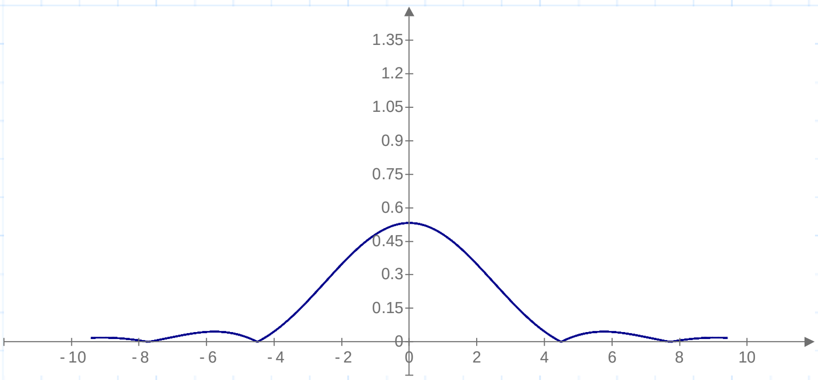
$$X(\omega) := \left(\frac{1}{\sqrt{2\cdot\pi}}\right)\int_{-a}^{a}\left(1 - t^2\right)\cdot e^{-j\cdot\omega\cdot t}\,dt$$ The Fourier Transform Formula

$$X(\omega) := \left(\frac{1}{\sqrt{2\cdot\pi}}\right)\int_{-a}^{a}\left(1 - t^2\right)\cdot e^{-j\cdot\omega\cdot t}\,dt \rightarrow \frac{\sqrt{2}\cdot(2\cdot\sin(\omega) - 2\cdot\omega\cdot\cos(\omega))}{\sqrt{\pi}\cdot\omega^3}$$ Answer



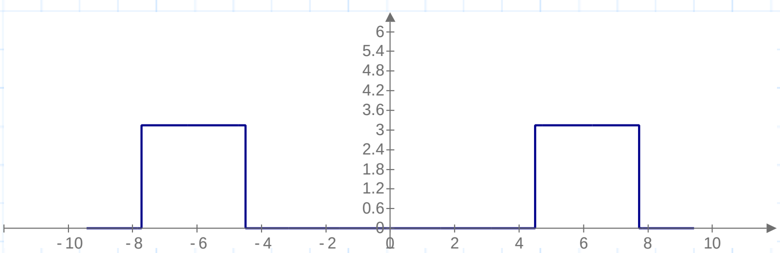
$X(\omega)$

ω



$|X(\omega)|$

ω



$\arg(X(\omega))$

ω

Next examples are from Chp 7 Linear Systems and Signals 2nd ed B.P. Lathi. This is an indepth/comprehensive treatment on the subject, highly recommended. For undergraduates.

Example (Not very clean but makes for learning)
Find the Fourier transform of (e^-(at)) u(t)?
Product of exponential and step function.

clear(x) clear(u) $t := 0$ $a := 0$

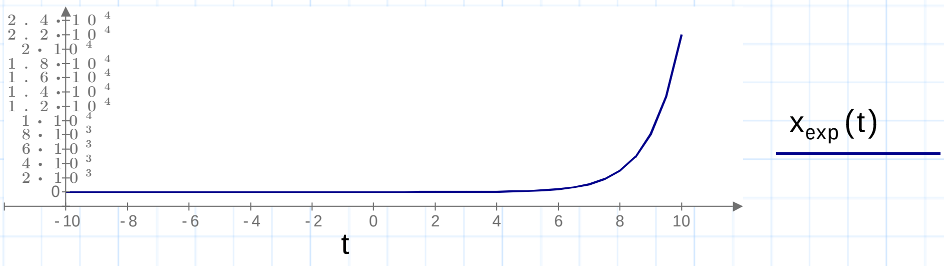
$u(t) := \Phi(t)$ step function u(t) set equal to heavyside function for Prime-Mathcad
step function in conditional statement will not work in the integral calculation for Fourier transform

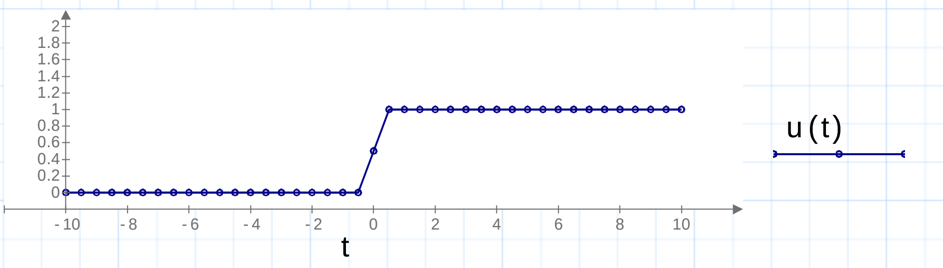
$a := -1$ <--- this sets value of 'a' in exp. function
Change the value of 'a' above to +ve and -ve, see the changes in the plots. When
(a) <0 then (e^-(at)) u(t) = infinity, (set plot y-axis upper scale to 1000)
(a) >0 then (e^-(at)) u(t) = 0

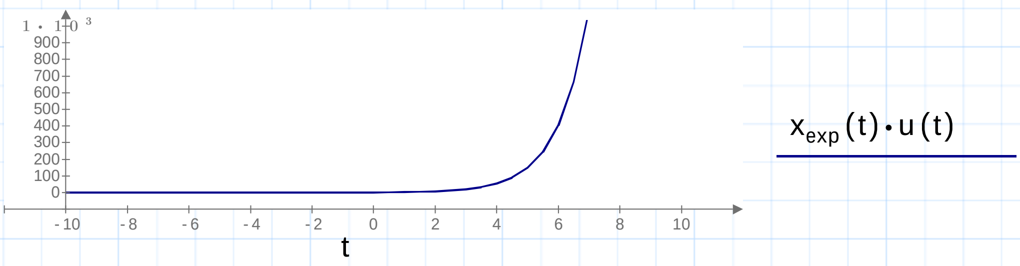
$t := -10, -9.5 .. 10$ defining a range for t

$x_{exp}(t) := e^{-a \cdot t}$

Plots of the original signals:







For the above (e^-(at)) u(t) plot see the red notes above to change the plot results

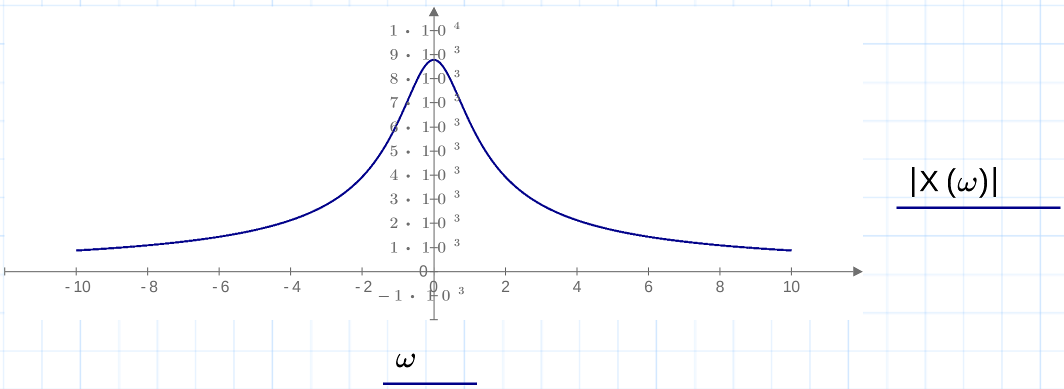
$j := \sqrt{-1}$ $\omega := -10 \cdot \pi, -9.999 \cdot \pi .. 10 \cdot \pi$ $X(\omega) := 0$ initialise X(w)

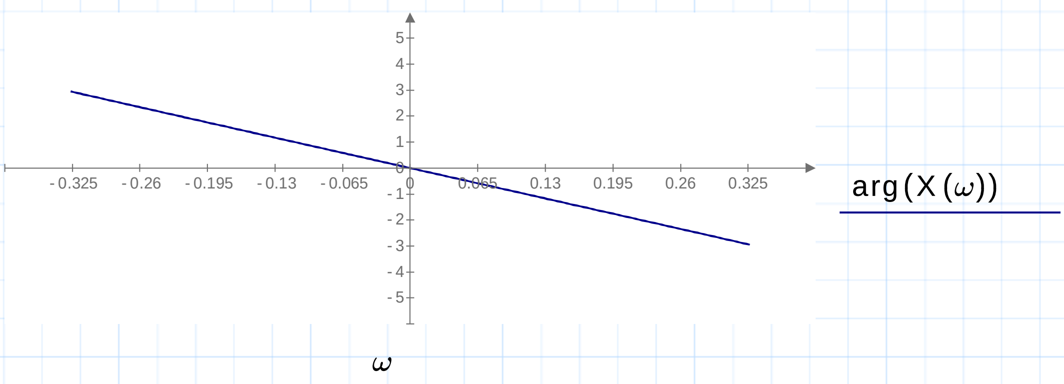
$$X(\omega) := \left(\frac{1}{\sqrt{2 \cdot \pi}}\right) \int_{-10}^{10} x_{exp}(t) \cdot u(t) \cdot e^{-j \cdot \omega \cdot t} \, dt$$ The Fourier Transform Formula

To calculate the integral set a = 1 or -1, the limits from 0 to 10 since there it it NOT equal to 0 here.

$$X(\omega) := \left(\frac{1}{\sqrt{2 \cdot \pi}}\right) \int_{-0}^{10} \left(x_{exp}(t) \cdot u(t)\right) \cdot e^{-j \cdot \omega \cdot t} \, dt \rightarrow \frac{\sqrt{2} \cdot \left(e^{10} \cdot e^{-(10i \cdot \omega)} - 1\right) \cdot (1 + \omega \cdot 1i)}{2 \cdot \sqrt{\pi} \cdot \left(\omega^2 + 1\right)}$$

The above integral result is not the textbook example answer, because its a theorectical example. However the calculated integral shows Prime can perform the calculation, the magnitude plot is close to the theorectical result in terms of the shape of the plot, the amplitude is high. Similarly phase angle plot is slightly different.

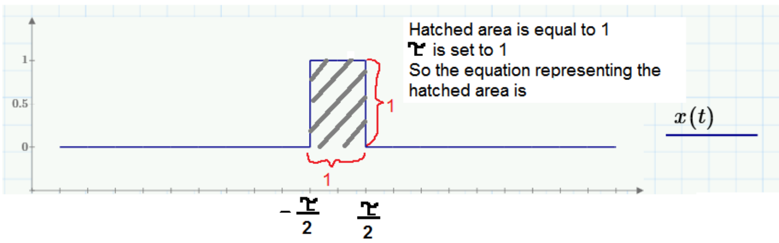




Example - Worked out for a solution, makes for a learning exercise, results may look accurate. The logic maybe close enough! Textbook results at end of example.

Define signal x(t) as a rectangular pulse (unit gate pulse) - rect(tau/t):

clear (x) **clear** (ω) **clear** (t) **clear** (n)



Sketch example for meeting the objective of the example problem.

$n_\tau := 1$ <----- enter 1 for width of signal of unit value horizontal axis

$pu_{scale} := \pi = 3.142$ the unit for the horizontal and vertical axis is pi 3.142

$$\tau_{delta} := \frac{\left(\frac{n_\tau \cdot (\pi)}{2} - \frac{(-n_\tau \cdot \pi)}{2}\right)}{pu_{scale}}$$

time domain signal width - tau axis horizontal

$\tau_{delta} = 1$

The unit gate pulse (rect(t)) is expanded by a factor tau along the horizontal axis, and therefore can be expressed as rect(t/tau)

$AmpV_{vrt} := n_\tau$ amplitude value - vertical axis is the voltage with a value equal to nt (ntau)

$x_{area\_t} := (\tau_{delta} \cdot AmpV_{vrt})$ time domain signal height - amplitude equal to tau value at horizontal axis

$x_{area\_t} = 1$ pulse area

now set the value of tau with respect to the plot of the signal above

$$\tau := \frac{n_\tau \cdot (\pi)}{2}$$

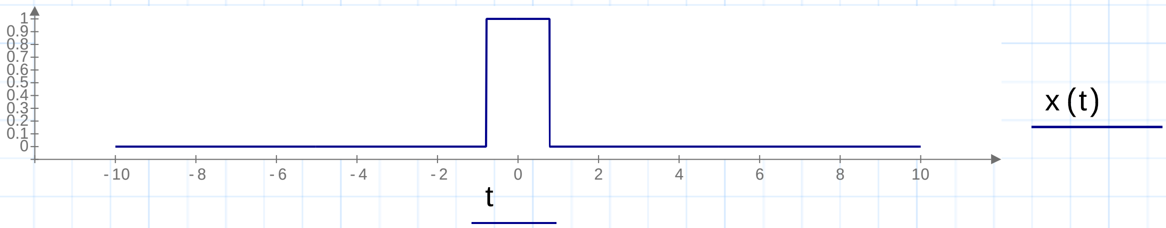
$n_t := 5$ time range - horizontal axis

$t := -10, -9.99 .. 10$

$x(t) := \text{if}\left(|t| < \frac{\tau}{2}, x_{area\_t}, 0\right)$ defining the function equal to x_area_t when |t|<tau/2, else 0,

Now we will focus just on the one rectangular pulse with a width of tau (horizontal axis) and V = 1 (vertical axis). Just the one rectangular pulse. This attempt is amateur but signals are formed on user needs, and the objective is to apply Fourier transforms in a mathematical software. You can form your own signals!

Plot in time domain this is the signal shape we seek



$j := \sqrt{-1}$

$\tau := \tau_{delta} = 1$

$n_w := n_t \cdot 2$

$$\omega := \frac{-(n_w \cdot \pi)}{\tau}, \frac{-((n_w - 0.01) \cdot \pi)}{\tau} .. \frac{(n_w \cdot \pi)}{\tau}$$

Equation of the signal to apply in the Fourier transform integral?

Our rectangular pulse area would make the euqation for the signal, which is equal to the value of the amplitude (area = 1, amplitude voltage = 1)

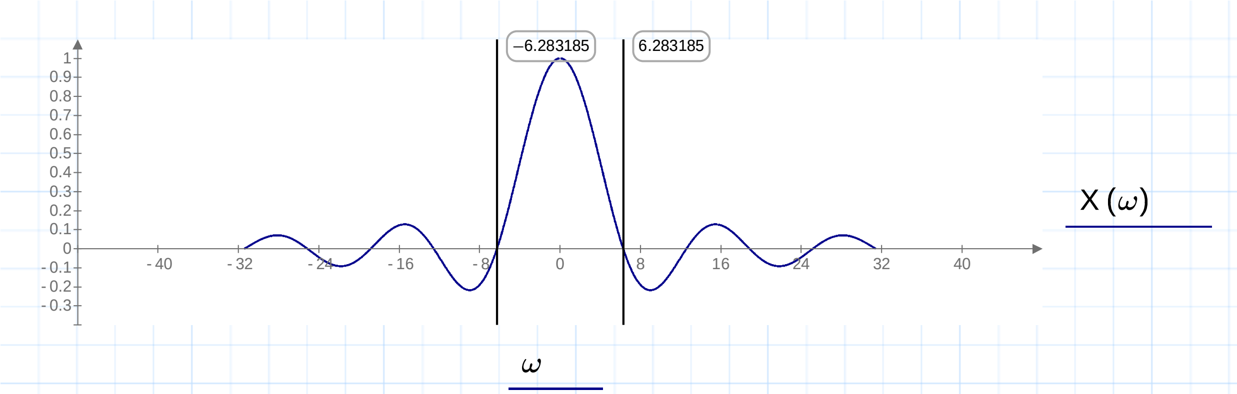
Hence we deduce rect(tau/t) = 1 = x_area_t = 1
our equation in the integral is rect(tau/t) = 1
so we let x = 1

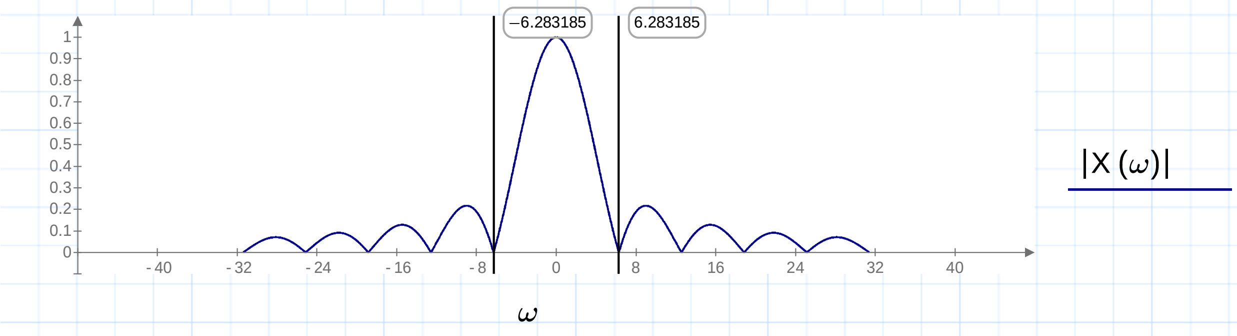
$x := 1$ equation of signal

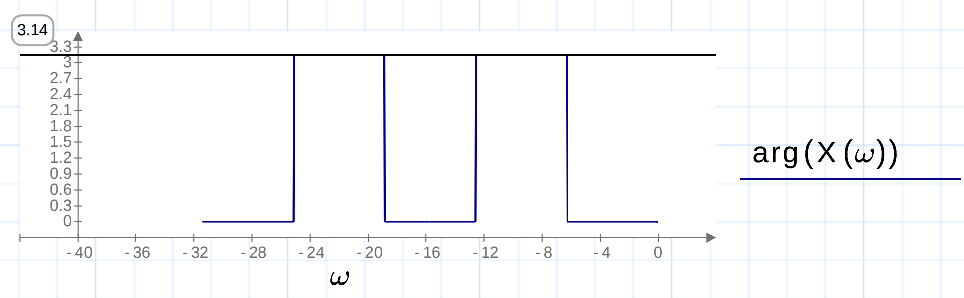
$$X(\omega) := \int_{\frac{-\tau}{2}}^{\frac{\tau}{2}} (x) \cdot e^{-j \cdot \omega \cdot t} \, dt$$ The Fourier Transform Formula

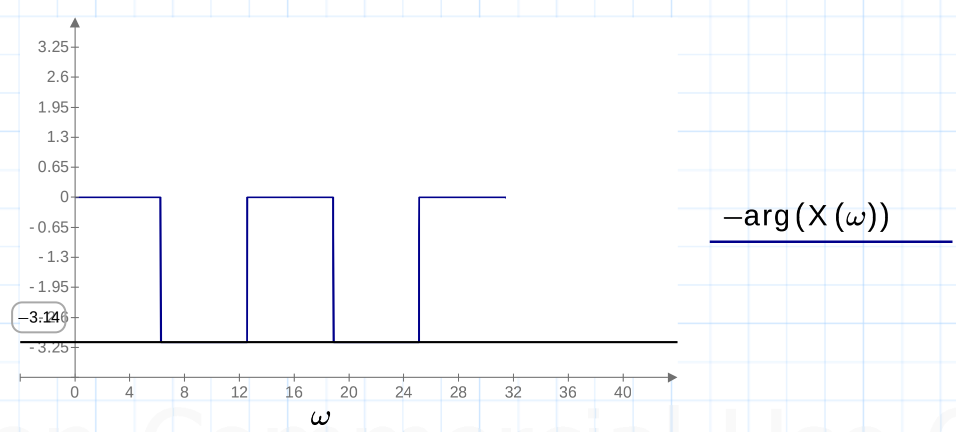
$$X(\omega) := \int_{\frac{-\tau}{2}}^{\frac{\tau}{2}} (x) \cdot e^{-j \cdot \omega \cdot t} \, dt \rightarrow \frac{2 \cdot \sin\left(\frac{\omega}{2}\right)}{\omega}$$ Answer

Sinc(t) = sin (t)/ t. Sinc(t) is an even function. Using L'Hopitals rule sinc(0)=1.
The above answer can be reduced to X(w) = (tau)sinc ((w tau)/2), where tau = 1.


–6.283185
6.283185
X (ω)
ω


–6.283185
6.283185
|X (ω)|
ω


3.14
arg(X (ω))
ω


–3.14
–arg(X (ω))
ω

Entered by: Karl S Bogha Dhaliwal - Grad Cert Power Systems Protection and Relaying Uni of Idaho. USA.
BSE - Arkansas State U 1990. BSc - USAO Oklahoma 1986.

Find the Fourier transform of $x(t) = \text{rect}(t/\tau)$ (Fig. 7.10a).

$$X(\omega) = \int_{-\infty}^{\infty} \text{rect}\left(\frac{t}{\tau}\right) e^{-j\omega t}\,dt$$

Since $\text{rect}(t/\tau) = 1$ for $|t| < \tau/2$, and since it is zero for $|t| > \tau/2$,

$$X(\omega) = \int_{-\tau/2}^{\tau/2} e^{-j\omega t}\,dt$$

$$= -\frac{1}{j\omega}\left(e^{-j\omega\tau/2} - e^{j\omega\tau/2}\right) = \frac{2\sin\left(\frac{\omega\tau}{2}\right)}{\omega}$$

$$= \tau\,\frac{\sin\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)} = \tau\,\text{sinc}\left(\frac{\omega\tau}{2}\right)$$

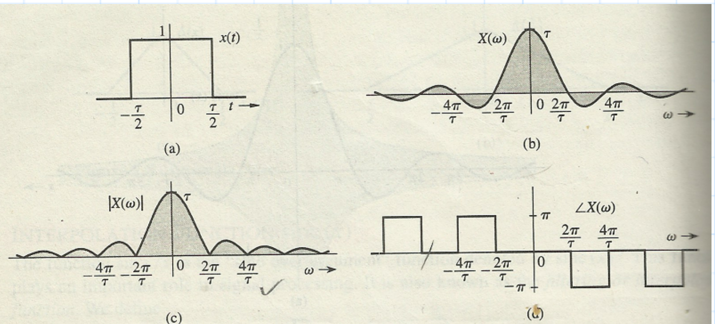


Figure 7.10 **(a)** A gate pulse $x(t)$, **(b)** its Fourier spectrum $X(\omega)$, **(c)** its amplitude spectrum $|X(\omega)|$, and **(d)** its phase spectrum $\angle X(\omega)$.

Therefore

$$\text{rect}\left(\frac{t}{\tau}\right) \Longleftrightarrow \tau\,\text{sinc}\left(\frac{\omega\tau}{2}\right) \tag{7.21}$$

Recall that $\text{sinc}(x) = 0$ when $x = \pm n\pi$. Hence, $\text{sinc}(\omega\tau/2) = 0$ when $\omega\tau/2 = \pm n\pi$; that is, when $\omega = \pm 2n\pi/\tau$, $(n = 1, 2, 3, \ldots)$, as depicted in Fig. 7.10b. The Fourier transform $X(\omega)$ shown in Fig. 7.10b exhibits positive and negative values. A negative amplitude can be considered to be a positive amplitude with a phase of $-\pi$ or π. We use this observation to plot the amplitude spectrum $|X(\omega)| = |\text{sinc}(\omega\tau/2)|$ (Fig. 7.10c) and the phase spectrum $\angle X(\omega)$ (Fig. 7.10d). The phase spectrum, which is required to be an odd function of ω, may be drawn in several other ways because a negative sign can be accounted for by a phase of $\pm n\pi$, where n is any odd integer. All such representations are equivalent.

Explanation provided above.

Applying the integral itself is not enough we need to have some idea of function and what its integral would produce. We need to study each type of signal function with respect to signal processing.

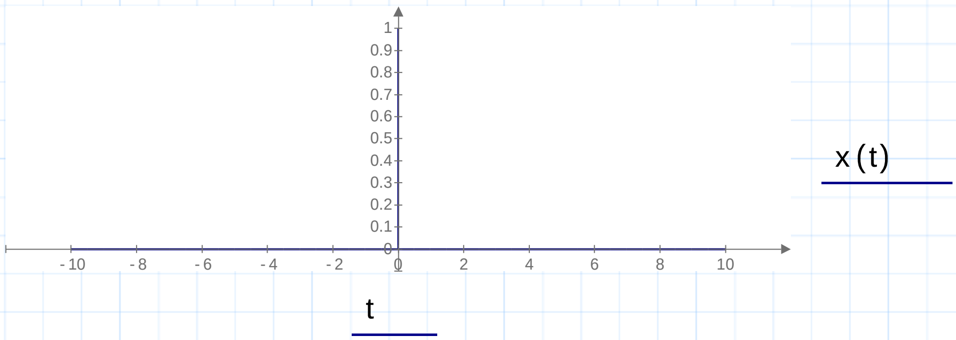
Example

Find the Fourier transform of the unit impulse delta(t)?

$\mathbf{clear}(x) \quad \mathbf{clear}(t)$

$x(t) := \mathbf{if}(t = 0, 1, 0)$ unit impulse function

$t := -10, -9.999 .. 10$ the interval has to be made small here it is 0.001 then the impulse of value 1 is seen as a straight line at time t = 0, otherwise a triangle shape is seen at t=0



The conditional signal cannot be placed in the Fourier transform equation.
The signal is 1 at time t=0, elsewhere it is 0. So the equation that best represents the function is x(t=0) = 1, which is really x=1.
So the Fourier transform of the unit impulse function = 1

$t := 0 \qquad \delta(t, 0) = 1$ This is Prime delta function for unit impulse delta(m,n) m=t time, when t= 0 t=n, and the function returns the value of 1. This function will not work in the integral either. Placing a 1 in the integral is not accurate because thats the output of the function not the function itself.

$\omega := -0.01 \cdot \pi, -0.009\,\pi .. 0.01 \cdot \pi$

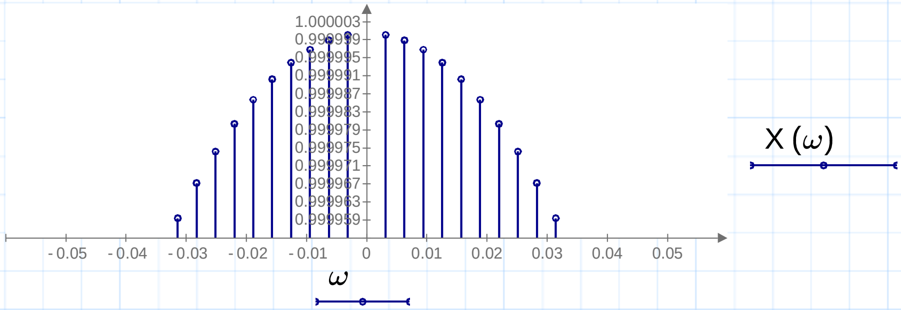
$$X(\omega) := \int_{\frac{-1}{2}}^{\frac{1}{2}} 1 \cdot e^{-j \cdot \omega \cdot t}\, dt$$

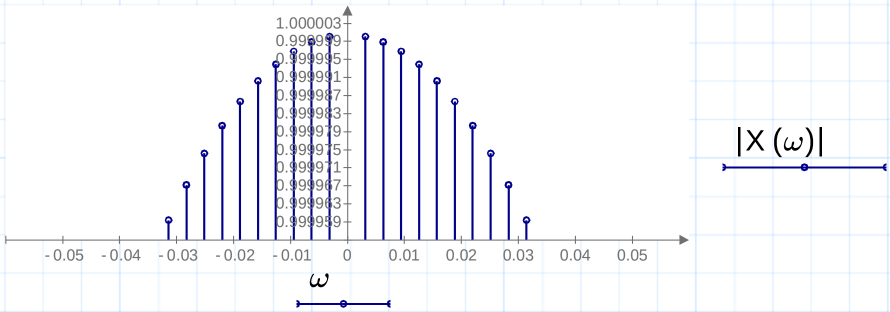
The Fourier Transform Formula
We place the constant 1 in the integral and set the upper and lower limits to 0.5 and -0.5 which would result with a value of 1 at t=0 in the plot, for limits of w=-0.01pi to 0.01pi.

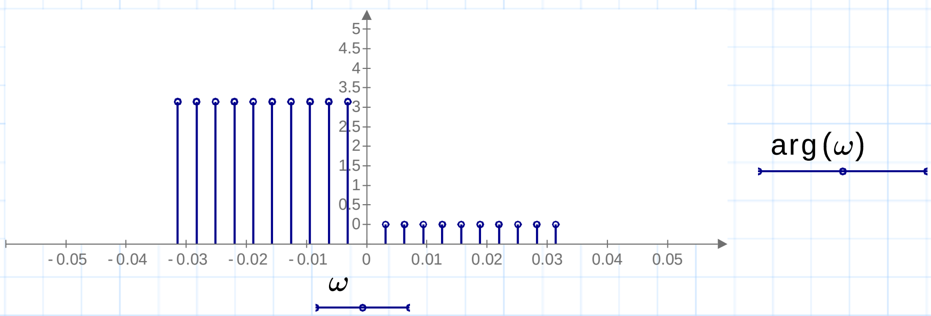
$$X(\omega) := \int_{\frac{-1}{2}}^{\frac{1}{2}} 1 \cdot e^{-j \cdot \omega \cdot t}\, dt \rightarrow \frac{2 \cdot \sin\left(\frac{\omega}{2}\right)}{\omega}$$

We did not expect to the se sinc function as the result of the integral

The plots show some output we can relate to the phase angle is zero on the positive w-axis. You may be able to do better wrt to the integral.







Example - Cosine

Find the Fourier transform of the everlasting sinusoid cos w0(t)

$t := -10, -9.999 .. 10$

$f_0 := 50$

$\omega_0 := 2 \cdot \pi \cdot f_0$ fundamental radian frequency

$A := 1$ ampliutude of the cosine signal

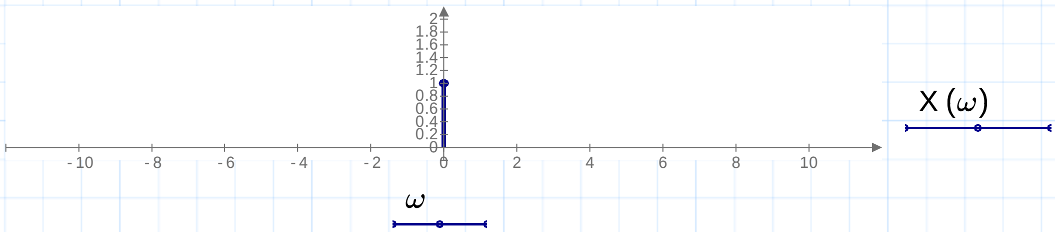
$w := -10 \cdot \omega_0, -9.99 \cdot \omega_0 .. 10 \cdot \omega_0$

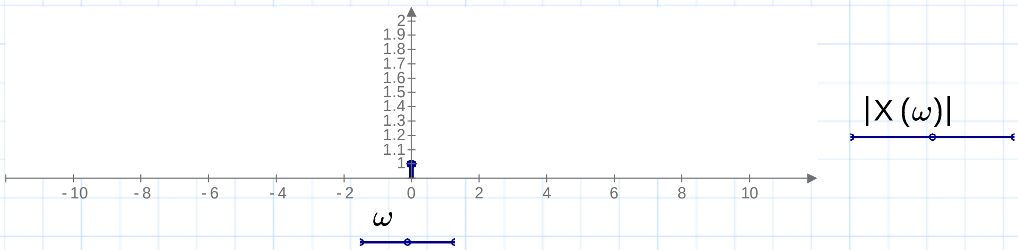
$x(t) := A \cdot \cos(\omega_0 \cdot t)$

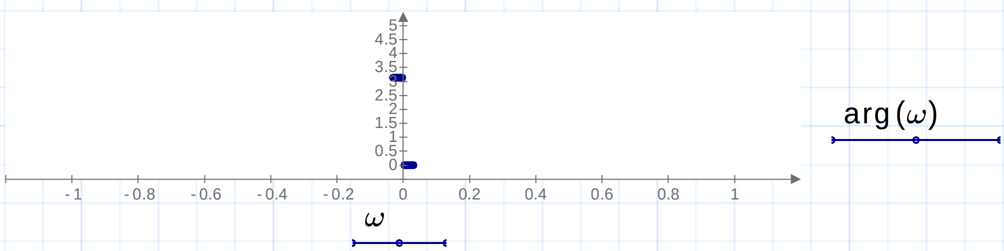
by setting the limits at 1 to -1 we get the result of 1

$$X(\omega) := \int_{-1}^{1} x(t) \cdot e^{-j \cdot \omega_0 \cdot t}\, dt \rightarrow 1$$

These plots look concurring but the logic is inaccurate. See the textbook explanation below the plots. WRONG!







Explanation provided below.

Find the Fourier transforms of the everlasting sinusoid $\cos \omega_0 t$ (Fig. 7.13a).

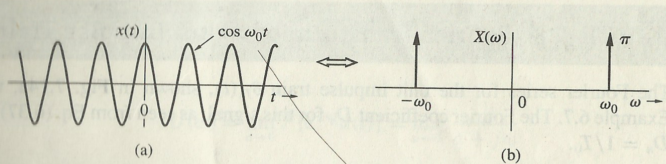


Figure 7.13 (a) A cosine signal and (b) its Fourier spectrum.

Recall the Euler formula

$$\cos \omega_0 t = \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})$$

Adding Eqs. (7.24a) and (7.24b), and using the foregoing result, we obtain

$$\cos \omega_0 t \Longleftrightarrow \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)] \qquad (7.25)$$

The spectrum of $\cos \omega_0 t$ consists of two impulses at ω_0 and $-\omega_0$, as shown in Fig. 7.13b. The result also follows from qualitative reasoning. An everlasting sinusoid $\cos \omega_0 t$ can be synthesized by two everlasting exponentials, $e^{j\omega_0 t}$ and $e^{-j\omega_0 t}$. Therefore the Fourier spectrum consists of only two components of frequencies ω_0 and $-\omega_0$.

TABLE 7.2 Fourier TransformsOperations

Operation	$x(t)$	$X(\omega)$
Scalar multiplication	$kx(t)$	$kX(\omega)$
Addition	$x_1(t)+x_2(t)$	$X_1(\omega)+X_2(\omega)$
Conjugation	$x^*(t)$	$X^*(-\omega)$
Duality	$X(t)$	$2\pi x(-\omega)$
Scaling (a real)	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{\omega}{a}\right)$
Time shifting	$x(t-t_0)$	$X(\omega)e^{-j\omega t_0}$
Frequency shifting (ω_0 real)	$x(t)e^{j\omega_0 t}$	$X(\omega-\omega_0)$
Time convolution	$x_1(t)*x_2(t)$	$X_1(\omega)X_2(\omega)$
Frequency convolution	$x_1(t)x_2(t)$	$\frac{1}{2\pi}X_1(\omega)*X_2(\omega)$
Time differentiation	$\frac{d^n x}{dt^n}$	$(j\omega)^n X(\omega)$
Time integration	$\int_{-\infty}^{t} x(u)\,du$	$\frac{X(\omega)}{j\omega}+\pi X(0)\delta(\omega)$

Here the focus is on signal processing so we are concerned with signals that are of use in applications, not solving problems in mathematics in general.

So the function to be Fourier transformed needs to have signal characteristics or properties for application and or processing in systems.

Inverse Fourier Transform

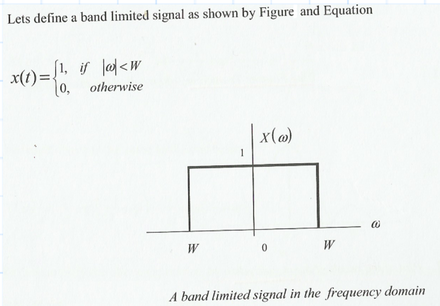
In our use of Fourier Transforms we want the result to be represented with respect to the radian frequency w. This is what we see in the horizontal axis. Frequency Domain.

Given the Fourier Transform of a signal (wrt to w) we want to find the original signal with respect to time.
Time Domain.

$$x(t)=\frac{1}{2\pi}\int_{-\frac{T}{2}}^{\frac{T}{2}} X(\omega)e^{j\omega t}d\omega$$

<--- Equation for the Inverse Fourier Transform

Example 3.8 - Chp 3 Freq. Domain Analysis



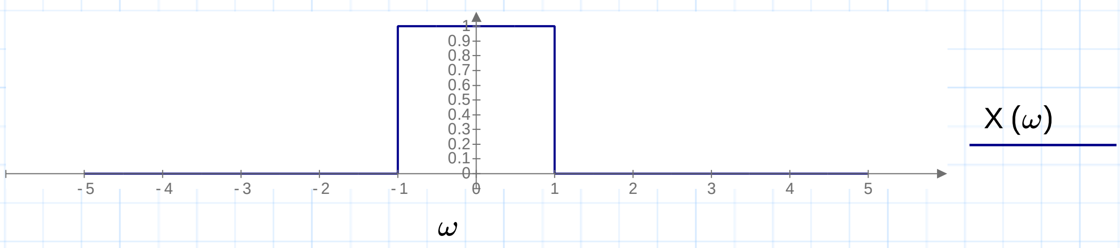
Apply Prime/Mathcad to find the inverse Fourier Transform of the above signal as follows:

Lets define w and X(w)

$W := 1$ defines the width from -W to W = 1

$\omega := -5, -4.999 .. 5$ Set a range

$X(\omega) := \text{if}(|\omega| < W, 1, 0)$ defining the function - this is correct



Now the inverse Fourier transform of X(w)

$j := \sqrt{-1}$ For the infinity sign ==> ctrl + Shift + z

$$x(t) := \left(\frac{1}{2 \cdot \pi}\right) \cdot \int_{-\infty}^{\infty} X(\omega) \cdot e^{j \cdot \omega \cdot t} \, d\omega$$ defining the Fourier transform equation

Look at the plot above for X(w) from the left side, coming from infinity to W, then W to -W, and finally from -W to infinity.

$$x(t) := \left(\frac{1}{2 \cdot \pi}\right) \cdot \int_{-\infty}^{-W} X(\omega) \cdot e^{j \cdot \omega \cdot t} \, d\omega + \left(\frac{1}{2 \cdot \pi}\right) \cdot \int_{-W}^{W} X(\omega) \cdot e^{j \cdot \omega \cdot t} \, d\omega + \left(\frac{1}{2 \cdot \pi}\right) \cdot \int_{W}^{\infty} X(\omega) \cdot e^{j \cdot \omega \cdot t} \, d\omega$$

The two side integrals produce the result of 0 in the plot so we concentrate on the middle integral which results in 1:

$X(\omega) := 1$

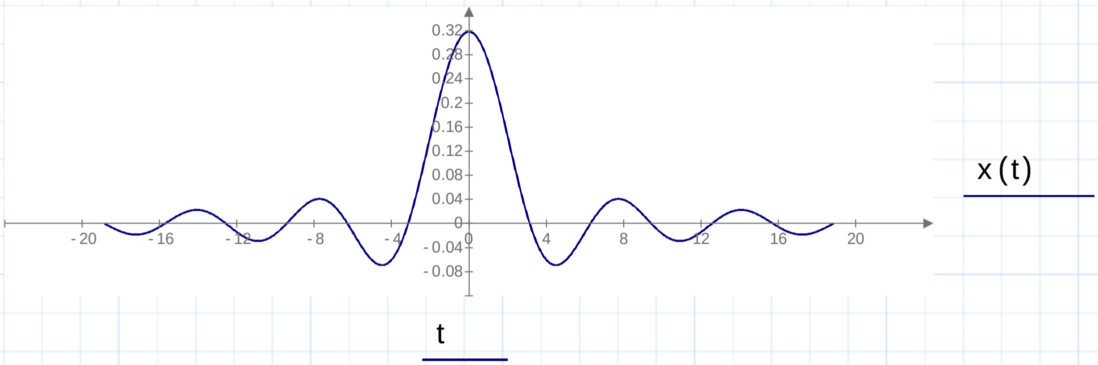
$$x(t) := \left(\frac{1}{2\cdot\pi}\right)\cdot\int_{-W}^{W} X(\omega)\cdot e^{j\cdot\omega\cdot t}\,d\omega$$

$$x(t) := \left(\frac{1}{2\cdot\pi}\right)\cdot\int_{-W}^{W} X(\omega)\cdot e^{j\cdot\omega\cdot t}\,d\omega \rightarrow \frac{\sin(t)}{\pi\cdot t}$$

the steps include exponential terms, euler's identity sine term and it solves for the sine term = sin(t)/((pi)t)

set t range for plot of x(t)

$t := -6\cdot\pi, -5.99\cdot\pi..6\cdot\pi$



The result of the inverse Fourier transform.

Example - Signals and Systems 2nd ed by B.P. Lathi

Find the inverse Fourier transform of del(w):

from - infinity to -0.0001 = 0, from 0 to 0 = 1, from 0.0001 to infinity = 0
so the value of the integral is at 0 so we use the middle intergral

clear (ω)

$$\delta(\omega) := \left(\frac{1}{2\cdot\pi}\right)\cdot\int_{-\infty}^{\infty} \delta(\omega)\cdot e^{-j\cdot\omega\cdot t}\,d\omega$$

at w = 0, the function = 1, elsewhere = 0

so evaluating manually

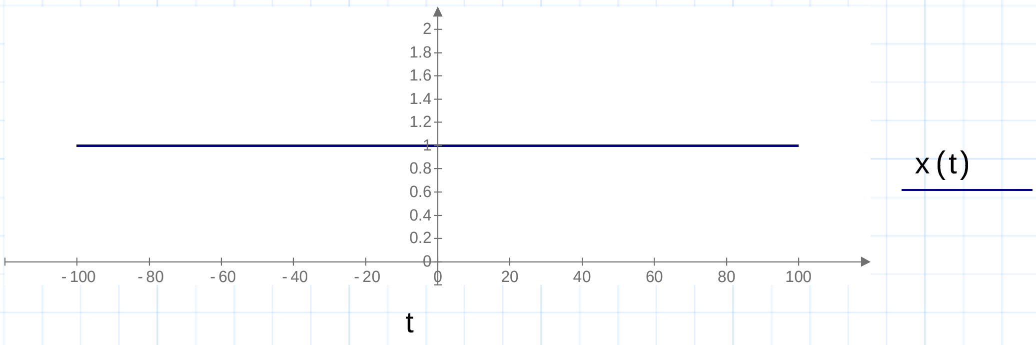
$\delta(w) := \left(\frac{1}{2 \cdot \pi}\right) \cdot 1$

$\delta(\omega) := \left(\frac{1}{2 \cdot \pi}\right)$

$2\ \pi \cdot \delta(\omega)$ = 1 is the amplitude of the signal

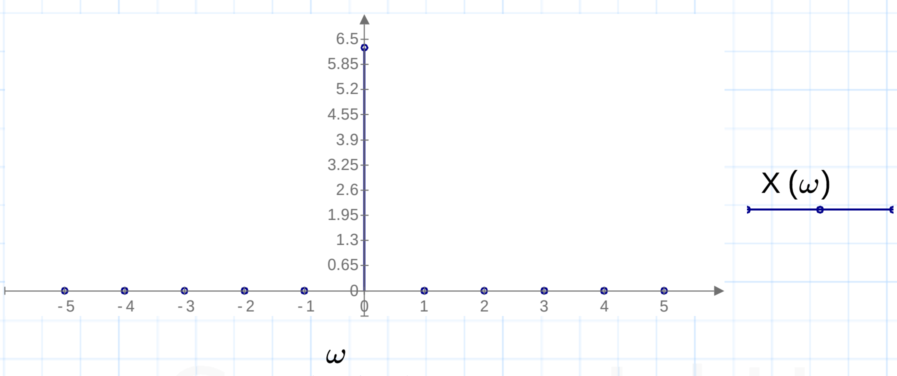
so x(t) = 2 pi del(w) = 1
x(t) = 1

$t := -100, -99..100$
$x(t) := 1$



Prime impulse function d(m,n) m=n = 1. When the value of m equal to n, spif n = 5 then del(m,5)=1 if m = 5

$\omega := -5, -4..5$
clear (δ)
$X(\omega) := 2 \cdot \pi \cdot \delta(0, \omega)$ when w = 0, then del(0.w)=1



Fourier Transform Spectra

Notes from Signals and Systems 2nd ed BP Lathi Publisher Oxford

7.1-1 Physical Appreciation of the Fourier Transform

In understanding any aspect of the Fourier transform, we should remember that Fourier representation is a way of expressing a signal in terms of everlasting sinusoids (or exponentials). The Fourier spectrum of a signal indicates the relative amplitudes and phases of sinusoids that are required to synthesize that signal. A periodic signal Fourier spectrum has finite amplitudes and exists at discrete frequencies (ω_0 and its multiples). Such a spectrum is easy to visualize, but the spectrum of an aperiodic signal is not easy to visualize because it has a continuous spectrum. The continuous spectrum concept can be appreciated by considering an analogous, more tangible phenomenon. One familiar example of a continuous distribution is the loading of a beam. Consider a beam loaded with weights $D_1, D_2, D_3, \ldots, D_n$ units at the uniformly spaced points $y_1, y_2, \ldots, y_n$, as shown in Fig. 7.5a. The total load W_T on the beam is given by the sum of these loads at each of the n points:

$$W_T = \sum_{i=1}^{n} D_i$$

Consider now the case of a continuously loaded beam, as depicted in Fig. 7.5b. In this case, although there appears to be a load at every point, the load at any one point is zero. This does not mean that there is no load on the beam. A meaningful measure of load in this situation is not the load at a point, but rather the loading density per unit length at that point. Let $X(y)$ be the loading density per unit length of beam. It then follows that the load over a beam length $\Delta y (\Delta y \to 0)$, at some point y, is $X(y)\Delta y$. To find the total load on the beam, we divide the beam into segments of interval $\Delta y (\Delta y \to 0)$. The load over the nth such segment of length Δy is $X(n\Delta y)\Delta y$. The total load W_T is given by

$$W_T = \lim_{\Delta y \to 0} \sum_{y_1}^{y_n} X(n\Delta y)\,\Delta y$$

$$= \int_{y_1}^{y_n} X(y)\,dy$$

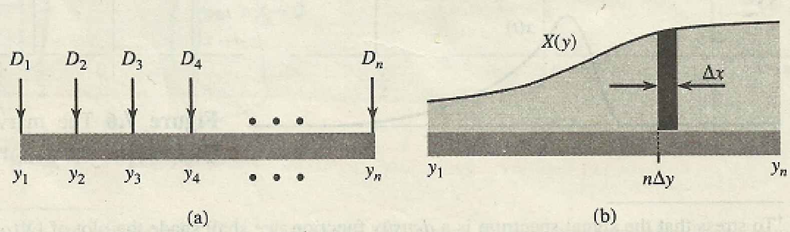


Figure 7.5 Weight-loading analogy for the Fourier transform.

CHAPTER 7 CONTINUOUS-TIME SIGNAL ANALYSIS: THE FOURIER TRANSFORM

The load now exists at every point, and y is now a continuous variable. In the case of discrete loading (Fig. 7.5a), the load exists only at the n discrete points. At other points there is no load. On the other hand, in the continuously loaded case, the load exists at every point, but at any specific point y, the load is zero. The load over a small interval Δy, however, is $[X(n\Delta y)]\,\Delta y$ (Fig. 7.5b). Thus, even though the load at a point y is zero, the relative load at that point is $X(y)$.

An exactly analogous situation exists in the case of a signal spectrum. When $x(t)$ is periodic, the spectrum is discrete, and $x(t)$ can be expressed as a sum of discrete exponentials with finite amplitudes:

$$x(t) = \sum_n D_n e^{jn\omega_0 t}$$

For an aperiodic signal, the spectrum becomes continuous; that is, the spectrum exists for every value of ω, but the amplitude of each component in the spectrum is zero. The meaningful measure here is not the amplitude of a component of some frequency but the spectral density per unit bandwidth. From Eq. (7.6b) it is clear that $x(t)$ is synthesized by adding exponentials of the form $e^{jn\Delta\omega t}$, in which the contribution by any one exponential component is zero. But the contribution by exponentials in an infinitesimal band $\Delta\omega$ located at $\omega = n\Delta\omega$ is $(1/2\pi)X(n\Delta\omega)\Delta\omega$, and the addition of all these components yields $x(t)$ in the integral form:

$$x(t) = \lim_{\Delta\omega\to 0} \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} X(n\Delta\omega)e^{(jn\Delta\omega)t}\,\Delta\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega \qquad (7.17)$$

Thus, $n\Delta\omega$ approaches a continuous variable ω. The spectrum now exists at every ω. The contribution by components within a band $d\omega$ is $(1/2\pi)X(\omega)\,d\omega = X(\omega)\,df$, where df is the bandwidth in hertz. Clearly, $X(\omega)$ is the *spectral density* per unit bandwidth (in hertz).† It also follows that even if the amplitude of any one component is infinitesimal, the relative amount of a component of frequency ω is $X(\omega)$. Although $X(\omega)$ is a spectral density, in practice it is customarily called the *spectrum* of $x(t)$ rather than the spectral density of $x(t)$. Deferring to this convention, we shall call $X(\omega)$ the Fourier spectrum (or Fourier transform) of $x(t)$.

†To stress that the signal spectrum is a *density* function, we shall shade the plot of $|X(\omega)|$ (as in Fig. 7.4b). The representation of $\angle X(\omega)$, however, will be a by a line plot, primarily to avoid visual confusion.

Example - With its understanding (Stem Plots)

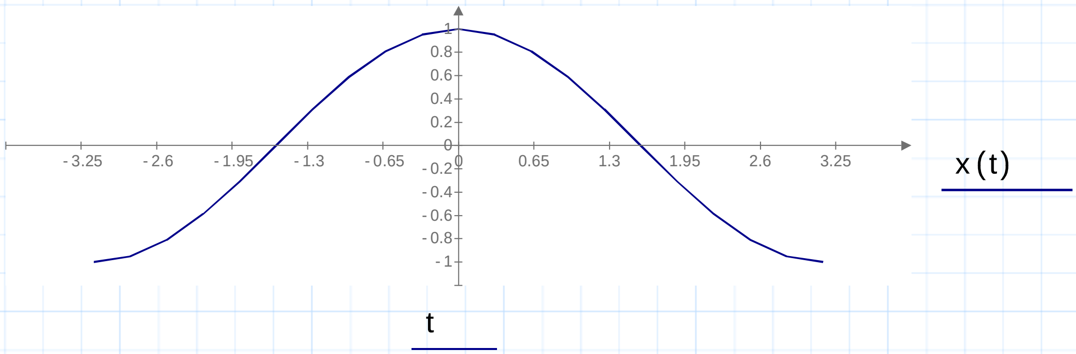
We can use the knowledge we gained from the previous section to perform spectral analysis in Mathcad.
Let the frequency of a signal be x(t) be

$\omega_0 := 1$

$x(t) := \cos(\omega_0 \cdot t)$

$t := -\pi, -0.9\cdot\pi .. \pi$

Plot of x(t):



Now the spectral which is the Fourier transform of x(t):

$x(t) := \cos(\omega_0 \cdot t)$ **clear** (x) **clear** (X)

$X(\omega) := \int_{-\pi}^{\pi} x(t) \cdot e^{-j \cdot \omega \cdot t} \, d\omega$

$X(\omega) := \int_{-\pi}^{\pi} x(t) \cdot e^{-j \cdot \omega \cdot t} \, d\omega$ Prime/Mathcad cannot evaluate this

manual evaluation: **clear** (X)

$X(\omega) := \pi \cdot (\mathrm{Dirac}(\omega - \omega_0) + \mathrm{Dirac}(\omega + \omega_0))$ Dirac is the Dirac Delta function

$X(\omega) := \pi \cdot (\delta(\omega - \omega_0)) + \pi \cdot (\delta(\omega + \omega_0))$ substituting Dirac for del function

Now lets plot the spectral by defining X(w) and d(w):

$\omega := -5, -4 .. 5$ $\omega_0 := 1$

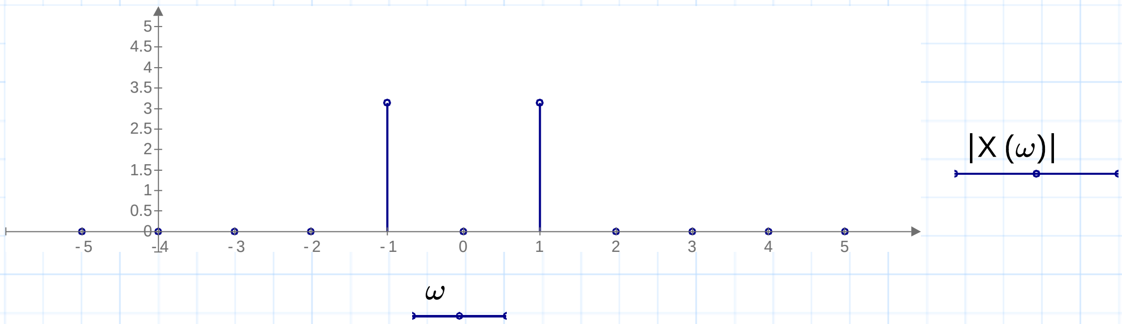
$\delta(\omega) := \mathbf{if}(\omega = 0, 1, 0)$

clear (X)

$X(\omega) := \pi \cdot (\delta(\omega - \omega_0)) + \pi \cdot (\delta(\omega + \omega_0))$

Now we can plot the spectral of x(t):

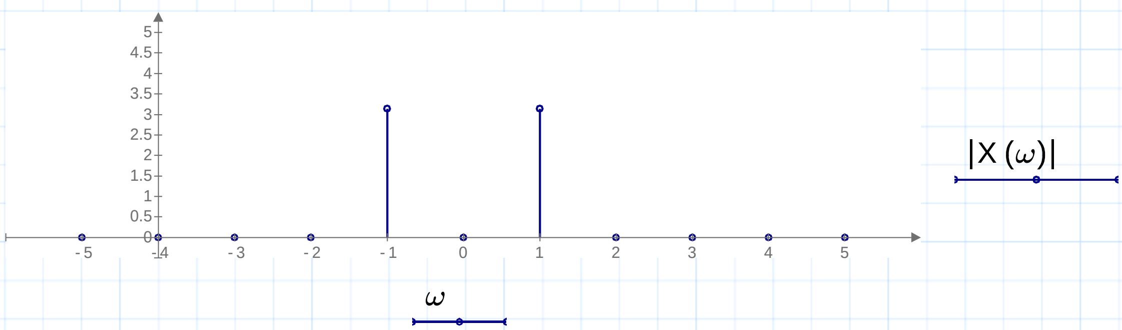
The spectral of x(t) - stem plot



Now let w0 = 2 where x(t) = cos (2 w0 t)

$x(t) := \cos(2 \cdot \omega_0 \cdot t)$
so w0 = 2
$w_0 := 2$
$\delta(\omega) := \text{if}(\omega = 0, 1, 0)$
clear (X)
$X(\omega) := \pi \cdot (\delta(\omega - \omega_0)) + \pi \cdot (\delta(\omega + \omega_0))$

The new spectral of x(t) - stem plot



Discrete Time Signals

Discrete Time Fourier Series:

Earlier examples the focus was on continous time signals, now the focus is on discrete time signals.

First we used Fourier series to approximate a continous time periodic signal as a sum of sine and cosine. By replacing sine and cosine with Euler's identity, the series changes to exponential Fourier series.

For discrete time signals we can use a similar approach to approximate discrete time periodic signals. In this example we learn more about discrete time Fourier series and how to use Mathcad to manage it.

A discrete periodic signal
x(n) = x(n + N)
can be approximated by using the discrete-time Fourier series equation.

$$x(n)=\sum_{k=0}^{N-1} c_k e^{\frac{jk2\pi n}{N}} \quad \text{where} \quad c_k=\frac{1}{N}\sum_{k=0}^{N-1} x(n) e^{\frac{-jk2\pi n}{N}}$$

and $n=0\cdots N-1$

Example - Discrete Time Fourier Series

Discrete time periodic signal shown below

Objective is to find an equation that represents this signal, x(n)?

Fist find the period of the signal?

Studying the signal just like a sunusoid signal, the period is 7.
From 0 to 3 = 1, then drops to 0 from 4 to 6, and there is space before 7 so the signal ends at 7, and picks back up again with a new period at 7

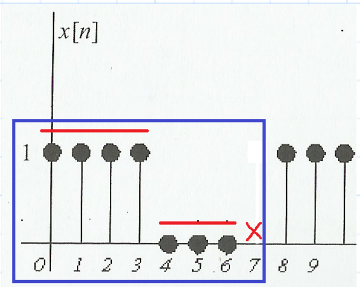
Instead of T, here in the discrete form we use N

$N := 7$ discrete period

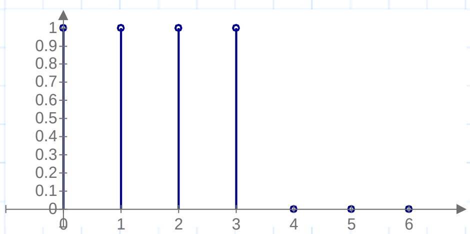
Set a range for n:
Here n is the number of points that represents a value for the signal, it does not do so at n=7, where the next period in the signal picks up, it is at n=N-1 at n=6

$n := 0, 1 .. N-1$

$x(n) := \text{if}(n \le 3, 1, 0)$ defining the signal



<---Correct for x(n) above



Discrete signal plotted

Now lets define Ck?

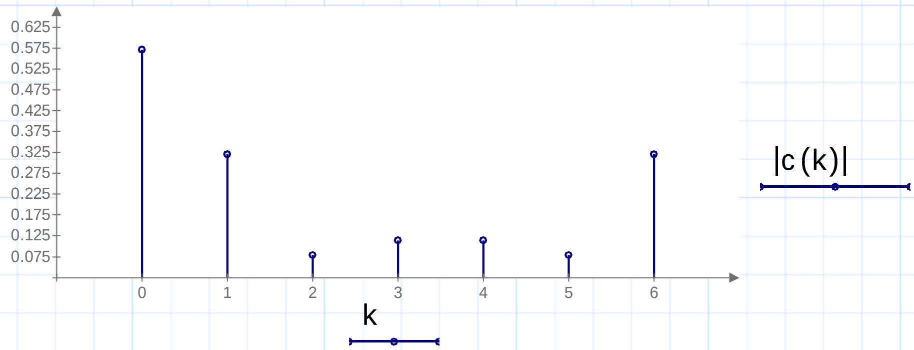
$k := 0 .. N-1$

$j := \sqrt{-1}$

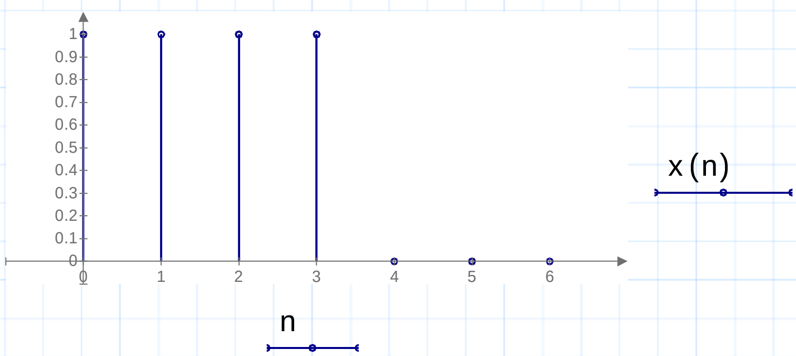
$$c(k) := \left(\frac{1}{N}\right) \cdot \sum_{n=0}^{N-1} x(n) \cdot e^{-\frac{(j \cdot k \cdot 2 \cdot \pi \cdot n)}{N}}$$

$$c(k) = \begin{bmatrix} 0.571429 \\ 0.071429 - 0.312949j \\ 0.071429 + 0.034398j \\ 0.071429 - 0.089569j \\ 0.071429 + 0.089569j \\ 0.071429 - 0.034398j \\ 0.071429 + 0.312949j \end{bmatrix}$$

The plot of the coefficients c(k) is not easy due to the narrow and wider range of the plot, however its magnitude can be plotted



next the plot of the equation x(n) defining the discrete Fourier series



The above plot of x(n) is the discrete signal which was provided to be solved at the beginning of the example. This satisfies the solution to the example.

How to make the signal periodic in the plot.
Extend the value of n from -10 to 10,
so several periods are seen like sinusoids in continous time?
This requires additional effort perhaps with a little programming with multiple if statements.

Discrete Fourier Transform DFT

The Discrete Fourier Transform

we use the Fourier integral to approximate a single pulse in continuous time. For discrete-time signals with N sample, we can use the Discrete Fourier Transform to approximate the signal. Let x_n be some sample of a signal, the Discrete Fourier Transform (DFT) of x_n can be evaluated by using Equ

$$x_k = \frac{1}{N}\sum_{k=0}^{N-1} x_n e^{-jk\left(\frac{2\pi}{N}\right)n}$$

We can use Prime (Mathcad) to compute the DFTs of xn from the following steps.

a).

Origin function to reset the subscript of the array to negative.
As we know, Prime can index arrays from both positive and negative.

clear(n)

$Orign := -50$ so we can enable negative tracking

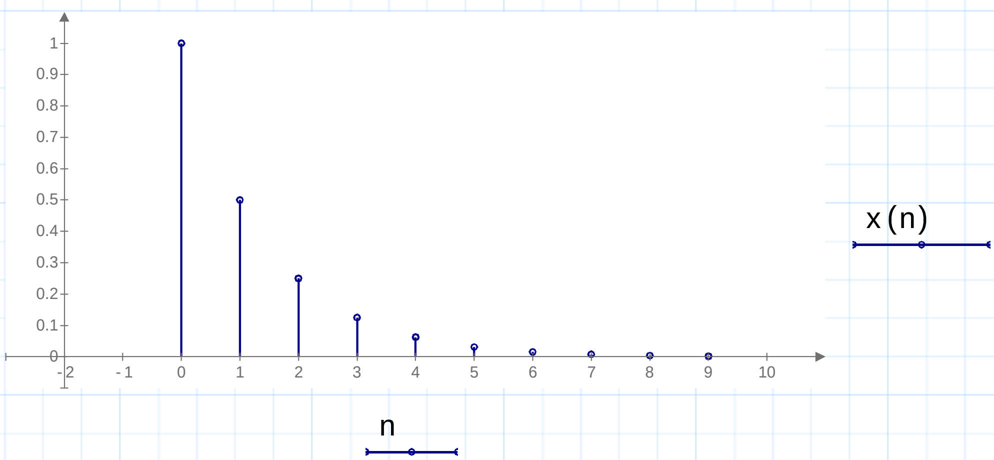
$N := 10$

$j := \sqrt{-1}$

$n := 0 .. N-1$

$u(n) := \mathbf{if}(n \geq 0, 1, 0)$ defining the step function

$x(n) := \left(\frac{1}{2}\right)^n \cdot u(n)$ defining the signal



b)
Compute the discrete Fourier transform (DFT) of x(n):

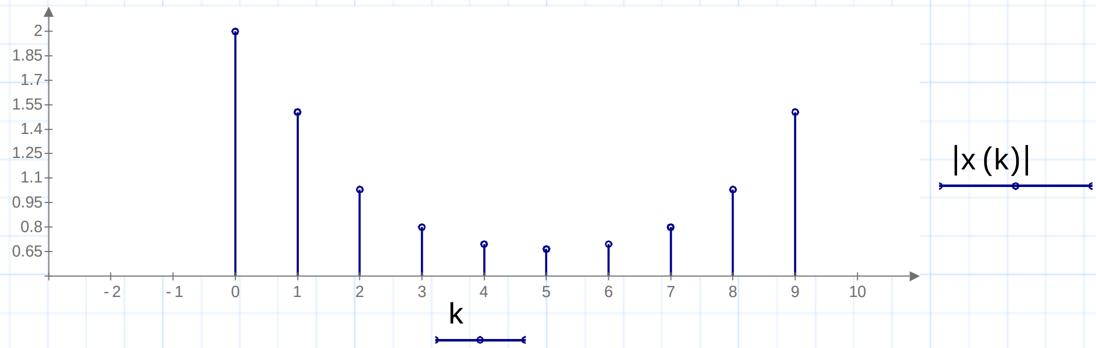
$k := 0 .. N-1$

$$x(k) := \sum_{n=0}^{N-1} x(n) \cdot e^{\frac{-j \cdot k \cdot 2 \cdot \pi \cdot n}{N}}$$

defining the equation of the transform
check the equation with your signals
and systems textbook (leaving out 1/N)

c).

Now plot x(k) which is the DFT of x(n)



Inverse Discrete Fourier Transform IDFT

The Inverse Discrete Fourier Transform

The inverse Discrete Fourier transform can be used to recover the signal x_n. The synthesis equation to compute the inverse Fourier transform is

$$x_n = \frac{1}{N} \sum_{k=0}^{N-1} x_k e^{jk\left(\frac{2\pi}{N}\right)n} \qquad \text{(Equ.3.18)}$$

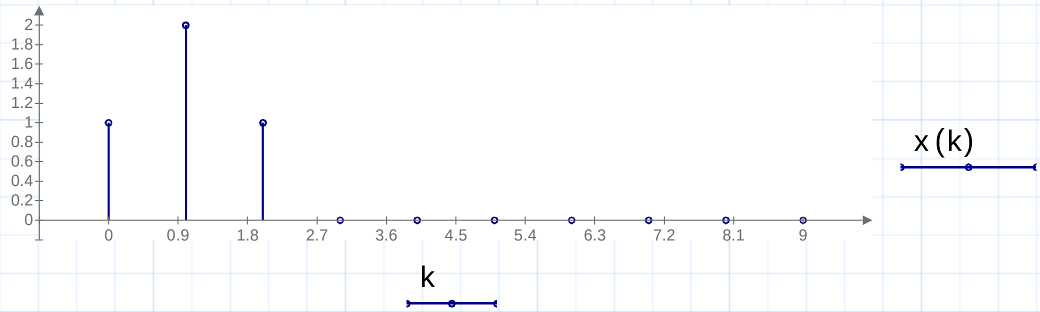
Where $n = 0 \cdots N-1$

Using MathCAD, we can compute the inverse DFT of a given signal; to do that, we must define both the signal and its range.

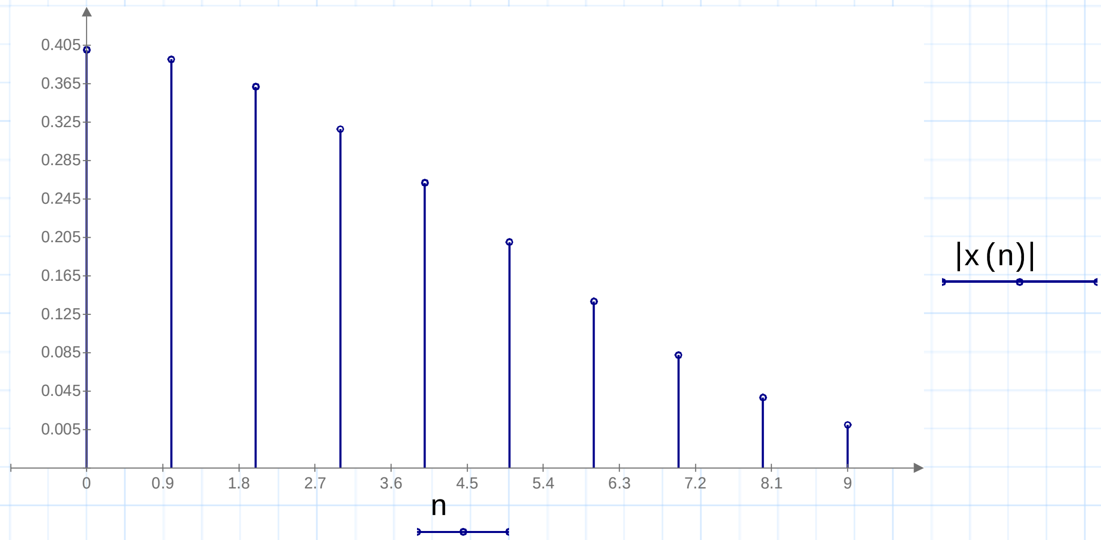
Example

$N := 10$
$k := 0 .. N-1$
$\delta(k) := \text{if}(k = 0, 1, 0)$ defining the unit sampling function - impulse function

$x(k) := \delta(k) + 2 \cdot \delta(k-1) + \delta(k-2)$ defining the transform signal



$$x(n) := \left(\frac{1}{N}\right) \cdot \sum_{k=0}^{N-1} x(k) \cdot e^{\frac{j \cdot k \cdot \pi \cdot n}{N}}$$ defining the inverse equation



Above is the IDFT plot

Discrete Time Fourier Transform

The Discrete Time Fourier Transform

The Discrete Time Fourier Transform (DTFT) can be used to compute the Fourier transform of an infinite length discrete time signal. The assumption is that the result of the summation of the infinite length must be converged. In other words, the infinite length sequence $x(n)$ must be absolutely summable, where $\sum_{n=-\infty}^{\infty} |x(n)| < \infty$. The Discrete Time Fourier Transform of a given signal $x(n)$ is denoted by $X(e^{j\omega})$ and can be calculated from Equ. In this section we show how to use MathCAD to compute the Discrete Time Fourier Transform of a given signal, both graphically and symbolically. Later, we will also show you how to compute the synthesis version of the DTFT, which is the inverse Discrete Time Fourier Transform of the sequence. The equation below is used to compute the DTFT of a given signal $x(n)$.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \qquad \text{Equ}$$

Example 3.20

$j := \sqrt{-1}$

$u(n) := \Phi(n)$ Prime built-in step function

$a := \frac{1}{2}$

$x(n) := a^n \cdot u(n)$ This is the signal we need to find its DTFT

$X(\omega) := \sum_{n=0}^{\infty} x(n) \cdot e^{-j \cdot \omega \cdot n}$ Defining the DTFT equation

Continuing with the requirements for a final plot:

$u(n) := \text{if}(n = 0, 1, 0)$

$a := \frac{1}{2}$

$x(n) := a^n \cdot u(n)$

next define the DTFT equation

$$X(\omega) := \sum_{n=0}^{\infty} a^n \cdot e^{-j \cdot \omega \cdot n}$$

$$X(\omega) \rightarrow \begin{bmatrix} -\frac{2}{e^{5i}-2} \\ -\frac{2}{e^{4i}-2} \\ -\frac{2}{e^{3i}-2} \\ -\frac{2}{e^{2i}-2} \\ -\frac{2}{e^{1i}-2} \\ 2 \\ -\frac{2}{e^{-1i}-2} \\ -\frac{2}{e^{-2i}-2} \\ -\frac{2}{e^{-3i}-2} \\ -\frac{2}{e^{-4i}-2} \\ -\frac{2}{e^{-5i}-2} \end{bmatrix}$$

Dividing the last term in the summation series above by (1/2)
- the summation is shown from -5 to 5

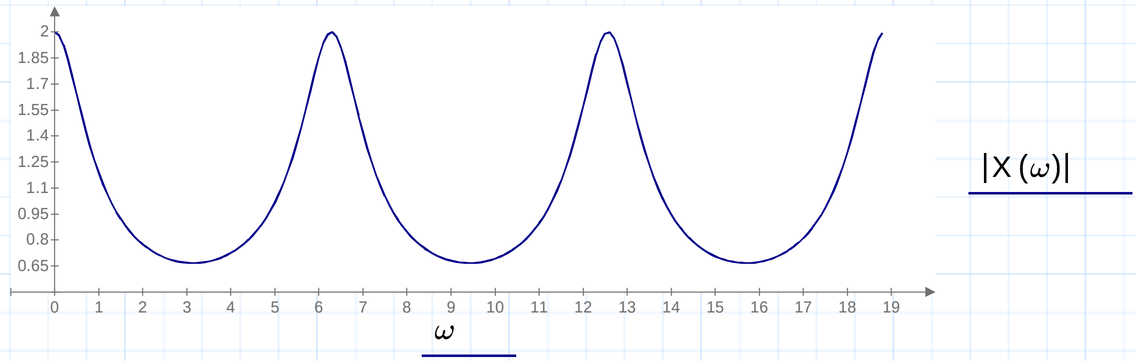
X(w) = -2/(e^-ni - 2)
= -1/(0.5e^-ni -1)
= -1 (-1 - 0.5e^-ni)
= 1/(1 - 0.5e^-ni)
= 1 / (1 - 0.5e^-njw)
X(w)= 1 / (1 - 0.5e^-jw)   n = 1

Now we set the range w

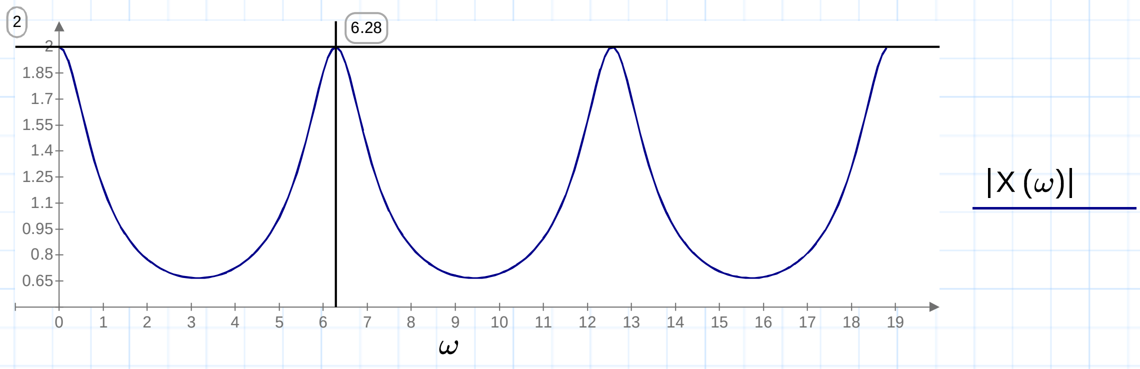
$\omega := 0, 0.1 .. 6 \cdot \pi$

clear (X)

$$X(\omega) := \frac{1}{\left(1 - \frac{1}{2} \cdot e^{(-j \cdot \omega)}\right)}$$



We had a signal in discrete time form x(n) and now we have it transformed into the frequency domain



The signal has a period of 2 pi (6.28)

Next with IDFT on next page

Inverse Discrete Time Fourier Transform

The Inverse Discrete Time Fourier Transform

In Example 3.20, we discuss how to compute the DTFT of an input sequence; in this section, we show how to use MathCAD to compute the inverse Discrete Time Fourier Transform. The inverse DTFT is computed from the following equation

$$x(n)=\frac{1}{2\pi}\int_{2\pi}X(\omega)e^{j\omega}d\omega \quad \text{Equ}$$

2 pi denotes that the intergration is carried out over one period, since Fourier transform is periodic, we can use Mathcad to compute the inverse DTFT of X(e^jw) both symbolically and numerically. Symbolically with Prime evaluation (--->) and numerically using the summation method.

Example 3.21

Find the Fourier transform of the rectangular pulse.

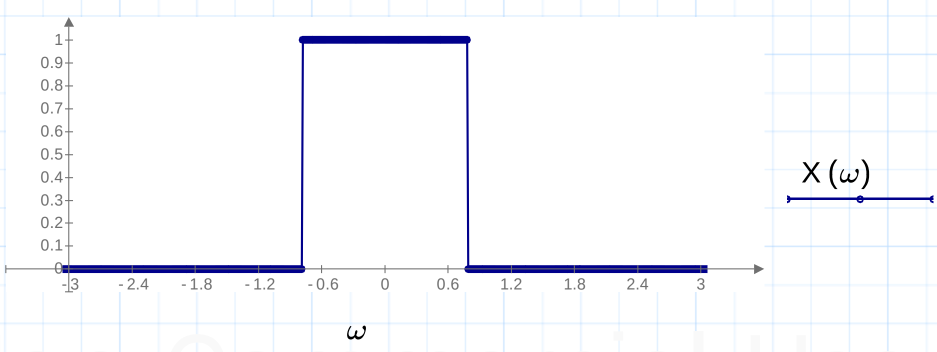
clear (X)

$\omega := -15, -14.99..15$

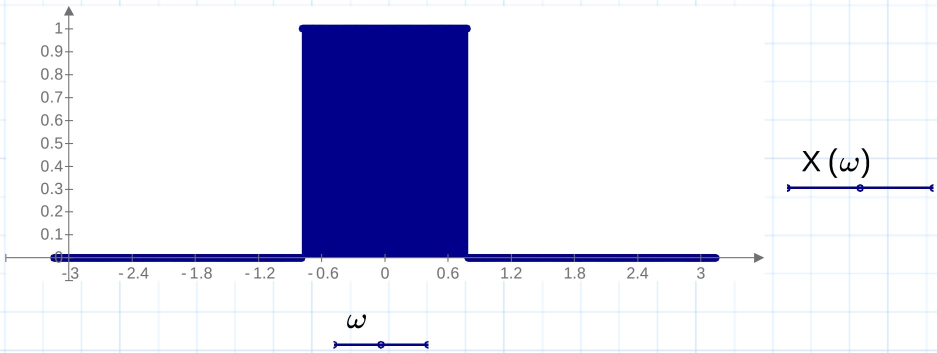
$$X(\omega) := \begin{cases} \text{if } \left(|\omega| \le \left(\frac{\pi}{4}\right)\right) \\ \quad X \leftarrow 1 \\ \text{else if } \left(\frac{\pi}{4} < |\omega| \le \pi\right) \\ \quad X \leftarrow 0 \end{cases}$$

These lines define the signal X(w) Note we cannot use the if(, ,) statement. Here an 'if then else' type scenario defines the signal

The plot range is set to zoom in on one period, w = -pi to pi in the if then else statement above, so we only get this in the plot



Stem plot of above signal with w set to -15 to 15, does the same result in the plot:
It is seen solid because the interval is 0.01 set in w at begining of the example



Now how do we show the periodicity of the signal:
We only have one period thus far, how to produce the periodicity of the signal?

Similar to example 3.20:

$j := \sqrt{-1}$ **clear** (x) **clear** (n)

$$x(n) := \left(\frac{1}{2\pi}\right) \cdot \int_{\left(\frac{-\pi}{4}\right)}^{\frac{\pi}{4}} 1 \cdot e^{j \cdot \omega \cdot n} \, d\omega$$ DTFT equation

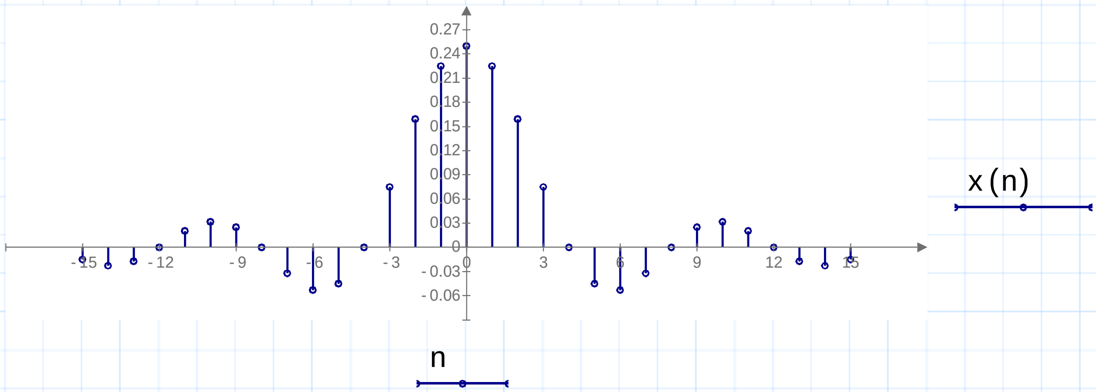
$$x(n) \rightarrow \frac{\sin\left(\frac{\pi \cdot n}{4}\right)}{\pi \cdot n}$$ Prime symbolic evaluated result

The result in exponential form:

x(n) = (1/2pi) ((-i/n)e^(0.25i pi n) + ((i/n)e^(-0.25 i pi n)).....exponential form

$n := -15, -14 .. 15$ Specify the value of n to plot the Inverse DTFT

Plot using the symbolic evaluation of x(n):



The plot of x(n) above matches expected results in the stem plot

EoF --- End of File