5
Analog Equivalent Digital Low Pass II° Order Filter

Introduction

A well known in the technical literature second order low pass active filter is here treated. The filter is realized using a common operational amplifier in a negative multiple feedback configuration. After a brief summary of the main results of the circuit analysis, the outputs to several signals among the most common and from an external file (Signal List.xmcd), are produced. This results are compared with the one obtained by the many algorithms implemented applying the z-transform to the output function. Since MATHCAD student edition is devoid of some functions, the program BCSA (Fourier series and signal bandwidth) was written to fill this gap. Four other programs (CANONIC2LP, SYNDIVC, BILINEAR, SYNDIVBL) corresponding to four different algorithms to calculate the filter sequential response, were realized and written in a new worksheet with the purpose to reuse them elsewhere and, last but not the least, to save worksheet's space as well.Thus one will see that, as the analog filter is effective, just is the digital one. The step to the practical application, using a DSP, should be very simple.

When saving or printing, disable Automatic Calculation (Mathcad 14 s. e.)

The subscript "gd" is the acronym of "Global Data.xmcd"
The subscript "fs" is the acronym of "Fourier series.xmcd"
The subscript "sl" is the acronym of "Signal List.xmcd"
The subscript "dp" is the acronym of "Dirac Pulse - formulae.xmcd"

Content:

References

- Reference:C:\new folder\global data.xmcd
- Reference:C:\new folder\Fourier Series.xmcd
- Reference:C:\new folder\Signals List.xmcd
- Reference:C:\new folder\Nichols Chart.xmcd
- Reference:C:\new folder\Dirac Pulse - formulas.xmcd
- Reference:C:\new folder\Pulse Train Data.xmcd
- Reference:C:\new folder\staircase pulse data.xmcd
- Reference:C:\new folder\staircase pulse train data.xmcd
- Reference:C:\new folder\staircase 2 pulse data.xmcd
- Reference:C:\new folder\staircase 2 pulse train data.xmcd
- Reference:C:\new folder\staircase 3 pulse data.xmcd
- Reference:C:\new folder\staircase 4 pulse data.xmcd
- Reference:C:\new folder\sawtooth pulse data.xmcd
- Reference:C:\new folder\sawtooth pulse train data.xmcd
- Reference:C:\new folder\rf pulse data.xmcd
- Reference:C:\new folder\FM data.xmcd
- Reference:C:\new folder\PM data.xmcd
- Reference:C:\new folder\programs.xmcd

Definitions and necessary constants

Scroll the slider, to change the filter's voltage gain

$k_{vg} :=$

$A_5 := -k_{vg} \cdot 1$ Voltage gain: $A_5 = -10$

Scroll the slider, to change the filter's pole Q factor

$k_{qf} :=$

$Q_5 := k_{qf} \cdot 0.1$ Pole Q factor: $Q_5 = 5.4$,

For $Q_5>1$ the frequency response of the active filter presents a resonance peak or an overshoot, while for $Q_5 \le 1$ it is a flat one.

Scroll the slider, to change the filter's bandwidth

$k_{cf} :=$

$Bw := k_{cf} \cdot 0.10 \cdot MHz$ Bandwidth: $Bw = 47.1 \cdot MHz$,

Calculation of f5

Step amplitude: $V_{pp} = 5V$

Pole frequency: $f_5 = 30.498 \cdot MHz$

Period: $T_5 := \frac{1}{f_5}$, $T_5 = 32.789 \cdot ns$

Pole Angular frequency: $\omega_5 := 2 \cdot \pi \cdot f_5$, $\omega_5 = 0.192 \cdot \frac{Grads}{sec}$,

Filter's bandwidth: $Bw = 47.1 \cdot MHz$

time constant: $\tau := \frac{1}{\omega_5}$, $\tau = 5.218 \cdot ns$

damping factor: $\zeta := \frac{\omega_5}{2 \cdot Q_5}$, $\zeta = 0.018 \cdot \frac{Grads}{sec}$, $\omega_5 = 2 \cdot \zeta \cdot Q_5$,

If $Q_5 = 0.5 \Rightarrow \zeta = \omega_5$,

Defined in "global data.xmcd": $dB3_{gd} = 20 \cdot \log(\sqrt{2})$; angular frequency units.

Chosen test signal period, to verify the filtering action of the system, it has been selected a test signal frequency outside the passband.

$T_{test} := \frac{1}{2} \cdot T_5$, i.e. a sub multiple of the 0 dB Voltage gain period.

$T_{test} = 16.394 \cdot ns$

Then the signal frequency $f_{test} := \frac{1}{T_{test}}$, $\boxed{f_{test} = 0.061 \cdot GHz}$: $Bw = 0.047 \cdot GHz$. $\omega_{test} := 2 \cdot \pi \cdot f_{test}$,

$\omega_{test} = 0.383 \cdot \frac{Grads}{sec}$ is higher than the cutoff angular frequency of the filter , $\frac{\omega_{test}}{\omega_5} = 2$,

As a result, the waveform at the filter output should be strongly attenuated.

Amplifier Gain: $A_5 = -10$, $A_{5dB} := 20 \cdot \log(|A_5|)$

$A_{5dB} = 20 \cdot dB$

5.1
II° Order Analog LOW PASS Digital Filter

The analog active filter chosen is the following (the power supply circuit isn't visible):

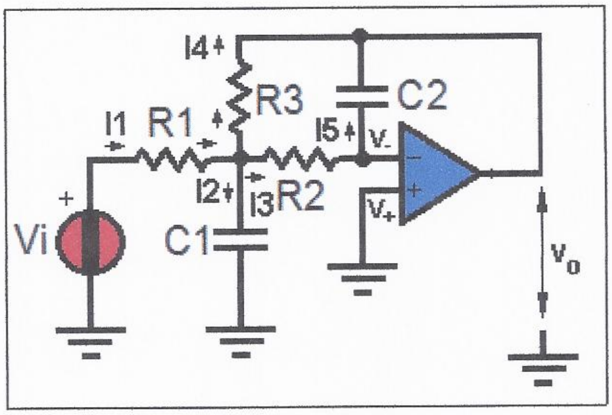


fig.:5.1.1

▶ Network Analysis

Network Analysis:

node admittance matrix:

$$Y = \begin{pmatrix} \frac{1}{R1} & -\frac{1}{R1} & 0 & 0 \\ -\frac{1}{R1} & \frac{1}{R1}+\frac{1}{R2}+\frac{1}{R3}+C1\cdot s & -\frac{1}{R2} & -\frac{1}{R3} \\ 0 & -\frac{1}{R2} & \frac{1}{R2}+C2\cdot s & -C2\cdot s \\ 0 & -\frac{1}{R3} & -C2\cdot s & \frac{1}{R3}+C2\cdot s \end{pmatrix} \qquad (5.1.1)$$

▶ L. T. I. System Analysis.

The general transfer function is expressed by the formula:

$$W(s) = \frac{-Y1(s) \cdot Y3(s)}{Y5(s) \cdot (Y1(s) + Y2(s) + Y3(s) + Y4(s)) + Y3(s) \cdot Y4(s)} \qquad (5.1.11)$$

Open loop voltage gain calculation.

The filter is composed by an ideal op amp in inverting configuration and a further feedback network as here below depicted:

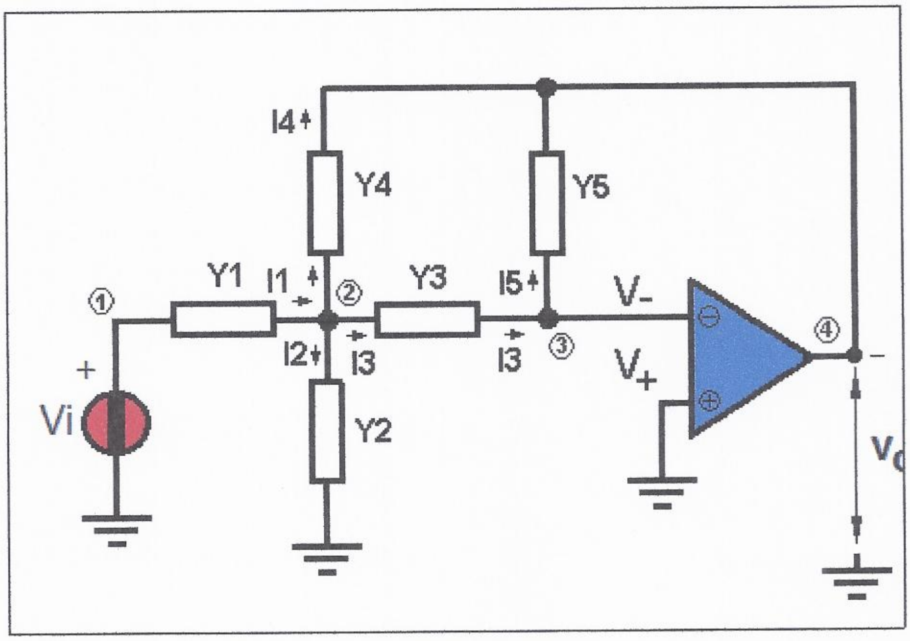


fig.:5.1.5

In terms of a feedback system, a new function could be defined for the direct branch of a new system but with the same t. f. :

$$G(s) = -\frac{Y1(s)}{Y4(s)},$$

and for the feedback branch: $H(s) = -\frac{Y5(s)}{Y3(s)} \cdot \left(\frac{Y1(s) + Y2(s) + Y3(s) + Y4(s)}{Y1(s)}\right),$

so that: $W_{lp}(s) = \frac{G(s)}{1 + G(s) \cdot H(s)},$

The corresponding open loop gain is: $GH(s) = \frac{Y5(s) \cdot (Y1(s) + Y2(s) + Y3(s) + Y4(s))}{Y3(s) \cdot Y4(s)}$

Other couple of functions, but describing a new system, can be found, i.e. :

$$G(s) = \frac{-Y1(s) \cdot Y3(s)}{Y5(s) \cdot (Y1(s) + Y2(s) + Y3(s) + Y4(s))}$$

and: $H(s) = -\frac{Y4(s)}{Y1(s)},$

$$GH(s) = \frac{Y3(s) \cdot Y4(s)}{Y5(s) \cdot (Y1(s) + Y2(s) + Y3(s) + Y4(s))}$$

In fact, substituting the two last relations in the generic transfer function of a feedback system, you get:

$$\frac{G(s)}{1+G(s)\cdot H(s)} = \frac{-Y1(s)\cdot Y3(s)}{Y5(s)\cdot(Y1(s)+Y2(s)+Y3(s)+Y4(s))\cdot(1+GH(s))},$$ and simplifying results:

$$W_{lp}(s) = \frac{G(s)}{1+G(s)\cdot H(s)} = \frac{-Y3(s)\cdot Y1(s)}{(Y2(s)+Y3(s)+Y4(s)+Y1(s))\cdot Y5(s)+Y3(s)\cdot Y4(s)},$$

that is the already known transfer function derived earlier.
But now determine the open loop gain of the given filter. Start from its circuit:

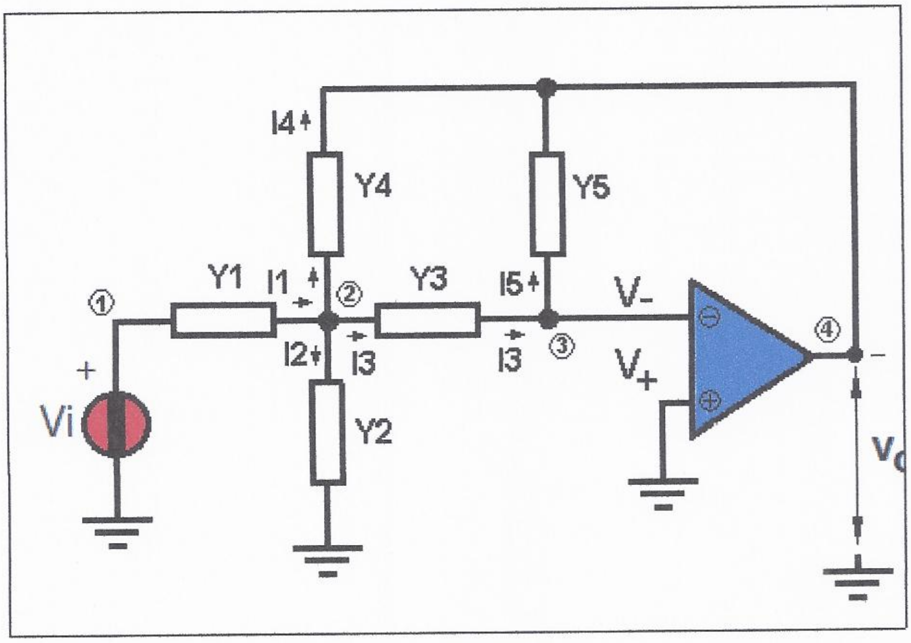


fig.:5.1.5'

The feedback network can be redrawn as follows:

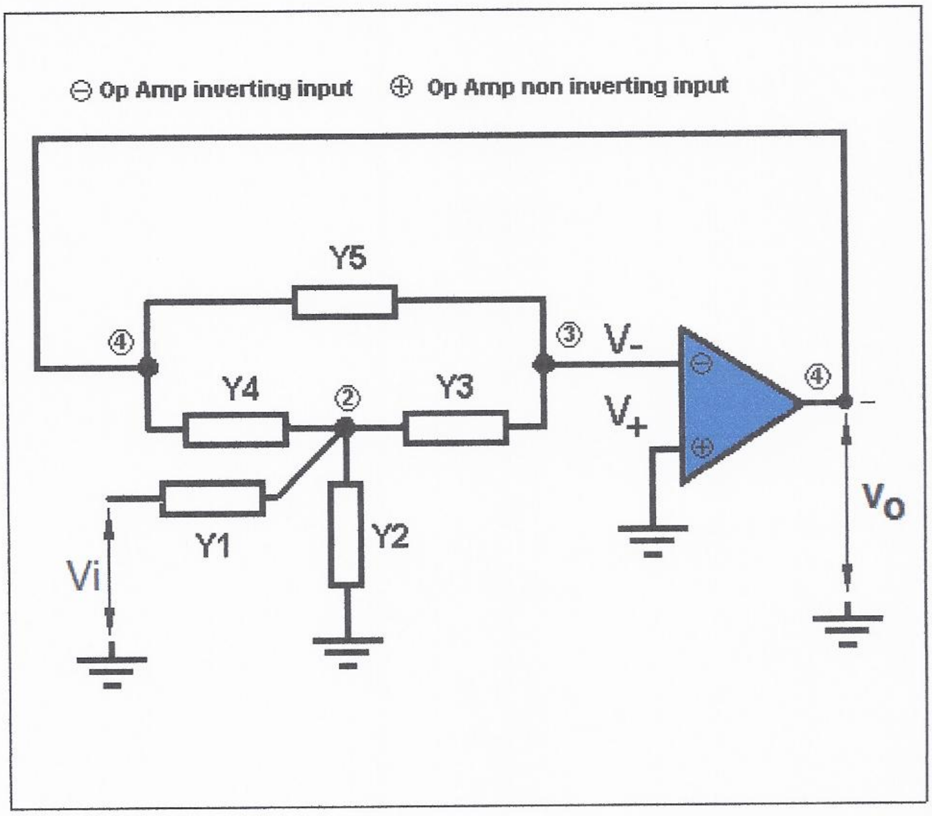


fig.:5.1.6

Let shift the bridged T along the loop, the network assumes the following form, although the topology of the network is unchanged:

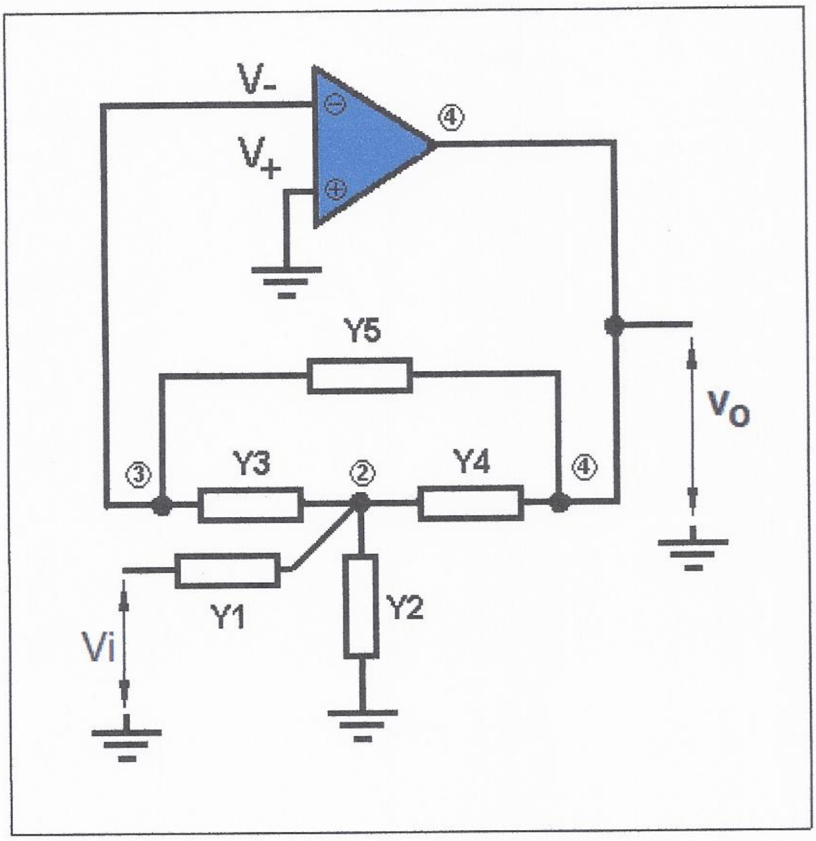


fig.:5.1.7

Now apply the known procedure to determine the open loop gain, namely: remove the effect of the input generator (Vi=0= input shorted), choose a cut on the loop and apply an independent voltage test source V_T the right side of the cut . It has been chosen the following cut:

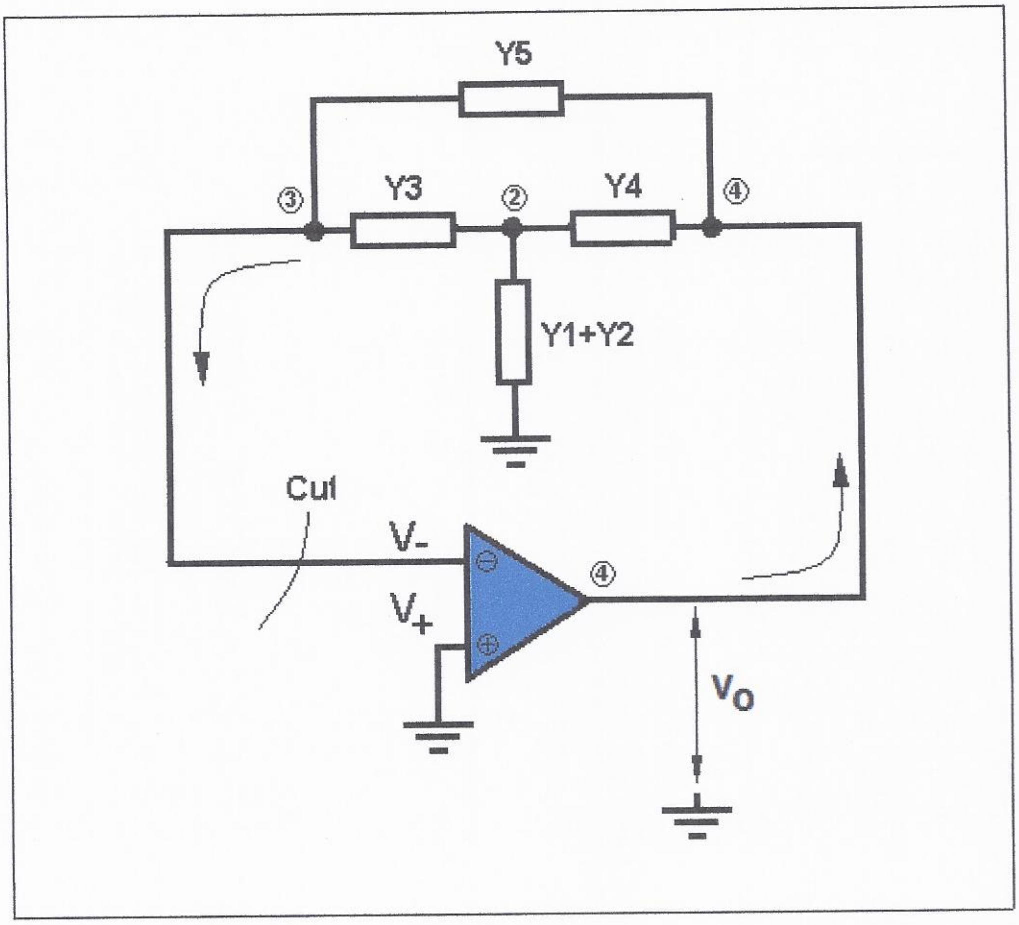


fig.:5.1.8

The voltage generator V_T has been placed at the input of the op amp, here represented with its equivalent circuit:

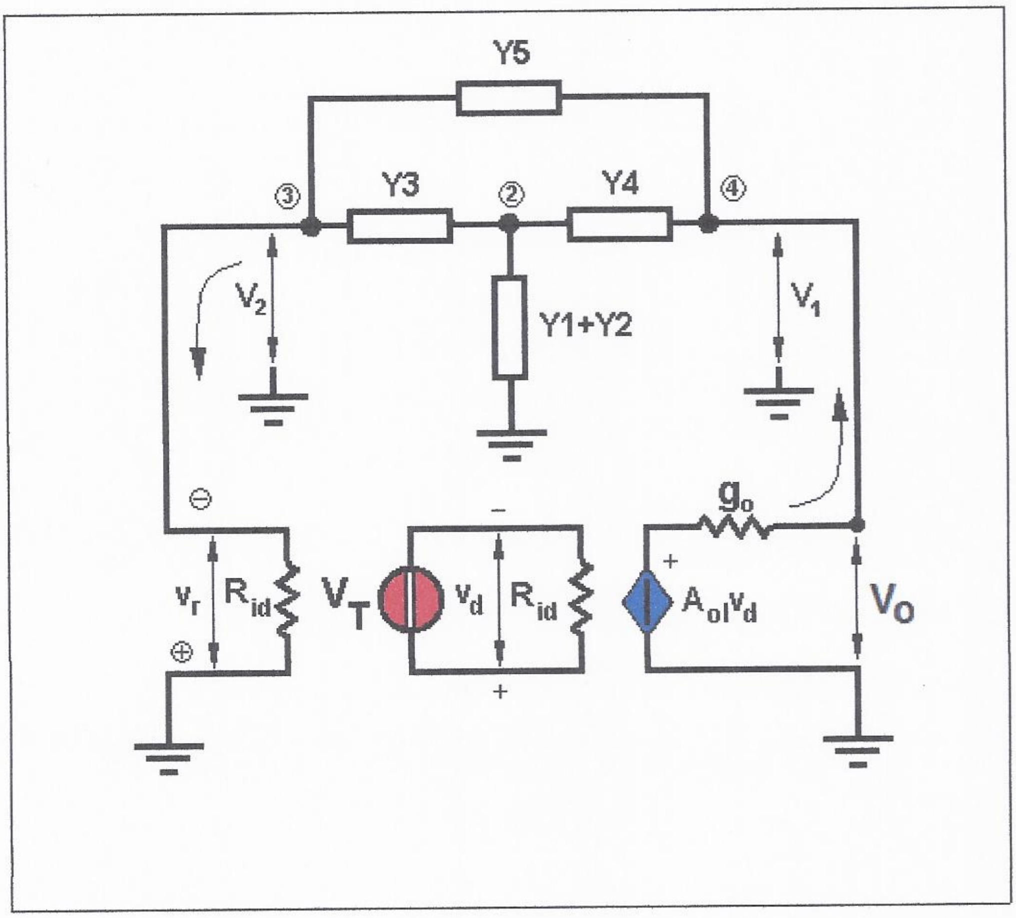


fig.:5.1.9

To simplify the circuit analysis, it has been placed: $R_{id}=\infty\Omega$, $r_o=0\Omega$. The equivalent circuit now is:

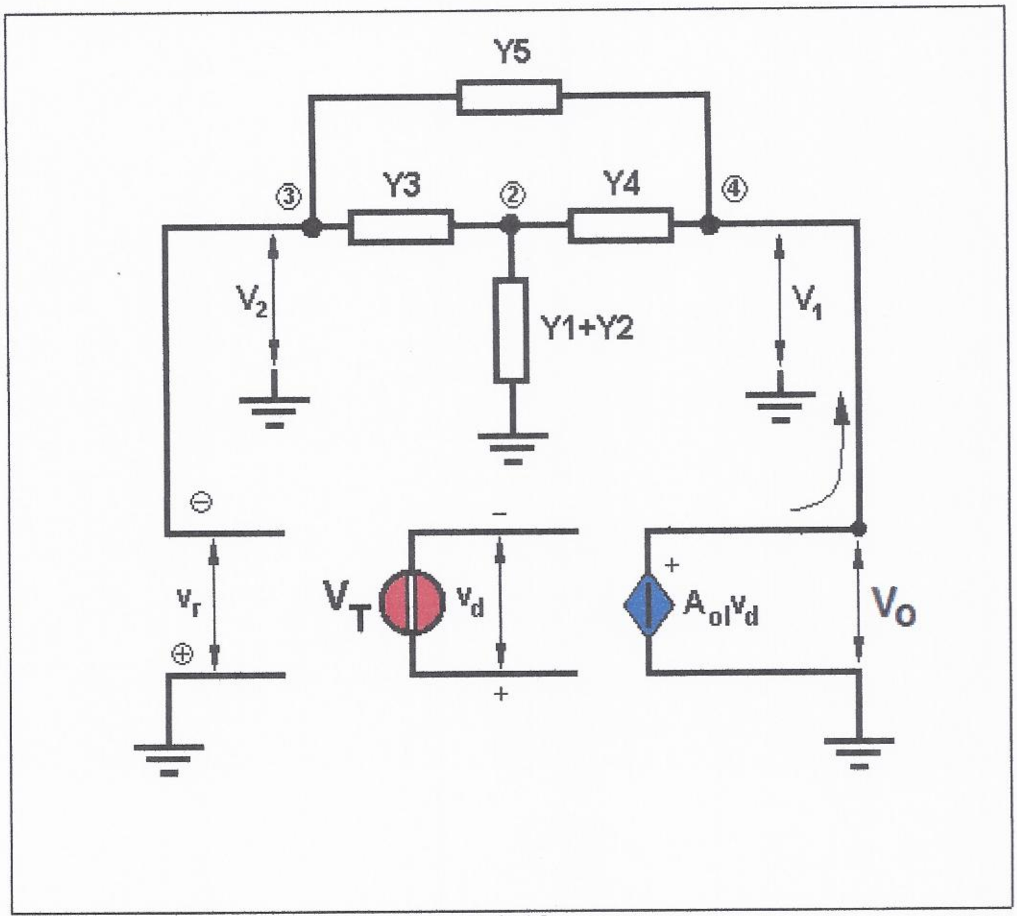


fig.:5.1.10

Computation of V2 by inspection

$$v_r = V2 = V3 + V12$$

$$YT = (Y1 + Y2) \;||\; [Y4 + (Y5 \;||\; Y3)]$$

$$IT = \frac{V1}{ZT} = V1 \cdot YT = V1 \cdot [(Y1 + Y2) \;||\; [Y4 + (Y5 \;||\; Y3)]]$$

$$IT = I4 + I5$$

$$I4 = \frac{Y4}{Y4 + Y5 \;||\; Y3} \cdot IT$$

$$I53 = \frac{Y5 \;||\; Y3}{Y5 \;||\; Y3 + Y4} \cdot IT$$

$$V3 = Z3 \cdot I53 = \frac{I53}{Y3} = \frac{Y5 \;||\; Y3}{Y5 \;||\; Y3 + Y4} \cdot \frac{IT}{Y3}$$

$$V12 = \frac{IT}{Y1 + Y2} = \frac{V1 \cdot [(Y1 + Y2) \;||\; [Y4 + (Y5 \;||\; Y3)]]}{Y1 + Y2}$$

$$v_r = V3 + V12 = \frac{Y5 \;||\; Y3}{Y5 \;||\; Y3 + Y4} \cdot \frac{IT}{Y3} + \frac{IT}{Y1 + Y2}$$

$$v_r = \left(\frac{Y5 \;||\; Y3}{Y5 \;||\; Y3 + Y4} \cdot \frac{1}{Y3} + \frac{1}{Y1 + Y2}\right) \cdot [V1 \cdot [(Y1 + Y2) \;||\; [Y4 + (Y5 \;||\; Y3)]]]$$

$$v_r = \frac{V1\cdot[(Y1+Y2+Y3+Y4)\cdot Y5 + Y3\cdot Y4]}{(Y1+Y2+Y3+Y4)\cdot Y5 + Y3\cdot(Y1+Y2+Y4)}$$ v_I

▲ Computation of V2 by inspection

The open loop gain results to be:

$$\frac{v_r}{V_T} = -\frac{[(Y1+Y2+Y3+Y4)\cdot Y5 + Y3\cdot Y4]\cdot A_{ol}}{(Y1+Y2+Y3+Y4)\cdot Y5 + Y3\cdot(Y1+Y2+Y4)} \quad (5.1.12)$$

▶

Explicit calculation of the open loop voltage gain. The admittances are:

$$Y1(s) = \frac{1}{R1},$$
$$Y2(s) = s\cdot C1,$$
$$Y3(s) = \frac{1}{R2},$$
$$Y4(s) = \frac{1}{R3},$$
$$Y5(s) = s\cdot C2.$$

Substituting into the general form (5.1.12), gets:

$$-\frac{[(Y1+Y2+Y3+Y4)\cdot Y5 + Y3\cdot Y4]\cdot A_{ol}}{(Y1+Y2+Y3+Y4)\cdot Y5 + Y3\cdot(Y1+Y2+Y4)} \left| \begin{array}{l} \text{substitute}, Y1 = \frac{1}{R1}, Y2 = s\cdot C1, Y3 = \frac{1}{R2} \\ \text{substitute}, Y4 = \frac{1}{R3}, Y5 = s\cdot C2 \\ \text{collect}, s, A_{ol} \end{array} \right. \rightarrow$$

$$GH_5(s) = -\left[\frac{C1\cdot C2\cdot R1\cdot R2\cdot R3\cdot s^2 + (C2\cdot R1\cdot R2 + C2\cdot R1\cdot R3 + C2\cdot R2\cdot R3)\cdot s + R1}{C1\cdot C2\cdot R1\cdot R2\cdot R3\cdot s^2 + (C1\cdot R1\cdot R3 + C2\cdot R1\cdot R2 + C2\cdot R1\cdot R3 + C2\cdot R2\cdot R3)\cdot s + R1 + R3}\cdot A_{ol}\right]$$

$$\boxed{GH_5(s) = -\left[\frac{s^2 + \frac{[R1\cdot(R2+R3) + R2\cdot R3]}{C1\cdot R1\cdot R2\cdot R3}\cdot s + \frac{1}{(C1\cdot C2\cdot R2\cdot R3)}}{s^2 + \frac{[C2\cdot(R1\cdot R2 + R1\cdot R3 + R2\cdot R3) + C1\cdot R1\cdot R3]}{(C1\cdot C2\cdot R1\cdot R2\cdot R3)}\cdot s + \frac{R1+R3}{C1\cdot C2\cdot R1\cdot R2\cdot R3}}\cdot A_{ol}\right]} \quad (5.1.13)$$

▶ Calculation of the Input and output resistances

Transfer function calculation

For the proposed filter, here reported:

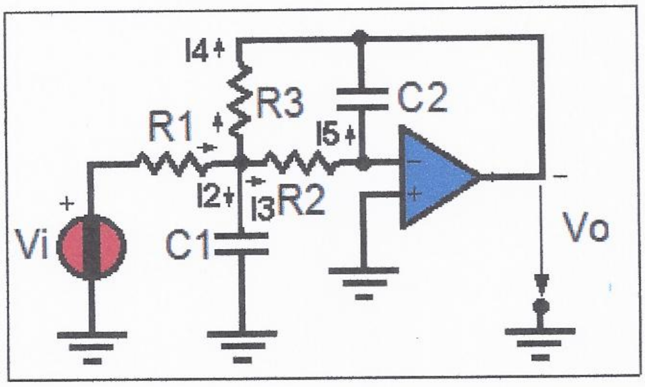


fig.:5.1.18

the admittances are:

$$Y1(s) = \frac{1}{R1},$$
$$Y2(s) = s \cdot C1,$$
$$Y3(s) = \frac{1}{R2},$$
$$Y4(s) = \frac{1}{R3},$$
$$Y5(s) = s \cdot C2.$$

Substituting into the general form (5.1.11), gets:

$$W_{lp}(s) := \frac{-Y1 \cdot Y3}{Y5 \cdot (Y1 + Y2 + Y3 + Y4) + Y3 \cdot Y4} \left| \begin{array}{l} \text{substitute}, Y1 = \frac{1}{R1}, Y2 = s \cdot C1, Y3 = \frac{1}{R2}, Y4 = \frac{1}{R3} \\ \text{substitute}, Y5 = s \cdot C2 \\ \text{collect}, s \end{array} \right. \rightarrow$$

The resulting transfer function, then is:

$$W_{lp}(s) = -\frac{R3}{C1 \cdot C2 \cdot R1 \cdot R2 \cdot R3 \cdot s^2 + (C2 \cdot R1 \cdot R2 + C2 \cdot R1 \cdot R3 + C2 \cdot R2 \cdot R3) \cdot s + R1}$$

Collecting the term: $C1 \cdot C2 \cdot R1 \cdot R2 \cdot R3$, the t. f. becomes:

$$\boxed{W_{lp}(s) = \frac{-\frac{1}{R1 \cdot R2 \cdot C1 \cdot C2}}{s^2 + s \cdot \frac{1}{C1} \cdot \left(\frac{1}{R1} + \frac{1}{R2} + \frac{1}{R3}\right) + \frac{1}{R2 \cdot R3 \cdot C1 \cdot C2}}} \qquad (5.1.14)$$

In order for this transfer function takes the form of the standard low-pass active filter, below rewritten:

$$\boxed{W_{lp}(s) = \frac{A_5 \cdot \omega_5^2}{s^2 + 2 \cdot \zeta \cdot s + \omega_5^2}} \qquad (5.1.15)$$

one must place into (5.1.29):

a) $$A_5 \cdot \omega_5^2 = \frac{-1}{R1 \cdot R2 \cdot C1 \cdot C2}$$

b) $\omega_5{}^2 = \dfrac{1}{R2 \cdot R3 \cdot C1 \cdot C2}$ $\qquad \omega_5 = \dfrac{1}{\sqrt{R2 \cdot R3 \cdot C1 \cdot C2}}$

c) $2 \cdot \zeta = \dfrac{\omega_5}{Q_5} = \dfrac{1}{C1} \cdot \left(\dfrac{1}{R1} + \dfrac{1}{R2} + \dfrac{1}{R3} \right)$

▶ Calculations A5 and Q5

In summary, it can be written: d) $\omega_5 = \dfrac{1}{\sqrt{R2 \cdot R3 \cdot C1 \cdot C2}}$, (5.1.16)

e) $A_5 = -\dfrac{R3}{R1}$,

f) $Q_5 = \dfrac{1}{\sqrt{\dfrac{C2}{C1}} \cdot \left(\dfrac{\sqrt{R2 \cdot R3}}{R1} + \sqrt{\dfrac{R3}{R2}} + \sqrt{\dfrac{R2}{R3}} \right)}$,

g) $\zeta = \dfrac{1}{2} \cdot \left[\dfrac{1}{C1} \cdot \left(\dfrac{1}{R1} + \dfrac{1}{R2} + \dfrac{1}{R3} \right) \right]$

A further condition is obtained by observing that the relationships $R1 = \dfrac{R3}{|A_5|}$ and $R2 = \dfrac{1}{\omega_5{}^2 \cdot R3 \cdot C1 \cdot C2}$ can be replaced in the expression c) $\dfrac{\omega_5}{Q_5} = \dfrac{1}{C1} \cdot \left(\dfrac{1}{R1} + \dfrac{1}{R2} + \dfrac{1}{R3} \right)$, namely:

$$\frac{\omega_5}{Q_5} = \frac{1}{C1} \cdot \left(\frac{1}{\dfrac{R3}{|A_5|}} + \frac{1}{\dfrac{1}{\omega_5{}^2 \cdot R3 \cdot C1 \cdot C2}} + \frac{1}{R3} \right),$$

which yields:

$$\frac{\omega_5}{Q_5} - C2 \cdot R3 \cdot \omega_5{}^2 - \frac{|A_5| + 1}{C1 \cdot R3} = 0,$$

it is a quadratic equation in the unknown R3:

$$R3^2 + \frac{|A_5| + 1}{C1 \cdot C2 \cdot \omega_5{}^2} - \frac{R3}{C2 \cdot Q_5 \cdot \omega_5} = 0,$$

whose roots are:

$A_5 := A_5 \qquad Q_5 := Q_5 \quad \omega_5 := \omega_5 \qquad C1 := C1 \quad C2 := C2$

$$R3^2 + \frac{|A_5| + 1}{C1 \cdot C2 \cdot \omega_5{}^2} - \frac{R3}{C2 \cdot Q_5 \cdot \omega_5} \ \Bigg|\ \begin{matrix} \text{solve}, R3 \\ \text{simplify}, \text{max} \end{matrix} \rightarrow \begin{pmatrix} \dfrac{\sqrt{-\dfrac{4 \cdot C2 \cdot Q_5{}^2 - C1 + 4 \cdot C2 \cdot Q_5{}^2 \cdot |A_5|}{C1}} + 1}{2 \cdot C2 \cdot Q_5 \cdot \omega_5} \\ -\dfrac{\sqrt{-\dfrac{4 \cdot C2 \cdot Q_5{}^2 - C1 + 4 \cdot C2 \cdot Q_5{}^2 \cdot |A_5|}{C1}} - 1}{2 \cdot C2 \cdot Q_5 \cdot \omega_5} \end{pmatrix}$$

hence:

$$R3 = \frac{1}{2 \cdot C2 \cdot Q_5 \cdot \omega_5} \cdot \left[\frac{1 + \sqrt{1 - \frac{4 \cdot C2 \cdot Q_5^2 \cdot (|A_5| + 1)}{C1}}}{1 - \sqrt{1 - \frac{4 \cdot C2 \cdot Q_5^2 \cdot (|A_5| + 1)}{C1}}} \right] \qquad (5.1.17)$$

it gives a further condition in order that R3 takes real values, namely:

$$1 - \frac{4 \cdot C2 \cdot Q_5^2 \cdot (|A_5| + 1)}{C1} \geq 0$$

$$\boxed{\frac{C2}{C1} \leq \frac{1}{4 \cdot Q_5^2 \cdot (|A_5| + 1)}} \qquad \boxed{\frac{C1}{C2} \geq 4 \cdot Q_5^2 \cdot (|A_5| + 1)}$$

$$\boxed{4 \cdot Q_5^2 \cdot (|A_5| + 1) = 1.283 \times 10^3}$$

results: $\boxed{C1 \geq 4 \cdot Q_5^2 \cdot (|A_5| + 1) \cdot C2}$ (5.1.18)

$$Q_5 = 5.4 \qquad A_5 = -10$$

Poles of the transfer function **($j = \sqrt{-1}$):**

$$WDen := denom\left(W_{lp}(s)\right) \text{ coeffs}, s \rightarrow \blacksquare \qquad 2\cdot\zeta = 0.035\cdot\frac{Grads}{sec}$$

Search of the poles of $W_{lp}(s)$: poles := polyroots(WDen)$^{\blacksquare}$, $\qquad WDen = \blacksquare$

$$A_5 := A_5 \qquad s := s \qquad \omega_5 := \omega_5 \qquad \zeta := \zeta$$

denominator polynomial: $\quad p(s) := s^2 + 2\cdot\zeta\cdot s + \omega_5^{\,2}$

$$\omega_5 = 0.192\cdot\frac{Grads}{sec} \qquad poles := p(s)\ \left|\begin{matrix}\text{solve}, s\\ \text{simplify}\end{matrix}\right. \rightarrow \begin{pmatrix}\sqrt{\zeta^2 - \omega_5^{\,2}} - \zeta \\ -\zeta - \sqrt{\zeta^2 - \omega_5^{\,2}}\end{pmatrix} \qquad (5.1.19)$$

$$\zeta = 0.018\cdot\frac{Grads}{sec} \qquad poles = \begin{pmatrix}-0.018 + 0.191j \\ -0.018 - 0.191j\end{pmatrix}\cdot\frac{Grads}{sec}$$

The pulsation ω_5 is the geometrical mean of the magnitude of the poles:

$$\omega_5 = \sqrt{|poles_0|\cdot|poles_1|}$$

$$\omega_5 = 0.192\cdot\frac{Grads}{sec} \qquad \sqrt{|poles_0|\cdot|poles_1|} = 0.192\cdot\frac{Grads}{sec}$$

For $\omega=\omega_5$ results:

$$\lim_{\omega \to \omega_5} \left(20\cdot\log\left(|W(j\cdot\omega)|\right)\right) = 20\cdot\log\left(\frac{|A_5|\cdot\omega_5}{2\cdot\zeta}\right) = 20\cdot\log\left(|A_5|\cdot Q_5\right)$$

$$20\cdot\log\left(\frac{|A_5|\cdot\omega_5}{2\cdot\zeta}\right) = 34.648 \qquad 20\cdot\log\left(|A_5|\cdot Q_5\right) = 34.648$$

Hence for $\omega=\omega_5$ results: $|A_5|\cdot Q_5 = 1 \qquad 20\cdot\log\left(|A_5|\right) = -20\cdot\log\left(Q_5\right)$

About Sensitivity

Sensitivity definition given the performance $\mathcal{P}$ and the parameter x_i:

$$S_{x.i} = \frac{x_i}{\mathcal{P}} \cdot \frac{\partial}{\partial x_i} \mathcal{P} = \frac{\partial}{\partial \ln x_i} \ln(\mathcal{P})$$

5.1.1) Calculation of the circuit performance A_5

The filter project could be developed even considering the desired sensitivity of the various parameters.

In this particular case, the performance is the voltage gain: $\mathcal{P} = A_5 = \frac{-R3}{R1}$

Sensitivity Calculation

$$\left|\frac{\Delta A_5}{A_5}\right| = |S_{R1}| \cdot \frac{|\Delta R1|}{R1} + |S_{R3}| \cdot \frac{|\Delta R3|}{R3} = \frac{|\Delta R1|}{R1} + \frac{|\Delta R3|}{R3}$$

$$\boxed{S_{A5} = \left|\frac{\Delta A_5}{A_5}\right| = \frac{|\Delta R1|}{R1} + \frac{|\Delta R3|}{R3}} \qquad (5.1.1.1)$$

5.1.2) Calculation of the sensitivity of the circuit performance ω_5

$$S_{x.i} = \frac{x_i}{\mathcal{P}} \cdot \frac{\partial}{\partial x_i} \mathcal{P} = \frac{\partial}{\partial \ln x_i} \ln(\mathcal{P})$$

$$\mathcal{P} = \omega_5 = \frac{1}{\sqrt{R2 \cdot R3 \cdot C1 \cdot C2}}$$

▶ Sensitivity Calculation

$$\boxed{S_{\omega 5} = \frac{1}{2} \cdot \left(\frac{|\Delta C1|}{C1} + \frac{|\Delta C2|}{C2} + \frac{|\Delta R1|}{R1} + \frac{|\Delta R2|}{R2} \right)} \qquad (5.1.2.1)$$

$$\frac{|\Delta C1|}{C1} + \frac{|\Delta C2|}{C2} = 2 \cdot S_{\omega 5} - \left(\frac{|\Delta R1|}{R1} + \frac{|\Delta R2|}{R2} \right)$$

5.1.3) Calculation of the sensitivity of the circuit performance Q

$$S_{x.i} = \frac{x_i}{\mathcal{P}} \cdot \frac{\partial}{\partial x_i} \mathcal{P} = \frac{\partial}{\partial \ln x_i} \ln(\mathcal{P})$$

$$Q_5 = \frac{1}{\sqrt{\frac{C2}{C1}} \cdot \left(\frac{\sqrt{R2 \cdot R3}}{R1} + \sqrt{\frac{R3}{R2}} + \sqrt{\frac{R2}{R3}} \right)}$$

▶ Sensitivity Calculation

$$S_{Q.5} = \frac{1}{2} \cdot \left(\frac{|\Delta C1|}{C1} + \frac{|\Delta C2|}{C2} \right) + \frac{1}{1 + R1 \cdot \left(\frac{1}{R3} + \frac{1}{R2} \right)} \cdot \frac{|\Delta R1|}{R1} + \frac{1}{2} \cdot \left| \frac{R1 \cdot (R2 - R3) + R2 \cdot R3}{R1 \cdot (R2 + R3) + R2 \cdot R3} \right| \cdot \frac{|\Delta R2|}{R2} \ldots + \left| \frac{1}{R3 \cdot \left(\frac{1}{R1} + \frac{1}{R2} \right) + 1} - \frac{1}{2} \right| \cdot \frac{|\Delta R3|}{R3}$$

(5.1.3.1)

if: $\frac{|\Delta C1|}{C1} = \frac{|\Delta C2|}{C2} = \frac{|\Delta R1|}{R1} = \frac{|\Delta R2|}{R2} = \frac{|\Delta R3|}{R3} = rtol$

$$S_{Q.5} = \left[\left| \frac{1}{R3 \cdot \left(\frac{1}{R1} + \frac{1}{R2} \right) + 1} - \frac{1}{2} \right| + \frac{1}{\left| R1 \cdot \left(\frac{1}{R2} + \frac{1}{R3} \right) + 1 \right|} + \frac{\left| R2 \cdot \left(\frac{1}{R1} + \frac{1}{R3} \right) - 1 \right|}{2 \cdot \left| R2 \cdot \left(\frac{1}{R1} + \frac{1}{R3} \right) + 1 \right|} + 1 \right] \cdot rtol$$

Summary

$$S_{A.5} = \left| \frac{\Delta A_5}{A_5} \right| = \frac{|\Delta R1|}{R1} + \frac{|\Delta R3|}{R3} = 2 \cdot rtol \qquad (5.1.3.2)$$

$$S_{\omega.5} = \frac{1}{2} \cdot \left(\frac{|\Delta C1|}{C1} + \frac{|\Delta C2|}{C2} + \frac{|\Delta R1|}{R1} + \frac{|\Delta R2|}{R2} \right) = 2 \cdot rtol \qquad (5.1.3.3)$$

$$S_{Q.5} = \left[\left| \frac{1}{R3 \cdot \left(\frac{1}{R1} + \frac{1}{R2} \right) + 1} - \frac{1}{2} \right| + \frac{1}{\left| R1 \cdot \left(\frac{1}{R2} + \frac{1}{R3} \right) + 1 \right|} + \frac{\left| R2 \cdot \left(\frac{1}{R1} + \frac{1}{R3} \right) - 1 \right|}{2 \cdot \left| R2 \cdot \left(\frac{1}{R1} + \frac{1}{R3} \right) + 1 \right|} + 1 \right] \cdot rtol \qquad (5.1.3.4)$$

Example (5.1.3.1-1)

are given: $rtol := 5.0\%$,

$R1 := 0.047 \cdot k\Omega$, or $R1 := 120 \cdot \Omega$

$C1 := 1.70 \cdot nF$,

from the definition of A_5 is: $R3 := -A_5 \cdot R1$,

from the definition of damping factor, is derived: $R2 := \dfrac{1}{2 \cdot C1 \cdot \zeta - \left(\dfrac{1}{R1} + \dfrac{1}{R3}\right)}$,

therefore, in order that R2>0, must be: $C1 > \dfrac{1}{2 \cdot \zeta} \cdot \left(\dfrac{1}{R1} + \dfrac{1}{R3}\right)$, $\quad \dfrac{1}{2 \cdot \zeta} \cdot \left(\dfrac{1}{R1} + \dfrac{1}{R3}\right) = 0.258 \cdot nF$

namely: $R2 = 19.546 \cdot \Omega$.

From the definition of ω_5, is derived: $C2 := \dfrac{1}{C1 \cdot R2 \cdot R3 \cdot \omega_5^{\,2}}$,

$C2 = 0.683 \cdot pF$,

$$\frac{\omega_5}{2 \cdot \zeta} = 5.4 \qquad \frac{\omega_5}{2 \cdot Q_5} = 17.743 \cdot \frac{Mrads}{sec}$$

$$R1 = 0.12 \cdot k\Omega \qquad 4 \cdot Q_5^{\,2} \cdot \left(\left|A_5\right| + 1\right) \cdot C2 = 0.876 \cdot nF$$

$$R2 = 19.546\,\Omega \qquad C1 = 1.7 \cdot nF$$

$$R3 = 1.2 \cdot k\Omega$$

Voltage gain: $\quad -\dfrac{R3}{R1} = -10$

Pole Q factor: $\quad \dfrac{1}{\sqrt{\dfrac{C2}{C1}} \cdot \left(\dfrac{\sqrt{R2 \cdot R3}}{R1} + \sqrt{\dfrac{R3}{R2}} + \sqrt{\dfrac{R2}{R3}}\right)} = 5.4$

Geometric mean of the poles and the resulting frequency:

$$\frac{1}{\sqrt{R2 \cdot R3 \cdot C1 \cdot C2}} = 0.192 \cdot \frac{Grads}{sec} \qquad \omega_5 = 0.192 \cdot \frac{Grads}{sec}$$

The previous findings (R1, R2, R3), after a slight modification of one or more of them, may be used as a guess for a new solution (not always found, also after many attempts) of the system, as follows:

Given (5.1.3.5)

$$\omega_5 = \frac{1}{\sqrt{R2 \cdot R3 \cdot C1 \cdot C2}}$$

$$A_5 = -\frac{R3}{R1}$$

$$Q_5 = \frac{1}{\sqrt{\dfrac{C2}{C1}} \cdot \left(\dfrac{\sqrt{R2 \cdot R3}}{R1} + \sqrt{\dfrac{R3}{R2}} + \sqrt{\dfrac{R2}{R3}}\right)}$$

$$\zeta = \frac{1}{2}\cdot\left[\frac{1}{C1}\cdot\left(\frac{1}{R1}+\frac{1}{R2}+\frac{1}{R3}\right)\right]$$

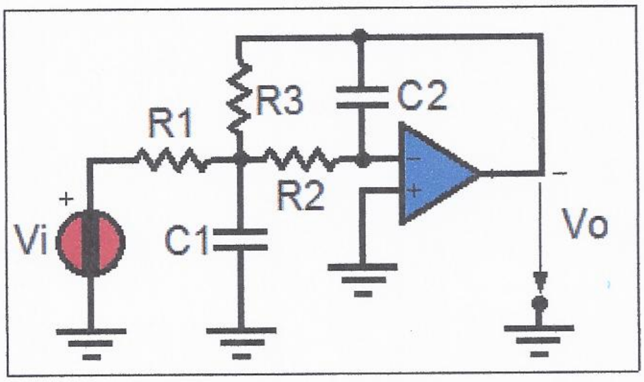
$R1 > 0.0\cdot\Omega$

$R2 > 0.0\cdot\Omega$

$R3 > 0.0\cdot\Omega$

$Rx := Find(R1, R2, R3)$

$$Rx = \begin{pmatrix} 120 \\ 19.546 \\ 1.2\times 10^{3} \end{pmatrix}\cdot\Omega \qquad C1 = 1.7\cdot nF \qquad C2 = 6.83\times 10^{-4}\cdot nF$$



$A_5 = -10$

$\omega_5 = 0.192\cdot\frac{Grads}{sec}$

$2\cdot\zeta = 0.035\cdot\frac{Grads}{sec}$

fig.:5.1.19

Sensitivity for this example

$$S_{A.5} = \left|\frac{\Delta A_5}{A_5}\right| \qquad S_{A.5} := (rtol + rtol) \qquad rtol = 5\cdot\% \qquad (5.1.3.6)$$

$$S_{\omega.5} = \frac{|\Delta\omega_5|}{|\omega_5|} \qquad S_{\omega.5} := \frac{1}{2}\cdot(rtol + rtol + rtol + rtol) \qquad (5.1.3.7)$$

$$S_{Q.5} := \left[\left|\frac{1}{R3\cdot\left(\frac{1}{R1}+\frac{1}{R2}\right)+1} - \frac{1}{2}\right| + \frac{1}{\left|R1\cdot\left(\frac{1}{R2}+\frac{1}{R3}\right)+1\right|} + \frac{\left|R2\cdot\left(\frac{1}{R1}+\frac{1}{R3}\right)-1\right|}{2\cdot\left|R2\cdot\left(\frac{1}{R1}+\frac{1}{R3}\right)+1\right|} + 1\right]\cdot rtol$$

results: $S_{A.5} = 10\cdot\%$ $\qquad S_{\omega.5} = 10\cdot\%$ $\qquad S_{Q.5} = 9.862\cdot\%$ $\qquad$ (5.1.3.8)

5.1.4 NYQUIST DIAGRAM

Op Amp open loop voltage gain: $A_{ol} := 10^5$

After the following substitution:

$$GH_5(s) := -\frac{[(Y1 + Y2 + Y3 + Y4)\cdot Y5 + Y3\cdot Y4]\cdot A_{ol}}{(Y1 + Y2 + Y3 + Y4)\cdot Y5 + Y3\cdot(Y1 + Y2 + Y4)} \quad \left| \begin{array}{l} \text{substitute}, Y1 = \frac{1}{R1}, Y2 = s\cdot C1, Y3 = \frac{1}{R2} \\ \text{substitute}, Y4 = \frac{1}{R3}, Y5 = s\cdot C2 \end{array} \right. \rightarrow$$

results the already found relation (5.1.13), her rewritten:

$$GH_5(s) := -\frac{s^2 + \frac{[R1\cdot(R2 + R3) + R2\cdot R3]}{C1\cdot R1\cdot R2\cdot R3}\cdot s + \frac{1}{(C1\cdot C2\cdot R2\cdot R3)}}{s^2 + \frac{[C2\cdot[R1\cdot(R2 + R3) + R2\cdot R3] + C1\cdot R1\cdot R3]}{(C1\cdot C2\cdot R1\cdot R2\cdot R3)}\cdot s + \frac{R1 + R3}{C1\cdot C2\cdot R1\cdot R2\cdot R3}}\cdot A_{ol} \quad (5.1.13')$$

Place:

$$\omega_5 = \frac{1}{\sqrt{R2\cdot R3\cdot C1\cdot C2}} \qquad \omega_1 := \frac{[R1\cdot(R2 + R3) + R2\cdot R3]}{C1\cdot R1\cdot R2\cdot R3}$$

$$\omega_2 = \frac{R1\cdot(R2 + R3) + R2\cdot R3}{C1\cdot R1\cdot R2\cdot R3} + \frac{1}{C2\cdot R2} = \omega_1 + \frac{1}{R2\cdot C2}$$

$$\omega_3 := \sqrt{\frac{R1 + R3}{R1}\cdot \omega_5^2} \qquad \omega_2 := \omega_1 + \frac{1}{R2\cdot C2}$$

$$\omega_5 = 0.192\cdot\frac{Grads}{sec} \qquad \omega_1 = 0.035\cdot\frac{Grads}{sec}$$

$$\omega_2 = 74.946\cdot\frac{Grads}{sec} \qquad \omega_3 = 0.636\cdot\frac{Grads}{sec} \qquad \omega_3^2 = 0.404\cdot\left(\frac{Grads}{sec}\right)^2$$

$s := s$

so that one can write:

$$\boxed{GH_5(s) := -\frac{s^2 + \omega_1\cdot s + \omega_5^2}{s^2 + \omega_2\cdot s + \omega_3^2}\cdot A_{ol}} \quad (5.1.4.2)$$

$$\omega := \frac{\omega_1}{10^1}, \frac{\omega_1}{10^1} + \frac{\left(2\cdot 10^2\cdot\omega_2 - \frac{\omega_1}{10^1}\right)}{10^4} .. 2\cdot 10^2\cdot\omega_2 \qquad \frac{\omega_1}{10^1} = 3.549\times 10^{-3}\cdot\frac{Grads}{sec}$$

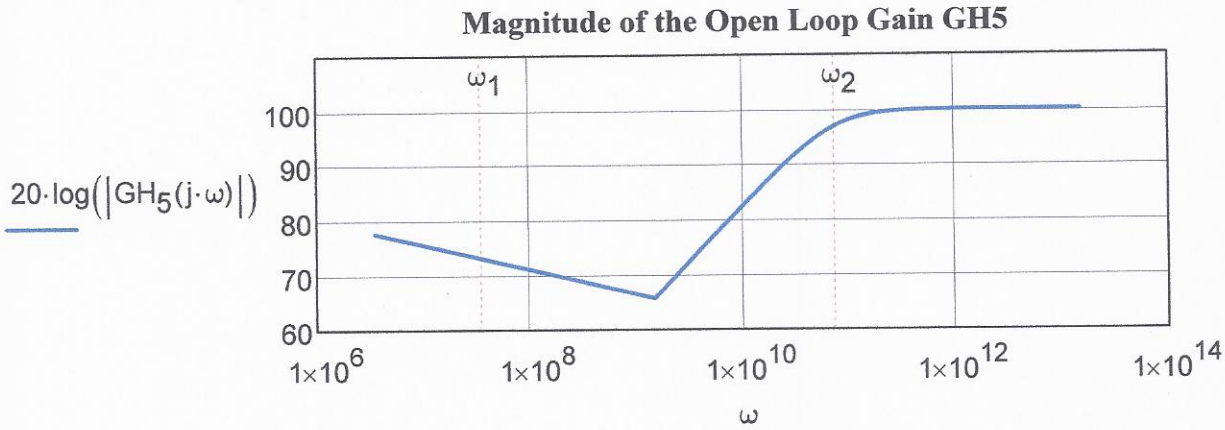


fig.:5.1.4.1

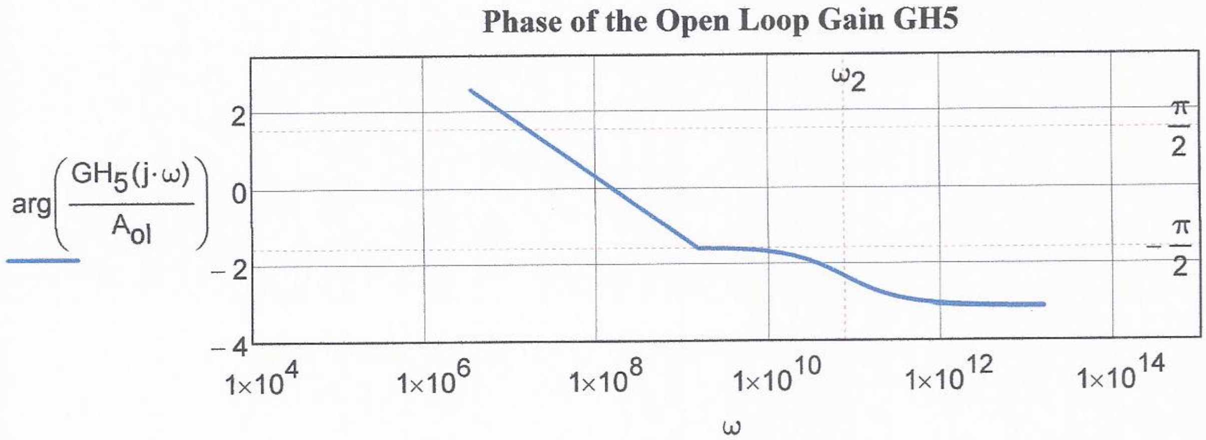


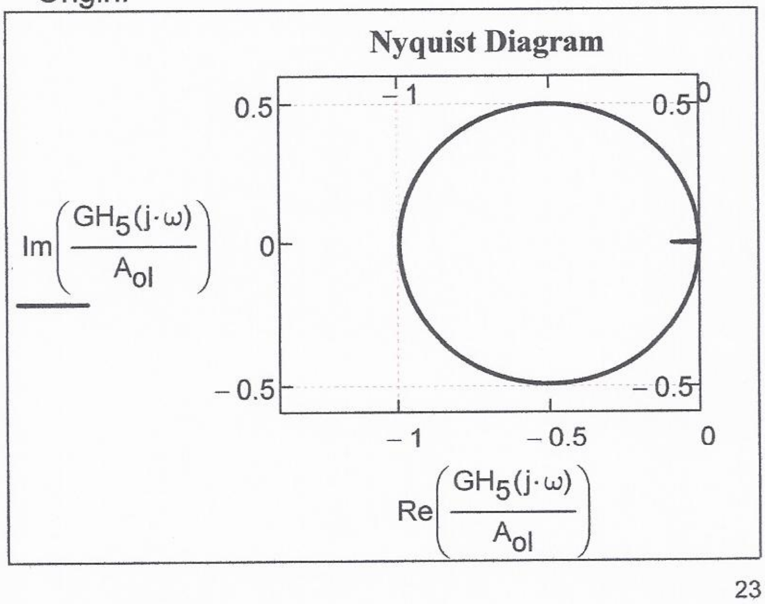
fig.:5.1.4.2

$$\omega := -2\cdot 10^2\cdot\omega_2, -2\cdot 10^2\cdot\omega_2 + \frac{2\cdot 10^2\cdot\omega_2 + 2\cdot 10^2\cdot\omega_2}{10^5} \;..\; 2\cdot 10^2\cdot\omega_2$$

To let see the inner loop of the Nyquist diagram it has been defined the following new interval:

$$\omega_x := -4\cdot 10^{-4}\cdot\omega_2, -4\cdot 10^{-4}\cdot\omega_2 + \frac{4\cdot 10^{-4}\cdot\omega_2 + 4\cdot 10^{-4}\cdot\omega_2}{10^5} \;..\; 4\cdot 10^{-4}\cdot\omega_2$$

The Nyquist diagram of the normalized open loop gain is composed by two circles, both tangent to the Origin:



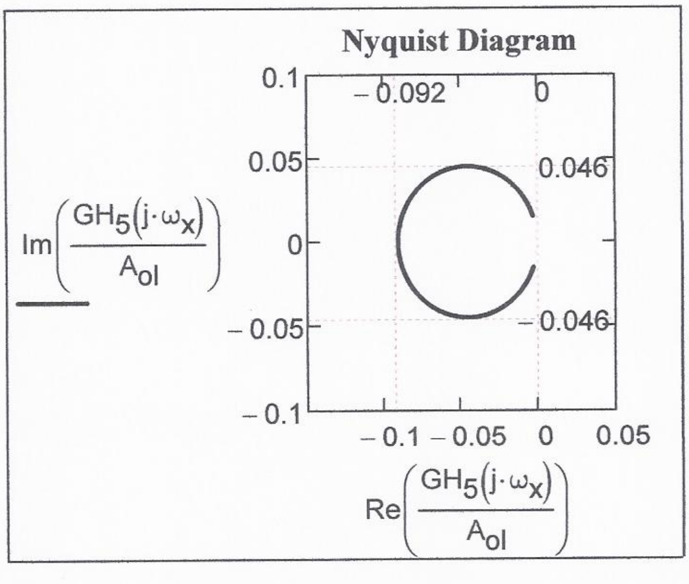


fig.:5.1.4.3

Desensitization:

$$D(s) = 1 + GH_5(s) = \frac{\left[[Y5\cdot(Y1+Y2+Y3+Y4)+Y3\cdot Y4]\cdot\left(1-A_{ol}\right)+Y3\cdot(Y1+Y2)\right]}{[(Y1+Y2+Y3+Y4)\cdot Y5+Y3\cdot(Y1+Y2+Y4)]} \quad (5.1.4.3)$$

$$\frac{1}{D(s)} = \frac{[(Y1+Y2+Y3+Y4)\cdot Y5+Y3\cdot(Y1+Y2+Y4)]}{\left[[Y5\cdot(Y1+Y2+Y3+Y4)+Y3\cdot Y4]\cdot\left(1-A_{ol}\right)+Y3\cdot(Y1+Y2)\right]} \quad (5.1.4.4)$$

$$W(s) = \lim_{A_{ol}\to\infty} \frac{(Y1+Y2+Y3+Y4)\cdot Y5+Y3\cdot(Y1+Y2+Y4)}{[Y5\cdot(Y1+Y2+Y3+Y4)+Y3\cdot Y4]\cdot\left(\frac{1-A_{ol}}{A_{ol}}\right)+\frac{Y3\cdot(Y1+Y2)}{A_{ol}}}$$

$$W(s) = -\frac{(Y1+Y2+Y3+Y4)\cdot Y5+Y3\cdot(Y1+Y2+Y4)}{Y5\cdot(Y1+Y2+Y3+Y4)+Y3\cdot Y4}$$

5.1.5 Nichols chart

From the analysis of a generic L.T.I. system with negative feedback, can be drawn some conclusion summarized here below:

NEGATIVE FEEDBACK SYSTEM

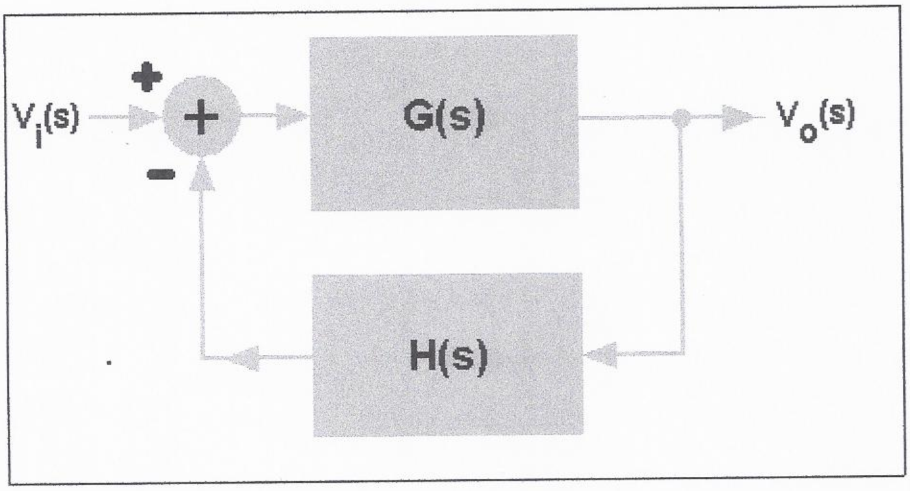


fig.:5.1.5.1

System transfer function with negative feedback:

$$G_f(s) = \frac{G(s)}{1 + G(s)\cdot H(s)} \qquad 5.1.5.1$$

System's parameters variation: desensitization.
Ratio of the relative gain variations with and without feedback:

$$\left| \frac{\frac{\Delta G_f(s)}{G_f(s)}}{\frac{\Delta G(s)}{G(s)}} \right| = \left| \frac{1}{D(s)} \right| \qquad 5.1.5.2$$

Low frequency voltage gain in dB: $W_{lpp} := 20\cdot\log\left(\left|A_5\right|\right)$ $\qquad W_{lpp} = 20$

Open loop gain($GH_5(s) = G(s)\cdot H(s) = -\frac{s^2 + \omega_1\cdot s + \omega_5^2}{s^2 + \omega_2\cdot s + \omega_3^2}\cdot A_{ol}$**)**:

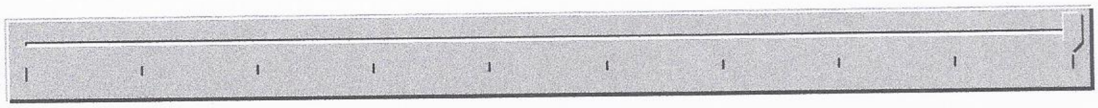
$$GH_5(s) := -\frac{s^2 + \omega_1\cdot s + \omega_5^2}{s^2 + \omega_2\cdot s + \omega_3^2}\cdot A_{ol} \qquad 5.1.5.3$$

In the following Nichols chart, the magnitude in dB of the open loop gain is in **purple red** depicted. The **red scarlet** line refers to the Magnitude in dB of the **open loop gain = 0dB**. The yellow arc is related to $\phi = \pm\frac{\pi}{2}$.

$$\phi(\omega) = \arg(G(j\cdot\omega)\cdot H(j\cdot\omega)) \qquad \alpha_{nch5}(\omega) = \arg(G(j\cdot\omega)\cdot H(j\cdot\omega)) \qquad 5.1.5.4$$

Scroll The Slider to Zoom In or Out The Nichols Chart

$mnp_{nch} :=$



Nichol's chart lower left corner: $m_{nch5} := 1.5$

Nichol's chart lower right corner: $n_{nch} := 0.0$

Nichol's chart upper left corner: $p_{nch} := mnp_{nch} \cdot 10 \qquad p_{nch} = 100$

$$\alpha_{nch5} := -\pi \cdot m_{nch5} \cdot 1.1, -\pi \cdot m_{nch5} \cdot 1.1 + \frac{2 \cdot \pi \cdot m_{nch5}}{10^4} .. m_{nch5} \cdot \pi$$

The values of the *gain margin* and the *phase margin* can be deduced observing the graph. Therefore have a measure of the degree of stability of the system with feedback.

$\omega\_x := 4 \cdot 10^0 \cdot \omega_2 \qquad GH_{init} := 20 \log\left[\left|GH_5[j \cdot (-\omega\_x)]\right|\right] \qquad GH_{end} := 20 \log\left[\left|GH_5[j \cdot (\omega\_x)]\right|\right]$

$\omega_1 = 0.035 \cdot \frac{Grads}{sec} \qquad \omega := -\omega\_x, -\omega\_x + \frac{\omega\_x + \omega\_x}{10^5 \cdot 2} .. \omega\_x$

$\omega_{test} = 0.383 \cdot \frac{Grads}{sec} \qquad \omega_2 = 74.946 \cdot \frac{Grads}{sec} \qquad Q_5 = 5.4$

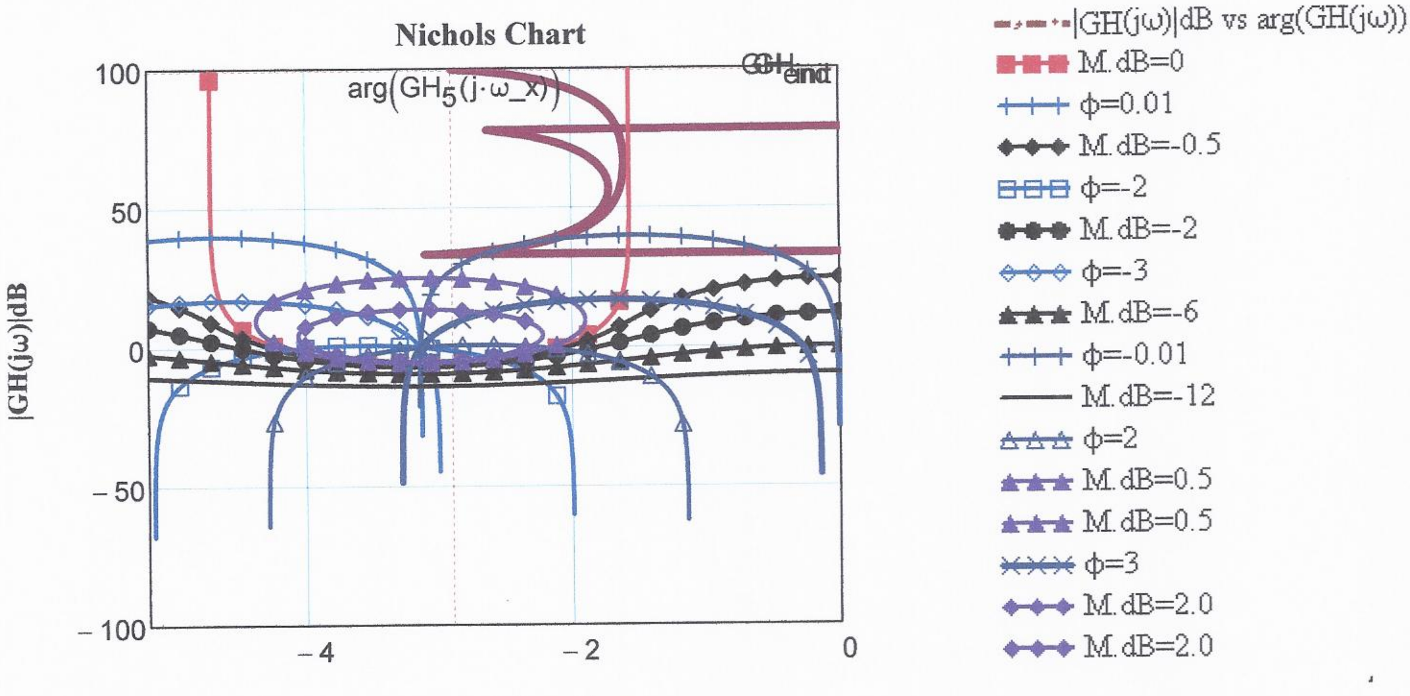


fig.:5.1.5.2

The *gain margin* is defined as $g_{mg} = \frac{1}{\left|GH_5(j \cdot \omega_x)\right|}$ and is calculated at the frequency ω_x where $\arg(GH(j \cdot \omega_x)) = -\pi$ or π. (For amplifiers it should be $g_{mg} \geq 4$)

The *phase margin,* on the other hand, is defined as $ph_{mg} = \pi - \left|\arg(GH_5(j \cdot \omega_y))\right|$ calculated at the frequency ω_y where $\left|GH(j \cdot \omega_y)\right| = 1$. (For amplifiers it should be $ph_{mg} \geq \frac{\pi}{3}$)

5.1.6 Four particular cases of the transfer function .

1a) R1 = R2 = R3 = R, and C1 = C2 = C,

2a) R2 = R3 = R, and C1 = C2 = C,

3a) R1 = R2 = R, and C1 = C2 = C,

4a) R1 = R2 = R3 = R,

1a) R1=R2=R3=R, and C1=C2=C

Substituting in the transfer function, $W_{lp}(s) = \dfrac{-\dfrac{1}{R1\cdot R2\cdot C1\cdot C2}}{s^2 + s\cdot\dfrac{1}{C1}\cdot\left(\dfrac{1}{R1}+\dfrac{1}{R2}+\dfrac{1}{R3}\right)+\dfrac{1}{R2\cdot R3\cdot C1\cdot C2}}$,

it becomes: $$W_{lp}(s) = \frac{-\dfrac{1}{R^2\cdot C^2}}{s^2 + s\cdot\dfrac{3}{C\cdot R}+\dfrac{1}{R^2\cdot C^2}} \qquad (5.1.6.1)$$

$$A_5\cdot\omega_5^{\,2} = -\frac{1}{R^2\cdot C^2} \qquad R3 = R = \frac{1}{2\cdot C\cdot Q_5\cdot\omega_5}\cdot\left[\frac{1+\sqrt{1-4\cdot Q_5^{\,2}\cdot\left(|A_5|+1\right)}}{1-\sqrt{1-4\cdot Q_5^{\,2}\cdot\left(|A_5|+1\right)}}\right] \qquad (5.1.6.2)$$

condition in order that R takes real values $|A_5| = 1$

$$\frac{1}{2\cdot\sqrt{2}} = 0.354 \qquad Q_5 \le \frac{1}{2\cdot\sqrt{|A_5|+1}} = \frac{1}{2\cdot\sqrt{2}}$$

In summary, it can be written: a) $$\omega_5 = \frac{1}{\sqrt{R2\cdot R3\cdot C1\cdot C2}} = \frac{1}{\sqrt{R\cdot R\cdot C\cdot C}} = \frac{1}{R\cdot C}, \qquad (5.1.6.3)$$

b) $$A_5 = -\frac{R3}{R1} = -1, \qquad (5.1.6.4$$

c) $$Q_5 = \frac{1}{\sqrt{\dfrac{C2}{C1}}\cdot\left(\dfrac{\sqrt{R2\cdot R3}}{R1}+\sqrt{\dfrac{R3}{R2}}+\sqrt{\dfrac{R2}{R3}}\right)} = \frac{1}{3}, \qquad (5.1.6.5)$$

d) $$\zeta = \frac{1}{2}\cdot\left[\frac{1}{C1}\cdot\left(\frac{1}{R1}+\frac{1}{R2}+\frac{1}{R3}\right)\right] = \frac{3}{2\cdot R\cdot C}, \qquad (5.1.6.6)$$

e) $$\frac{\omega_5}{2\cdot\zeta} = \frac{1}{3}. \qquad (5.1.6.7$$

2a) R2=R3=R, and C1=C2=C

$$A_5 \cdot \omega_5^2 = -\frac{1}{R1 \cdot R \cdot C \cdot C} \qquad \boxed{W_{lp}(s) = \frac{-\frac{1}{R1 \cdot R \cdot C^2}}{s^2 + s \cdot \frac{1}{C} \cdot \left(\frac{1}{R1} + \frac{2}{R}\right) + \frac{1}{R^2 \cdot C^2}}} \quad (5.1.6.8)$$

$$R = \frac{1}{2 \cdot C \cdot Q_5 \cdot \omega_5} \cdot \begin{bmatrix} 1 + \sqrt{1 - 4 \cdot Q_5^2 \cdot (|A_5| + 1)} \\ 1 - \sqrt{1 - 4 \cdot Q_5^2 \cdot (|A_5| + 1)} \end{bmatrix} \quad (5.1.6.9)$$

Condition in order that R3 takes real values:

$$\boxed{1 \geq 4 \cdot Q_5^2 \cdot (|A_5| + 1)} \quad (5.1.6.10)$$

$$Q_5 \leq \frac{1}{2 \cdot \sqrt{|A_5| + 1}} \qquad \frac{1}{2 \cdot \sqrt{|A_5| + 1}} = 0.151$$

In summary, it can be written: a) $\omega_5 = \frac{1}{\sqrt{R2 \cdot R3 \cdot C1 \cdot C2}} = \frac{1}{\sqrt{R \cdot R \cdot C \cdot C}} = \frac{1}{R \cdot C}$, (5.1.6.11)

b) $A_5 = -\frac{R3}{R1} = -\frac{R}{R1}$, (5.1.6.12)

c) $Q_5 = \frac{1}{\sqrt{\frac{C2}{C1}} \cdot \left(\frac{\sqrt{R2 \cdot R3}}{R1} + \sqrt{\frac{R3}{R2}} + \sqrt{\frac{R2}{R3}}\right)} = \frac{1}{\left(\frac{R}{R1} + 2\right)} = \frac{1}{2 - A_5}$,

that holds for $Q_5 < \frac{1}{2}$ or $A_5 < 2$. (5.1.6.13)

d) $\zeta = \frac{1}{2} \cdot \left[\frac{1}{C1} \cdot \left(\frac{1}{R1} + \frac{1}{R2} + \frac{1}{R3}\right)\right] = \frac{3}{2 \cdot R \cdot C}$, (5.1.6.14)

e) $\frac{\omega_5}{2 \cdot \zeta} = \frac{1}{3} = \frac{1}{2 - A_5}$. (5.1.6.15)

3a) R1=R2=R, and C1=C2=C

$$W_{lp}(s) = \frac{\frac{1}{R^2 \cdot C^2}}{s^2 + s \cdot \frac{1}{C} \cdot \left(\frac{2}{R} + \frac{1}{R3}\right) + \frac{1}{R \cdot R3 \cdot C^2}} \qquad (5.1.6.16)$$

$$R3 = \frac{1}{2 \cdot C \cdot Q_5 \cdot \omega_5} \cdot \left[\frac{1 + \sqrt{1 - 4 \cdot Q_5^2 \cdot (|A_5| + 1)}}{1 - \sqrt{1 - 4 \cdot Q_5^2 \cdot (|A_5| + 1)}}\right] \qquad (5.1.6.17)$$

condition in order that R3 takes real values

$$A_5 = -10 \qquad Q_5 \le \frac{1}{2 \cdot \sqrt{|A_5| + 1}} \qquad \frac{1}{2 \cdot \sqrt{|A_5| + 1}} = 0.151$$

In summary, it can be written:

a) $$\omega_5 = \frac{1}{\sqrt{R \cdot R3 \cdot C \cdot C}} = \frac{1}{C \cdot \sqrt{R \cdot R3}}, \qquad (5.1.6.18)$$

$$A_5 \cdot \omega_5^2 = \frac{A_5}{(R \cdot R3 \cdot C \cdot C)} = -\frac{1}{R^2 \cdot C^2}, \qquad (5.1.6.19)$$

b) $$A_5 = -\frac{R3}{R}, \qquad (5.1.6.20$$

c) $$Q_5 = \frac{1}{\sqrt{\frac{C}{C}} \cdot \left(\frac{\sqrt{R \cdot R3}}{R} + \sqrt{\frac{R3}{R}} + \sqrt{\frac{R}{R3}}\right)} = \frac{1}{\left(2 \cdot \sqrt{\frac{R3}{R}} + \sqrt{\frac{R}{R3}}\right)} = \frac{1}{\left(2 \cdot \sqrt{|A_5|} + \sqrt{\frac{1}{|A_5|}}\right)}, \qquad (5.1.6.2$$

d) $$\zeta = \frac{1}{2} \cdot \frac{1}{C} \cdot \left(\frac{1}{R} + \frac{1}{R} + \frac{1}{R3}\right) = \frac{1}{2 \cdot C} \cdot \left(\frac{2}{R} + \frac{1}{R3}\right), \qquad (5.1.6.22)$$

e) $$\frac{\omega_5}{2 \cdot \zeta} = \frac{1}{\sqrt{\frac{R}{R3}} + 2 \cdot \sqrt{\frac{R3}{R}}}. \qquad (5.1.6.23)$$

from b) it follows: $\frac{R3}{R} = -A_5 \qquad A_5 < 0 \qquad \frac{R3}{R} = |A_5|$

$$Q_5 = \frac{1}{2 \cdot \sqrt{|A_5|} + \frac{1}{\sqrt{|A_5|}}} \qquad (5.1.6.24)$$

$$\sqrt{|A_5|} = \begin{pmatrix} \frac{\sqrt{1 - 8 \cdot Q_5^2} + 1}{4 \cdot Q_5} \\ -\frac{\sqrt{1 - 8 \cdot Q_5^2} - 1}{4 \cdot Q_5} \end{pmatrix} \qquad \sqrt{|A_5|} = \frac{\sqrt{1 - 8 \cdot Q_5^2} + 1}{4 \cdot Q_5}$$

$$1 - 8 \cdot Q_5^2 > 0$$

condition in order that R3 takes real values $Q_5 \le \dfrac{1}{2\cdot\sqrt{|A_5|+1}}$ $\boxed{0 < Q_5 < \dfrac{1}{2\cdot\sqrt{2}}}$ (5.1.6.25)

$$\frac{1}{2\cdot\sqrt{2}} = 0.3536 \qquad \frac{1}{2\cdot\sqrt{|A_5|+1}} = 0.151$$

4a) R1=R2=R3=R

$$\boxed{W_{lp}(s) = \frac{\dfrac{1}{R^2\cdot C1\cdot C2}}{s^2 + s\cdot\dfrac{1}{C1}\cdot\dfrac{3}{R} + \dfrac{1}{R^2\cdot C1\cdot C2}}} \qquad (5.1.6.26)$$

$$R3 = R = \frac{1}{2\cdot C2\cdot Q_5\cdot\omega_5}\cdot\left[\begin{matrix}1+\sqrt{1-\dfrac{4\cdot C2\cdot Q_5^{\,2}\cdot\left(|A_5|+1\right)}{C1}}\\ 1-\sqrt{1-\dfrac{4\cdot C2\cdot Q_5^{\,2}\cdot\left(|A_5|+1\right)}{C1}}\end{matrix}\right] \qquad (5.1.6.27)$$

condition in order that R3 takes real values:

$$\boxed{\frac{C1}{C2} \ge 4\cdot Q_5^{\,2}\cdot\left(|A_5|+1\right)} \qquad Q_5 \le \frac{1}{2}\cdot\sqrt{\frac{C1}{C2\cdot\left(|A_5|+1\right)}} \qquad (5.1.6.28)$$

In summary, it can be written: a) $\omega_5 = \dfrac{1}{\sqrt{R2\cdot R3\cdot C1\cdot C2}} = \dfrac{1}{R\cdot\sqrt{C1\cdot C2}}$, (5.1.6.29)

b) $A_5 = -\dfrac{R3}{R1} = -1$, (5.1.6.

c) $Q_5 = \dfrac{1}{\sqrt{\dfrac{C2}{C1}}\cdot\left(\dfrac{\sqrt{R2\cdot R3}}{R1} + \sqrt{\dfrac{R3}{R2}} + \sqrt{\dfrac{R2}{R3}}\right)} = \dfrac{1}{3}\cdot\sqrt{\dfrac{C1}{C2}}$ (5.1.6.31

$$\zeta = \frac{\omega_5}{2\cdot Q_5} = \frac{\dfrac{1}{R\cdot\sqrt{C1\cdot C2}}}{2\cdot\dfrac{1}{\sqrt{\dfrac{C2}{C1}}\cdot 3}} = \frac{3\cdot\sqrt{\dfrac{C2}{C1}}}{2\cdot R\cdot\sqrt{C1\cdot C2}} = \frac{3}{2\cdot R\cdot C1} \qquad (5.1.6.32)$$

condition in order that R3 takes real values $Q_5 \le \dfrac{1}{2}\cdot\sqrt{\dfrac{C1}{C2\cdot\left(|A_5|+1\right)}}$ (5.1.6.33)

$$\frac{1}{2}\cdot\sqrt{\frac{C1}{C2\cdot\left(|A_5|+1\right)}} = 7.521 \qquad \frac{1}{3}\cdot\sqrt{\frac{C1}{C2}} = 16.631$$

5.1.7 Pulse response.
General case: R1≠R2≠R3, C1≠C2

Known values:

Voltage gain: $A_5 = -10$ Pole Q factor: $Q_5 = 5.4$,

Pole pulsation: $\omega_5 = 0.192 \cdot \frac{Grads}{sec}$

damping factor: $\zeta = 0.018 \cdot \frac{Grads}{sec}$

Graph of the pulse response:

Transfer function:

$$W_{lp}(s) := \begin{cases} \dfrac{A_5 \cdot \omega_5^2}{s^2 + 2 \cdot \zeta \cdot s + \omega_5^2} & \text{if } \zeta \neq \omega_5 \\ A_5 \cdot \dfrac{\omega_5^2}{(s + \omega_5)^2} & \text{otherwise} \end{cases} \qquad (5.1.7.1)$$

$A_5 := A_5 \quad s := s \quad a := a \quad \omega_5 := \omega_5 \qquad \zeta := \zeta$

Calculation of the pulse response as the inverse Laplace transform of the t. f.:

$$w(t) := \frac{A_5 \cdot \omega_5^2}{s^2 + 2 \cdot \zeta \cdot s + \omega_5^2} \left| \begin{array}{l} \text{invlaplace}, s, t \\ \text{rewrite}, \exp \\ \text{simplify}, \max \end{array} \right. \rightarrow$$

Dirac pulse response chosen:

Pulse response:

$$w(t) := A_5 \cdot \omega_5^2 \cdot t \cdot \text{sinc}\left(t \cdot \sqrt{\omega_5^2 - \zeta^2}\right) \cdot e^{-\zeta \cdot t} \cdot \Phi(t) \qquad (5.1.7.2)$$

Search of the minimum:

$$\frac{\partial}{\partial t} w(t) = 0 \text{ for t= } tx2 := \begin{cases} \dfrac{2 \cdot \text{atan}\left(\dfrac{\zeta - \omega_5}{\sqrt{\omega_5^2 - \zeta^2}}\right)}{\sqrt{\omega_5^2 - \zeta^2}} & \text{if } \zeta \neq \omega_5 \\ \dfrac{1}{\zeta} & \text{otherwise} \end{cases} , \qquad (5.1.7.3)$$

$tx2 = 7.747 \cdot ns$

Minimum: $w(tx2) = -1.67 \cdot \frac{Grads}{sec}$

Initial value theorem: $\lim_{t \to 0} f(t) = \lim_{s \to \infty} (s \cdot F(s))$, Final value theorem: $\lim_{t \to \infty} f(t) = \lim_{s \to 0} (s \cdot F(s))$

$A_5 := A_5 \quad s := s \quad a := a \quad \omega_5 := \omega_5$

$\zeta := \zeta$

$$\lim_{s \to \infty} \left(s \cdot \frac{A_5 \cdot \omega_5^2}{s^2 + 2 \cdot \zeta \cdot s + \omega_5^2} \right) \to 0 \qquad \omega_5^2 - \zeta^2 = 0.036 \cdot \left(\frac{Grads}{sec} \right)^2$$

$A_5 := A_5 \quad s := s \quad a := a \quad \omega_5 := \omega_5$

$\zeta := \zeta$

$$\lim_{s \to 0} \left(s \cdot \frac{A_5 \cdot \omega_5^2}{s^2 + 2 \cdot \zeta \cdot s + \omega_5^2} \right) \to 0$$

$tx2 = 7.747 \cdot ns$

$$t := -1 \cdot T_{test}, -1 \cdot T_{test} + \frac{20 \cdot T_5 + 1 \cdot T_{test}}{10000} .. 20 \cdot T_5$$

▶ Right Lower Corner Graph Control

Graph of the impulse response.

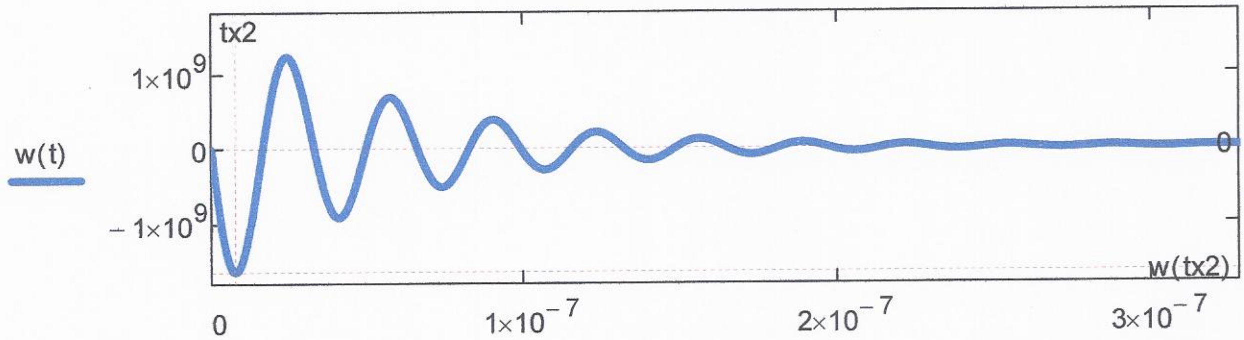


fig.:5.1.7.1

5.1.8 BODE PLOTS (Low Pass II° order):

For time harmonic signal place: s=jω and call the magnitude in dB of the frequency response as follows:

$$WlpdB(\omega) := 20 \cdot \log(|W_{lp}(j \cdot \omega)|) \qquad WlpdB(\omega_5) = 34.648$$

now proceed to its computing:

$A_5 := A_5 \quad s := s \quad a := a \quad \omega_5 := \omega_5 \quad \zeta := \zeta \quad \omega := \omega$

$$W_{lp\_}(\omega) := \frac{A_5 \cdot \omega_5^2}{s^2 + 2 \cdot \zeta \cdot s + \omega_5^2} \left| \begin{array}{l} substitute, s = j \cdot \omega \\ simplify, max \\ collect, \omega \end{array} \right. \to$$

$$W_{lp\_}(\omega) := \frac{\omega^2 - \omega_5^2 + 2j\cdot\zeta\cdot\omega}{4\cdot\zeta^2\cdot\omega^2 + \omega^4 - 2\cdot\omega^2\cdot\omega_5^2 + \omega_5^4}\cdot\omega_5^2\cdot A_5 \qquad (5.1.8.1)$$

Considering the poles, the transfer function can be rewritten as:

$$W_{lp\_}(\omega) := \frac{A_5\cdot\omega_5^2}{\left[j\cdot\omega - \left(\sqrt{\zeta^2 - \omega_5^2} - \zeta\right)\right]\cdot\left(j\cdot\omega + \zeta + \sqrt{\zeta^2 - \omega_5^2}\right)} \qquad (5.1.8.2)$$

Hence the magnitude of the frequency response in dB is:

$$A_5 = -|A_5| \quad WlpdB(\omega) := 20\cdot\log\left[|A_5|\cdot\omega_5^2\cdot\frac{\sqrt{\left(-\omega^2 + \omega_5^2\right)^2 + (-2\cdot\zeta\cdot\omega)^2}}{\omega^4 + \omega_5^4 + \omega^2\cdot\left(4\cdot\zeta^2 - 2\cdot\omega_5^2\right)}\right] \qquad (5.1.8.3)$$

The phase response is:

$$\varphi_s(\omega) := \pi - \operatorname{atan}\left[\frac{\omega}{-\left(\sqrt{\zeta^2 - \omega_5^2} - \zeta\right)}\right] - \operatorname{atan}\left[\frac{\omega}{\left(\zeta + \sqrt{\zeta^2 - \omega_5^2}\right)}\right] \qquad (5.1.8.4)$$

If Q_5>0.5 the frequency response presents a overshoot at:

$$\omega_{pick} := \begin{cases} \sqrt{\omega_5^2 - 2\cdot\zeta^2} & \text{if } \zeta \neq \omega_5 \wedge \omega_5 > \sqrt{2}\cdot\zeta \\ \omega_5 & \text{otherwise} \end{cases}$$

$$\omega_{pick} = 0.19\cdot\frac{\text{Grads}}{\text{sec}}$$

and the pick amplitude is

$$WlpdB_{pick} := \begin{cases} 20\cdot\log\left[\frac{\omega_5^2\cdot|A_5|}{2\cdot\sqrt{\zeta^2\cdot\left(\omega_5^2 - \zeta^2\right)}}\right] & \text{if } \zeta \neq \omega_5 \wedge \omega_5 > \sqrt{2}\cdot\zeta \\ 20\cdot\log(|A_5|) & \text{otherwise} \end{cases} \qquad (5.1.8.5)$$

pick amplitude $WlpdB_{pick} = 34.685$

Bandwidth calculation.

The bandwidth is given by the frequency at which the magnitude of the transfer function is $\frac{|A_5|}{\sqrt{2}}$, (s=jω),

namely:

$$|A_5|\cdot\omega_5^2\cdot\frac{\sqrt{\left(-\omega^2 + \omega_5^2\right)^2 + (-2\cdot\zeta\cdot\omega)^2}}{\omega^4 + \omega_5^4 + \omega^2\cdot\left(4\cdot\zeta^2 - 2\cdot\omega_5^2\right)} = \frac{|A_5|}{\sqrt{2}}$$

▶ bandwidth calculation

Bandwidth: $$Bw := \frac{\sqrt{\omega_5^2 - 2\cdot\zeta^2 + \sqrt{2}\cdot\sqrt{2\cdot\zeta^2\cdot\left(\zeta^2-\omega_5^2\right)+\omega_5^4}}}{2\cdot\pi} \qquad (5.1.8.6)$$

Knowing the bandwidth one can determine ω_5 vs. Q_5 and Bw: $Bw = 47.1\cdot MHz$

$$\omega_5 = 0.192\cdot\frac{Grads}{sec} \qquad \omega_5 := \sqrt{\frac{(Bw\cdot 2\cdot\pi)^2\cdot\left[\sqrt{4\cdot Q_5^2\cdot\left(2\cdot Q_5^2-1\right)+1}-2\cdot Q_5^2+1\right]}{2\cdot Q_5^2}} \qquad (5.1.8.7)$$

if Q_5=0.5, it results. $\omega_5 = 2\cdot\pi\cdot Bw\cdot\sqrt{\sqrt{2}+1}$

Knowing Q_5 and ω_5, one can obtain Bw:

$$Bw = \frac{\sqrt{2}\cdot\omega_5\cdot Q_5}{2\cdot\pi\cdot\sqrt{\sqrt{4\cdot Q_5^2\cdot\left(2\cdot Q_5^2-1\right)+1}-2\cdot Q_5^2+1}} \qquad (5.1.8.8)$$

Numerical result:

$$f_5 = 30.498\cdot MHz \qquad \frac{\sqrt{2}\cdot\omega_5\cdot Q_5}{2\cdot\pi\cdot\sqrt{\sqrt{4\cdot Q_5^2\cdot\left(2\cdot Q_5^2-1\right)+1}-2\cdot Q_5^2+1}} = 47.1\cdot MHz$$

$$Bw = 47.1\cdot MHz$$

If $\zeta=\omega_5$ results: $$Bw = \frac{\sqrt{\omega_5^2 - 2\cdot\omega_5^2 + \sqrt{2}\cdot\sqrt{2\cdot\omega_5^2\cdot\left(\omega_5^2-\omega_5^2\right)+\omega_5^4}}}{2\cdot\pi} = \frac{\omega_5}{2\cdot\pi}\cdot\sqrt{\sqrt{2}-1}$$

$\sqrt{\sqrt{2}-1} = 0.644$

For ω=Bw, the voltage gain in dB takes the value $WlpdB(2\cdot\pi\cdot Bw) = 16.99\cdot dB \qquad W_{lpp} - dB3_{gd} = 16.99$

Low frequency voltage gain: $W_{lpp} = 20\cdot dB \qquad \zeta = 0.018\cdot\frac{Grads}{sec} \qquad \omega_5 = 0.192\cdot\frac{Grads}{sec} \qquad Q_5 = 5.4$

The angular frequency for which WlpdB(ω)=0dB is:

$$\omega_{5dB0} := \sqrt{\sqrt{4\cdot\zeta^2\cdot\left(\zeta^2-\omega_5^2\right)+\omega_5^4\cdot\left(\left|A_5\right|\right)^2}-2\cdot\zeta^2+\omega_5^2}$$

$$\omega_{5dB0} = 0.635\cdot\frac{Grads}{sec} \qquad WlpdB\left(\omega_{5dB0}\right) = 0$$

If $\zeta = \omega_5$, the corresponding angular frequency for which WlpdB(ω)=0dB is $\omega_{5dB0} := \omega_5\cdot\sqrt{\left|A_5\right|-1}$

$$\zeta = 0.018\cdot\frac{Grads}{sec} \qquad \omega_5 = 0.192\cdot\frac{Grads}{sec} \qquad \omega_{5dB0} = 0.635\cdot\frac{Grads}{sec} \qquad WlpdB\left(\omega_{5dB0}\right) = 1.929\times 10^{-15}$$

System modes.

Knowing the transfer function: $W_{lp}(s) := \dfrac{A_5 \cdot \omega_5^{\,2}}{s^2 + 2 \cdot \zeta \cdot s + \omega_5^{\,2}}$ one can know which the system modes.

▶ Modes calculation

$$\textbf{modes} = \text{"Pseudoperiodics"} \qquad Q_5 = 5.4 \qquad (5.1.8.9)$$

Bode Plots

$$W1_{lp}(s) := \begin{cases} \dfrac{A_5 \cdot \omega_5^{\,2}}{s^2 + 2 \cdot \zeta \cdot s + \omega_5^{\,2}} & \text{if } \zeta \neq \omega_5 \\ \dfrac{A_5 \cdot \omega_5^{\,2}}{(s + \omega_5)^2} & \text{otherwise} \end{cases} \qquad (5.1.8.10)$$

$$\text{poles}^T = (-0.018 + 0.191j \quad -0.018 - 0.191j) \cdot \frac{\text{Grads}}{\text{sec}}$$

$$\left|\text{poles}_0\right| = 0.192 \cdot \frac{\text{Grads}}{\text{sec}}$$

$$\sqrt{\left|\text{poles}_0\right| \cdot \left|\text{poles}_1\right|} = 0.192 \cdot \frac{\text{Grads}}{\text{sec}} \qquad \omega_5 = 0.192 \cdot \frac{\text{Grads}}{\text{sec}}$$

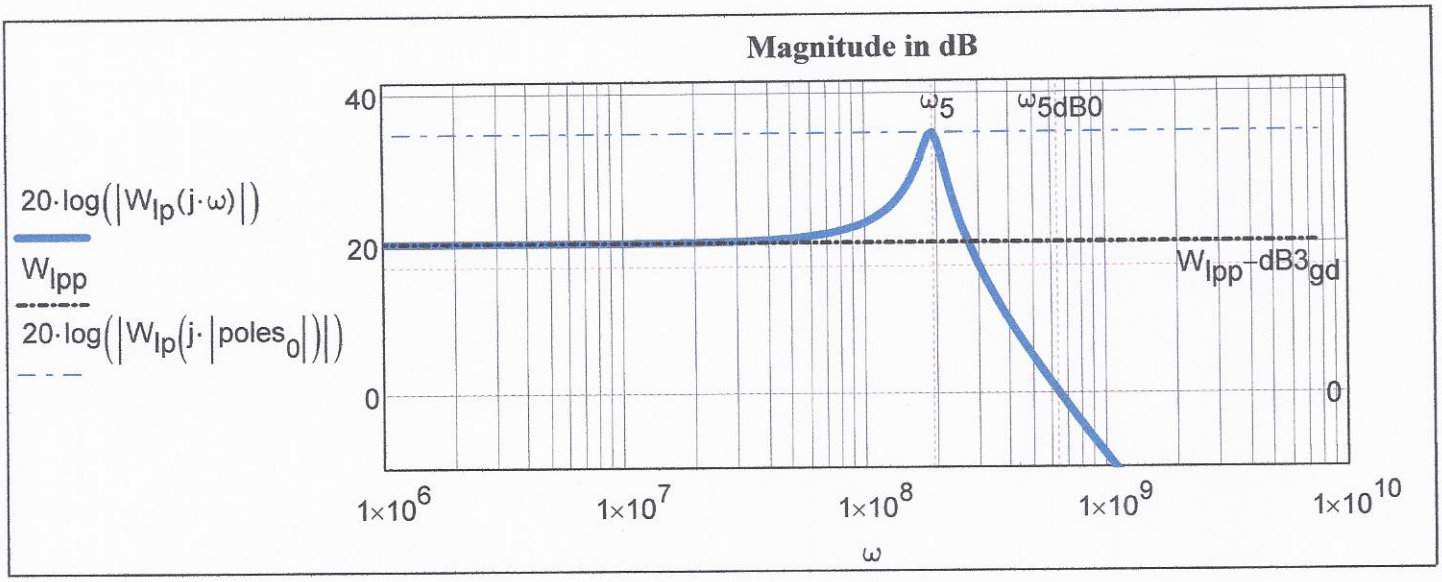
$$\omega := \frac{\omega_5}{1000}, \frac{\omega_5}{1000} + \frac{40 \cdot \omega_5 - \frac{\omega_5}{1000}}{1000} \,..\, 40 \cdot \omega_5$$

$\text{WlpdB}(\omega_5) = 34.648$ $\qquad W_{lpp} = 20 \cdot \text{dB}$

Pick amplitude of the frequency response if Q_5>0.5 $\text{rpeak} := 20 \cdot \log\left(\left|\dfrac{A_5 \cdot \omega_5}{2 \cdot \zeta}\right|\right)$

▶ Plots Control

$Q_5 = 5.4$



$Bw = 47.1 \cdot MHz$

fig.:5.1.8.1

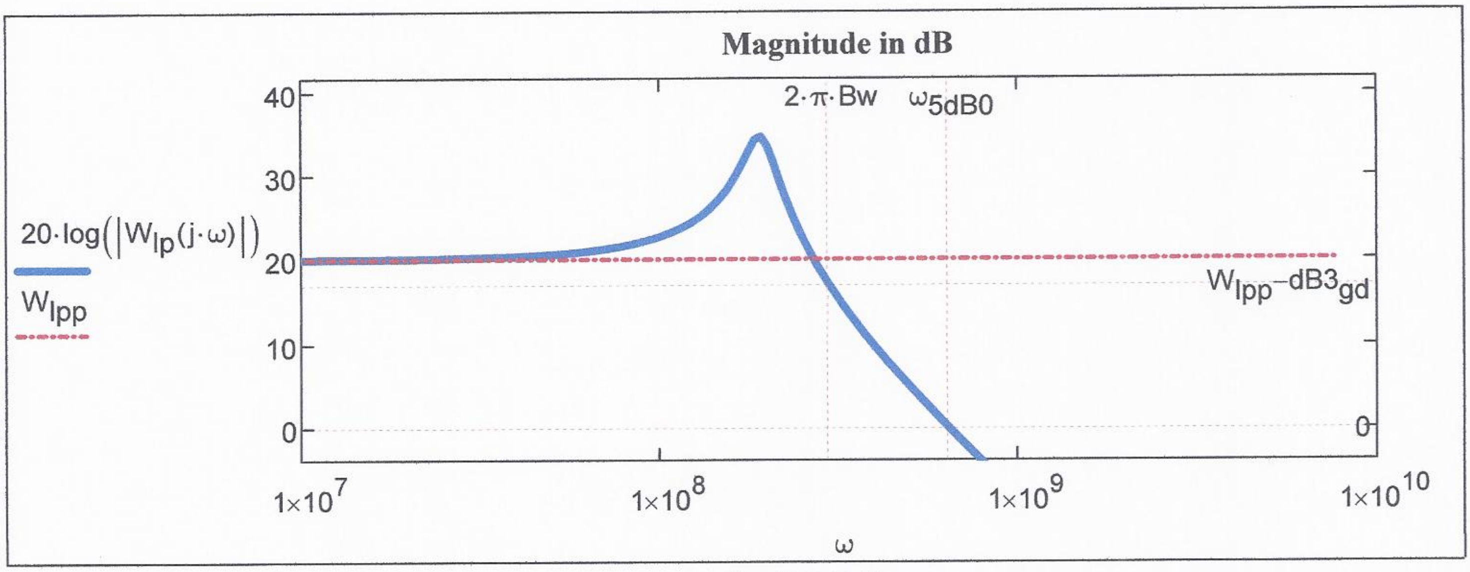


fig.:5.1.8.1'

pick amplitude $WlpdB_{pick} = 34.685$ $\qquad W_{lpp} = 20$

$\omega_{pick} = 0.19 \cdot \frac{Grads}{sec}$ $\qquad \omega_5 = 0.192 \cdot \frac{Grads}{sec}$

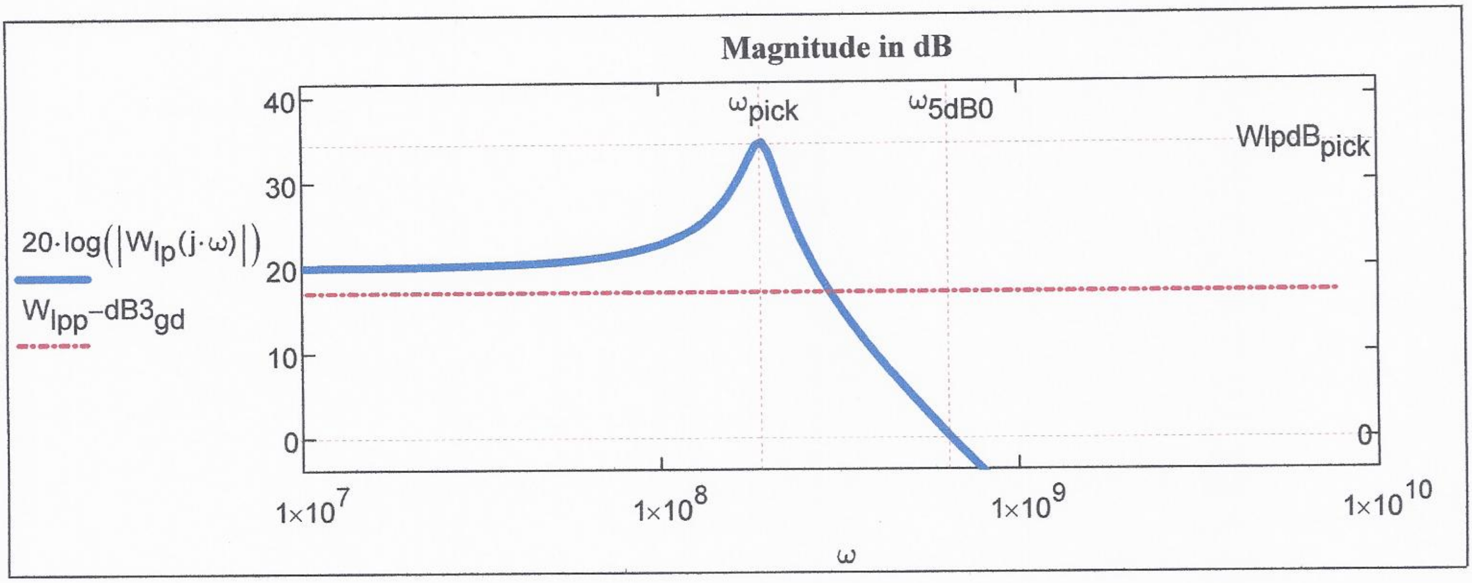


fig.:5.1.8.1"

$$\omega_{pick} = 0.19 \cdot \frac{Grads}{sec} \qquad \omega_5 = 0.192 \cdot \frac{Grads}{sec} \qquad \omega_5 - \omega_{pick} = 1.65 \cdot \frac{Mrads}{sec}$$

$$W_{lpp} - 2 \cdot dB3_{gd} = rpeak = 20 \cdot \log\left(\left|\frac{A_5 \cdot \omega_5}{2 \cdot \zeta}\right|\right)$$

$$W_{lpp} - 2 \cdot dB3_{gd} = 13.979$$

$$rpeak = 34.648$$

$$20 \cdot \log\left(\left|\frac{A_5 \cdot \omega_5}{2 \cdot \zeta}\right|\right) = 34.648$$

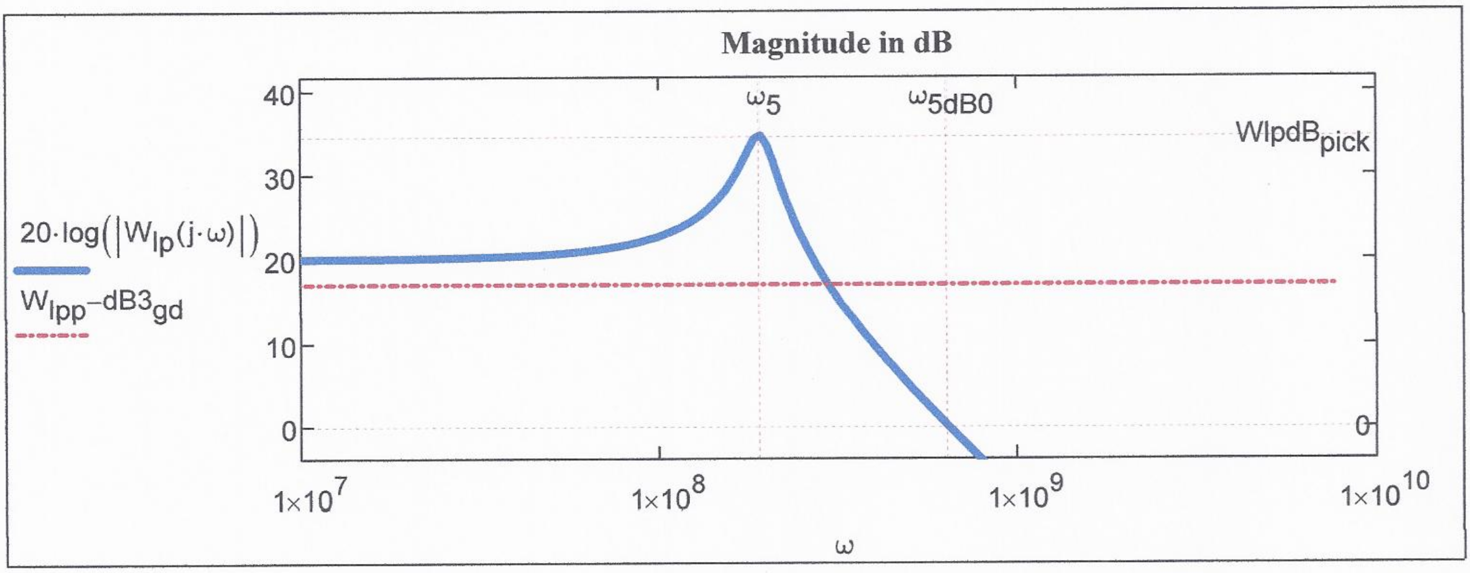


fig.:5.1.8.1'''

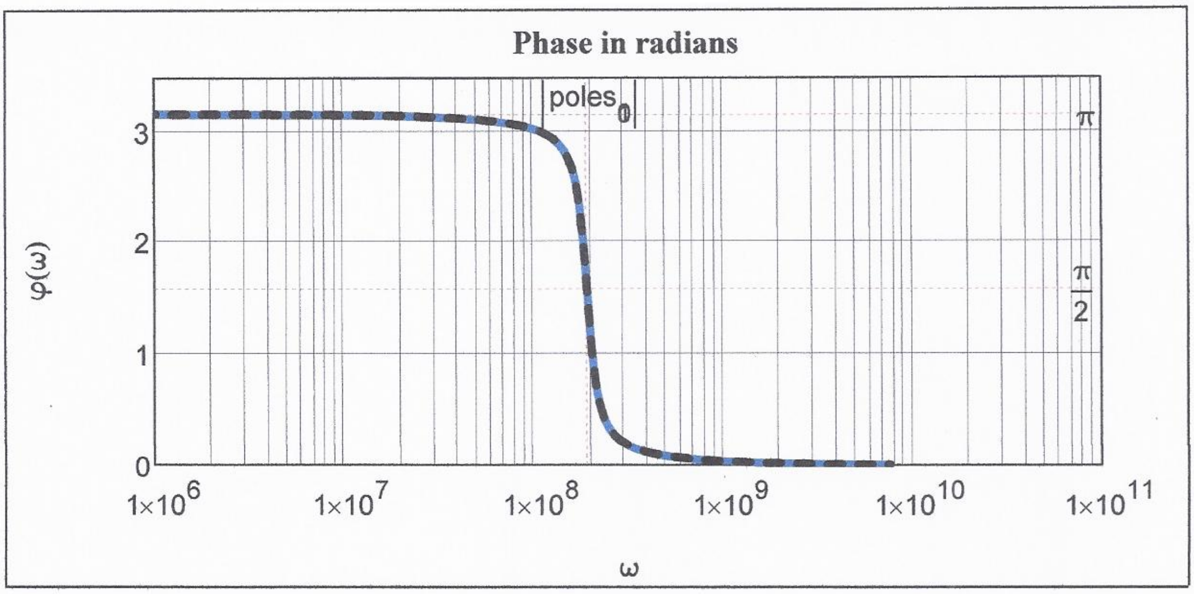


fig.:5.1.8.2

Knowing the poles of the transfer function, it is immediate to see the system stability:

▶ Stability Type

stability = " System Exponentially Stable"

5.2
ANALOG FILTER OUTPUT ANALYSIS

For a signal definition refer to the file "Signal List.xmcd"

Chosen period of the test signal, $T_{test} = 0.016 \cdot \mu s$. At the corresponding frequency, the voltage gain of the filter is $20 \cdot \log\left(\left|W_{lp}\left(j \cdot \omega_{test}\right)\right|\right) = 10.392 \cdot dB$. As seen the pulse response waveform is:

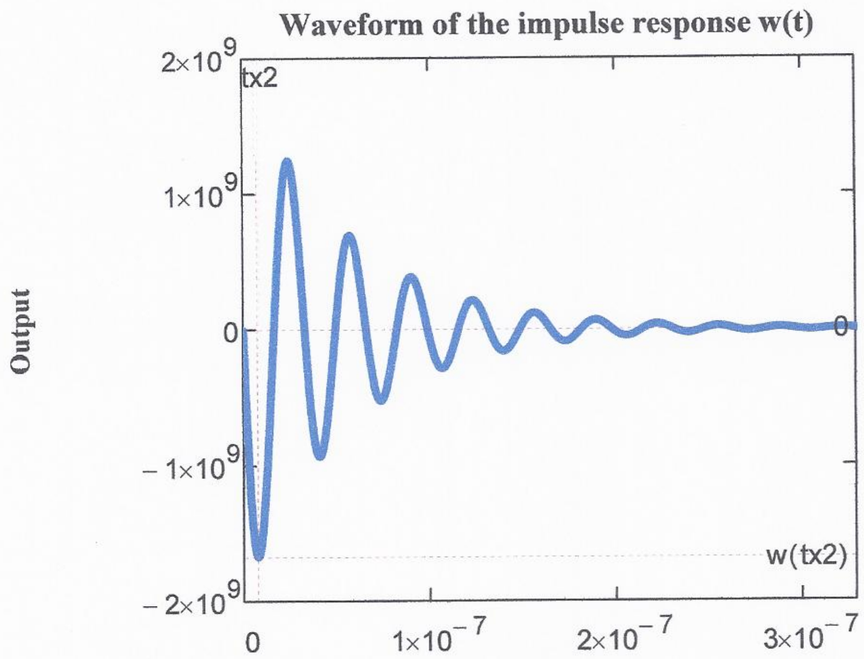


fig.:5.2.1

$$w(tx2) = -1.67 \cdot \frac{Grads}{sec}$$

5.2 ANALOG FILTER OUTPUT ANALYSIS

5.2.1 Voltage step response - Analytical solution

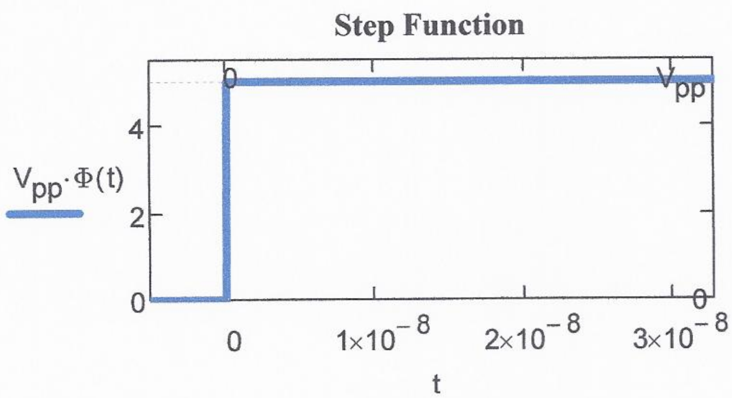


fig.:5.2.1.1

$A_5 := A_5 \quad \omega_5 := \omega_5$

$V_{pp} := V_{pp}$ $V_{pp} = 5V$

The evaluation is disabled because the result exceeds the page margins.

$$y_{sr}(t) := \frac{A_5 \cdot \omega_5^2}{s^2 + 2\cdot\zeta\cdot s + \omega_5^2} \cdot \frac{V_{pp}}{s} \left| \begin{array}{l} \text{invlaplace}, s \\ \text{simplify}, \max \\ \text{collect}, A_5 \cdot V_{pp}, e^{-\zeta \cdot t} \end{array} \right. \rightarrow$$

Step response:
define the function:

$$g_{sr}(t, A_5, \zeta, \omega_5) := -\left[\left(\cosh\left(t\cdot\sqrt{\zeta^2 - \omega_5^2}\right) + \frac{\zeta \cdot \sinh\left(t\cdot\sqrt{\zeta^2 - \omega_5^2}\right)}{\sqrt{\zeta^2 - \omega_5^2}}\right)\cdot e^{-\zeta\cdot t} - 1\right],$$

the output waveform is: $$y_{sr}(t) := A_5 \cdot V_{pp} \cdot \begin{cases} g_{sr}(t, A_5, \zeta, \omega_5)\cdot \Phi(t) & \text{if } \zeta \neq \omega_5 \\ \left[1 - e^{-t\cdot\omega_5}\cdot(t\cdot\omega_5 + 1)\right]\cdot\Phi(t) & \text{otherwise} \end{cases} \quad (5.2.1.1)$$

Calculation of the initial and final values of the output:

Initial value theorem: $\lim_{t \to 0} f(t) = \lim_{s \to \infty} (s\cdot F(s))$,

Final value theorem: $\lim_{t \to \infty} f(t) = \lim_{s \to 0} (s\cdot F(s))$

Output's Initial value:

Input signal: $V_i(s) = \frac{V_{pp}}{s}$ (5.2.1.2)

$$\lim_{s \to \infty} (s\cdot V_o(s)) = \lim_{s \to \infty} (s\cdot W(s)\cdot V_i(s)) = V_{pp} \cdot \lim_{s \to \infty} (W(s))$$

$A_5 := A_5$ $s := s$ $a := a$ $\omega_5 := \omega_5$ $\zeta := \zeta$

$$V_{pp} \cdot \lim_{s \to \infty} \left(\frac{A_5\cdot\omega_5^2}{s^2 + 2\cdot\zeta\cdot s + \omega_5^2}\right) = 0\cdot\text{volt}$$

Output's final value:

Input signal: $V_i(s) = \frac{V_{pp}}{s}$

$$\lim_{s \to 0} (s\cdot V_o(s)) = \lim_{s \to 0} (s\cdot W(s)\cdot V_i(s)) = V_{pp} \cdot \lim_{s \to 0} (W(s))$$

$A_5 := A_5$ $s := s$ $a := a$ $\omega_5 := \omega_5$ $\zeta := \zeta$

$$V_{pp} \cdot \lim_{s \to 0} \left(\frac{A_5\cdot\omega_5^2}{s^2 + 2\cdot\zeta\cdot s + \omega_5^2}\right) \rightarrow \left| \begin{array}{ll} A_5 \cdot V_{pp} & \text{if } \omega_5 \neq 0 \\ \text{undefined} & \text{otherwise} \end{array} \right.$$

Graph's controls

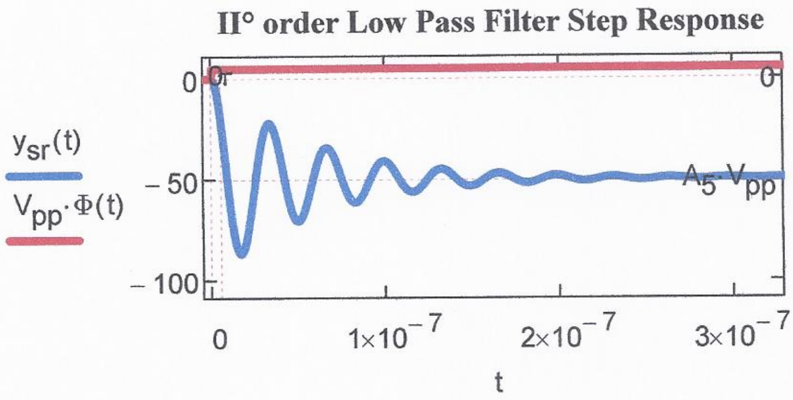


fig.:5.2.1.2

$$\zeta = 17.743\cdot\frac{Mrads}{sec}$$

$$\omega_5 = 191.627\cdot\frac{Mrads}{sec}$$

$V_{pp} = 5\times10^{3}\cdot mV$ $\quad f_5 = 30.498\cdot MHz$ $\quad Q_5 = 5.4$

Bode plots

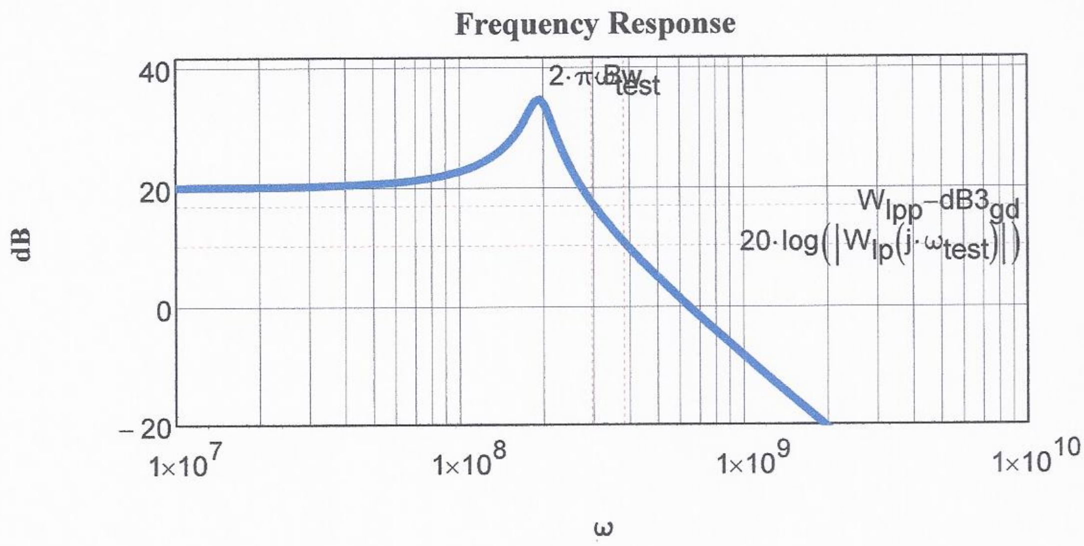


fig.:5.2.1.3

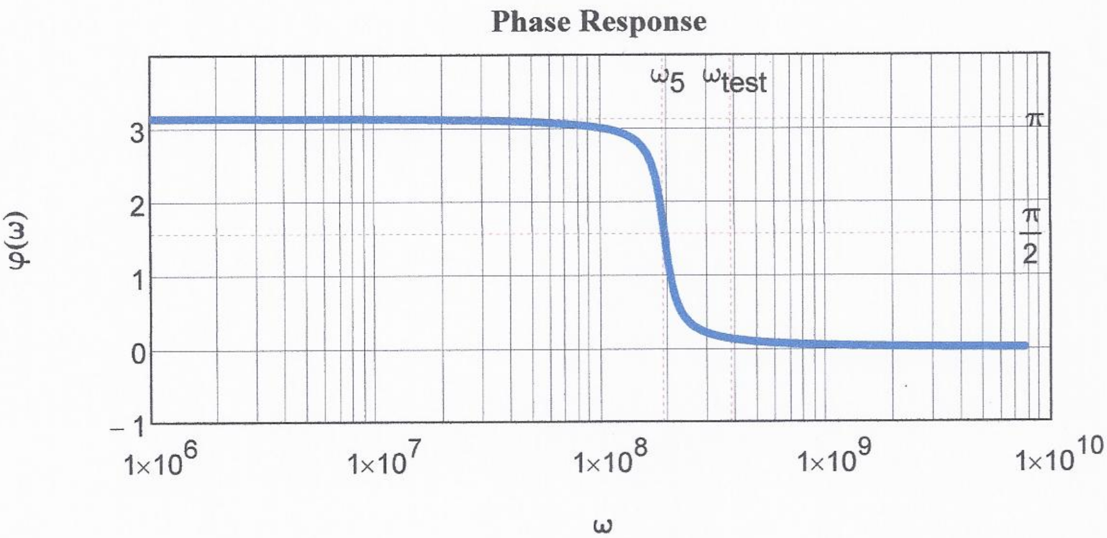


fig.:5.2.1.4

At the chosen test frequency, the voltage gain assumes the following value:

$$20 \cdot \log\left(\left|W_{lp}(j \cdot \omega_{test})\right|\right) = 10.392 \cdot dB \qquad 20 \cdot \log\left(\left|\frac{A_5 \cdot \omega_5}{2 \cdot \zeta}\right|\right) = 34.648 \cdot dB$$

while for $\omega=\omega_{5dB0}$ the voltage gain is 0dB, being: $\frac{|A_5| \cdot \omega_{5dB0}}{2 \cdot \zeta} = 178.944$

$$20 \cdot \log\left(\left|W_{lp}(j \cdot \omega_{5dB0})\right|\right) = 0 \cdot dB$$

Angular frequency for 0 dB Voltage gain: $\omega_{5dB0} = 0.635 \cdot \frac{Grads}{sec}$

Sampling of the step response

Signal frequency: $f_{test} = 60.997 \cdot MHz$,

arbitrary sampling frequency: $f_{sstp} := 10 \cdot f_{test}$, $f_{sstp} = 609.968 \cdot MHz$, (5.2.1.3)

sampling angular frequency: $\omega_{smp} := 2 \cdot \pi \cdot f_{sstp} \cdot$ $\omega_{smp} = 3.833 \cdot \frac{Grads}{sec}$,

sampling period: $T_{sstp} := \frac{1}{f_{sstp}}$, $T_{sstp} = 1.639 \cdot ns$,

generic pulse delay time: $\tau_5 := 0.4 \cdot T_{test}$

sampling time step: $nstp_k := \frac{k}{f_{sstp}}$, $N0 = 256$ (5.2.1.4)

$$\frac{N0}{f_{sstp}} \cdot f_5 = 12.8 \quad . \qquad (5.2.1.5)$$

sampling time step:

$nstp^T =$ (·µs)

	0	1	2	3	4	5	6
0	0	$1.639 \cdot 10^{-3}$	$3.279 \cdot 10^{-3}$	$4.918 \cdot 10^{-3}$	$6.558 \cdot 10^{-3}$	$8.197 \cdot 10^{-3}$	...

$N0 = 256$ $\qquad ysr_k := \frac{y_{sr}(nstp_k)}{volt}$ (5.2.1.6)

$T_5 = 0.033 \cdot \mu s$ $\qquad \tau = 5.218 \times 10^{-3} \cdot \mu s$

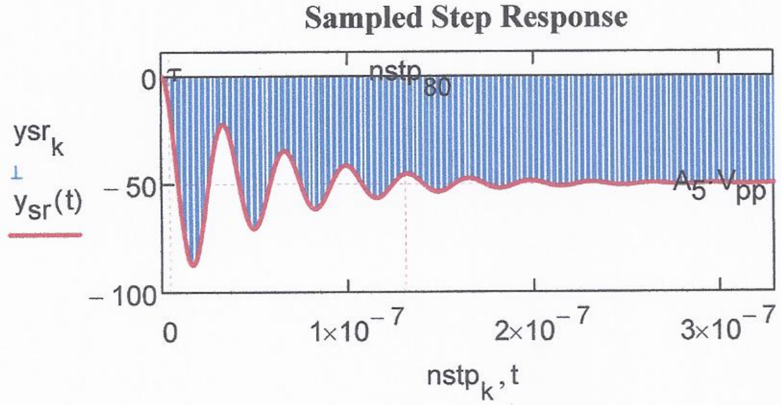


fig.:5.2.1.5

$$\zeta = 17.743 \cdot \frac{Mrads}{sec} \qquad \omega_5 = 0.192 \cdot \frac{Grads}{sec}$$

Samples:

$ysr^T =$

	0	1	2	3	4	5
0	0	-2.4	-9.192	-19.48	-32.089	...

sampling time step:

$nstp^T =$ $\cdot \mu s$

	0	1	2	3	4
0	0	$1.639 \cdot 10^{-3}$	$3.279 \cdot 10^{-3}$	$4.918 \cdot 10^{-3}$	...

Fourier Transform of the test signal

$$f_{test} = 0.061 \cdot GHz \qquad \frac{f_{sstp}}{f_{test}} = 10 \qquad \frac{N0}{f_{sstp}} \cdot \frac{1}{T_{test}} = 25.6$$

Fourier Transform: $Fy_{sr} := \text{fft}(ysr)$ (5.2.1.7)

$Fy_{sr}^T =$

	0	1	2	3	4
0	-796.601	3.421-0.761j	3.491-1.549j	3.615-2.392j	...

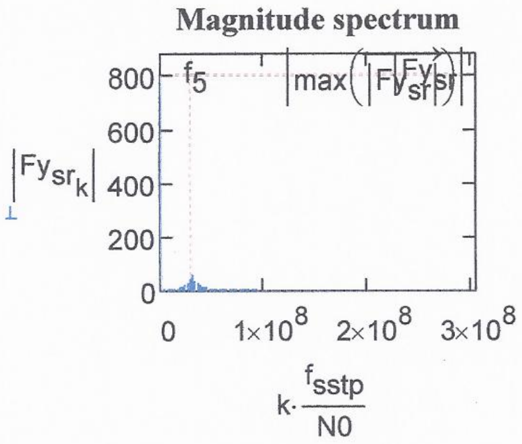


fig.:5.2.1.6

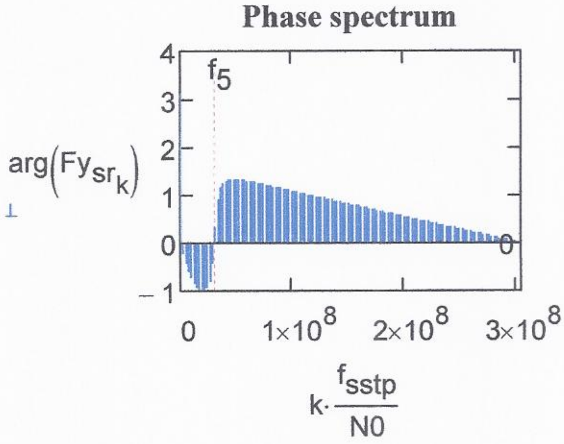


fig.:5.2.1.7

5.2 ANALOG FILTER OUTPUT ANALYSIS

5.2.2 Short Voltage Pulse response

Description of the waveform's parameters:

Definition: $V_4(t,\tau_5,\tau_{pw},V_{in}) = V_{in}\cdot \text{rect1}(t,\tau_5,\tau_{pw})$, $V_{in} = \dfrac{V_{pp}}{V}$ (5.2.2.1)

V4(t,τ_5, τ_{pw}, V_{in})=V4(time, Rising Edge , Pulse Width, Dimensionless Amplitude).

Pulse amplitude: $V_{pp} = 5\times 10^3\cdot mV$ Pulse width: $\tau_{pw} := T_5\cdot 20$ $\tau_{pw} = 655.772\cdot ns$

Pulse displacement from the origin: $\xi_{sl} := 0.8,$ $\tau_{pw} = \tau_{pw}\cdot(1-\xi_{sl}) + \xi_{sl}\cdot\tau_{pw}$, (5.2.2.2)

Time delay from the origin: $\tau_5 := -\tau_{pw}\cdot(1-\xi_{sl})$, risingedge = τ_5 , width = τ_{pw}.

Generic pulse definition defined in "Fourier Series.xmcd":

Input signal defined in "Test Signal.xmcd": $V_w(t) := V_4(t,\tau_5,\tau_{pw},V_{pp})$ (5.2.2.3)

Consider a Short Voltage Pulse delayed τ_5 seconds: $\tau_5 = -0.131\cdot \mu s$ $T_{sstp} = 1.639\cdot ns$

$$t := -2\cdot\tau_{pw}, -2\cdot\tau_{pw} + \frac{4\cdot\tau_{pw}}{5000} .. 2\cdot\tau_{pw}$$

$V_w(\tau_{pw}) = 0$

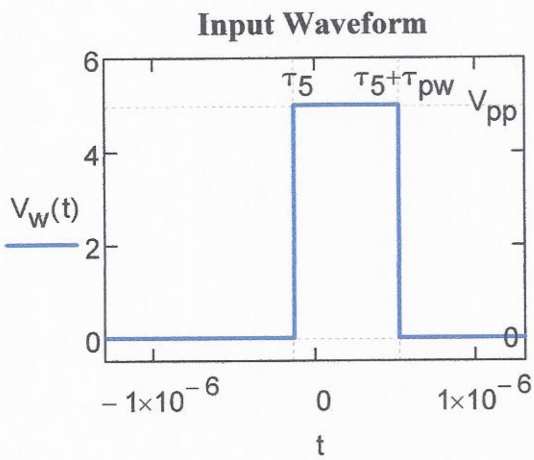


fig.:5.2.2.1

Consider now the same <u>*signal repeated periodically*</u>, with period $T_{vp} := 4\cdot(\tau_{pw}+\tau_5)$, in such a way that it is possible to calculate the bandwidth using the program BCSA defined in "Fourier Analysis.xmcd":

> Description of the program's parameters:
> BCSA(*Dimensionless signal name, relative error, polynomial degree, start time, signal period*)
> ***BCSA stands for "Bandwidth Calculation and Signal Analysis"***

$Sb_{vp0} := \text{BCSA}\left[V_w, rt_{gd}, 50, 0.0, 2\cdot(\tau_{pw}+\tau_5)\right]$ $rt_{gd} = 1\cdot\%$ (5.2.2.4)

▶ Bandwidth Calculation

Signal bandwidth: $B_{vp0} = 0.046\cdot GHz$ $f_{test} = 0.061\cdot GHz$

$Parseval_{vp0} = 24.899V^2$ Averagevp0 = 2.5V RMSvp0 = 3.536V

Sampling frequency: $f_{samp} = \frac{1}{T_{samp}} \geq 2 \cdot f_1$

Chosen sampling frequency (Nyquist rate): $f_{svp0} := 2 \cdot B_{vp0}$ $\quad f_{svp0} = 0.091 \cdot GHz$ (5.2.2.5)

$$T_{svp0} := \frac{1}{f_{svp0}} \qquad (5.2.2.6)$$

$$n_{svp0_k} := k \cdot T_{svp0} + \tau_5 \qquad \frac{N0}{f_{svp0}} \cdot \frac{1}{T_{test}} = 170.667$$

$V_{pp} = 5V$ Pulse sampling: $u44_k := V_w\left(n_{svp0_k}\right)$ (5.2.2.7)

Sampled Input

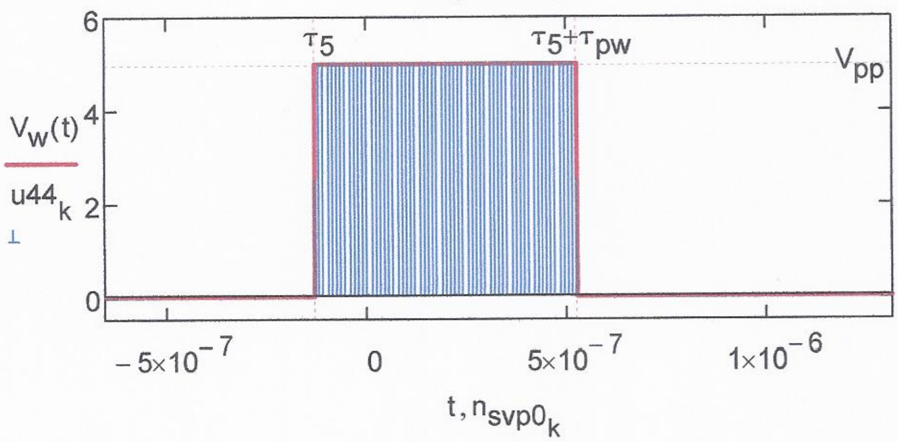


fig.:5.2.2.2

Filter response:

Input signal: $V_w(t) = V_4(t, \tau_5, \tau_{pw}, V_{pp}) = V_{pp} \cdot \text{rect1}(t, \tau_5, \tau_{pw})$ (5.2.2.8)

or: $V_4(t) = V_{pp} \cdot \left(\Phi(t - \tau_5) - \Phi(t - \tau_{pw} - \tau_5)\right)$ (5.2.2.9)

Laplace transform of the input signal: $V_4(s) = V_{pp} \cdot \left[\frac{e^{-\tau_5 \cdot s}}{s} - \frac{e^{-(\tau_5+\tau_{pw}) \cdot s}}{s}\right]$ (5.2.2.10)

$$V_4(s) = \frac{V_{pp}}{s} \cdot e^{-\tau_5 \cdot s} \cdot \left(1 - e^{-\tau_{pw} \cdot s}\right) \qquad (5.2.2.11)$$

Laplace transform of the output signal: $Y_{vp}(s) = W(s) \cdot V_4(s)$ where W(s) is the t. f.:

$$Y_{vp}(s) = \begin{cases} \left[\frac{V_{pp}}{s} \cdot e^{-\tau_5 \cdot s} \cdot \left(1 - e^{-\tau_{pw} \cdot s}\right) \cdot \frac{A_5 \cdot \omega_5^2}{s^2 + 2 \cdot \zeta \cdot s + \omega_5^2}\right] & \text{if } \zeta \neq \omega_5 \\ \left[\frac{V_{pp}}{s} \cdot e^{-\tau_5 \cdot s} \cdot \left(1 - e^{-\tau_{pw} \cdot s}\right) \cdot \frac{A_5 \cdot \omega_5^2}{(s + \omega_5)^2}\right] & \text{otherwise} \end{cases} \qquad (5.2.2.12)$$

1) First case: $\zeta \neq \omega_5$

Rewrite the Laplace transform of the response in this form:

$$Y_{vp}(s) = V_{pp} \cdot \frac{e^{-\tau_5 \cdot s}}{s} \cdot \frac{A_5 \cdot \omega_5^2}{s^2 + 2 \cdot \zeta \cdot s + \omega_5^2} - V_{pp} \cdot \frac{e^{-(\tau_5+\tau_{pw}) \cdot s}}{s} \cdot \frac{A_5 \cdot \omega_5^2}{s^2 + 2 \cdot \zeta \cdot s + \omega_5^2}$$

namely:

$$Y_{vp}(s) = V_{pp} \cdot A_5 \cdot \omega_5^2 \cdot \left[\frac{e^{-\tau_5 \cdot s}}{s \cdot \left(s^2 + 2 \cdot \zeta \cdot s + \omega_5^2\right)} - \frac{e^{-(\tau_5+\tau_{pw}) \cdot s}}{s \cdot \left(s^2 + 2 \cdot \zeta \cdot s + \omega_5^2\right)} \right] \qquad (5.2.2.13)$$

$$\text{and call } F(s) = \frac{1}{s \cdot \left(s^2 + 2 \cdot \zeta \cdot s + \omega_5^2\right)} \qquad (5.2.2.14)$$

results that: $$\frac{e^{-\tau_5 \cdot s}}{s \cdot \left(s^2 + 2 \cdot \zeta \cdot s + \omega_5^2\right)} = e^{-\tau_5 \cdot s} \cdot F(s) = \mathcal{L}\left(f\left(t - \tau_5\right)\right)$$

and: $$\frac{e^{-(\tau_5+\tau_{pw}) \cdot s}}{s \cdot \left(s^2 + 2 \cdot \zeta \cdot s + \omega_5^2\right)} = e^{-(\tau_5+\tau_{pw}) \cdot s} \cdot F(s) = \mathcal{L}\left[f\left[t - \left(\tau_5 + \tau_{pw}\right)\right]\right]$$

so that one can write as well:

$$Y_{vp}(s) = V_{pp} \cdot A_5 \cdot \omega_5^2 \cdot \left[e^{-\tau_5 \cdot s} \cdot F(s) - e^{-(\tau_5+\tau_{pw}) \cdot s} \cdot F(s)\right] \qquad (5.2.2.15)$$

Now calculate the inverse Laplace transform of F(s):

$$\text{call it: } f1(t) = \mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}\left[\frac{1}{s \cdot \left(s^2 + 2 \cdot \zeta \cdot s + \omega_5^2\right)}\right]$$

$$\frac{1}{s \cdot \left(s^2 + 2 \cdot \zeta \cdot s + \omega_5^2\right)} \ \begin{vmatrix} \text{invlaplace}, s \\ \text{simplify}, \text{max} \\ \text{collect}, e^{-\zeta \cdot t} \end{vmatrix} \rightarrow$$

results: $$f1(t) := \left[\left(-\cosh\left(t \cdot \sqrt{\zeta^2 - \omega_5^2}\right) - \frac{\zeta \cdot \sinh\left(t \cdot \sqrt{\zeta^2 - \omega_5^2}\right)}{\sqrt{\zeta^2 - \omega_5^2}}\right) \cdot e^{-\zeta \cdot t} + 1\right] \cdot \frac{1}{\omega_5^2} \qquad (5.2.2.16)$$

The output is: $$Y_{vp}(s) = V_{pp} \cdot A_5 \cdot \omega_5^2 \cdot \left[e^{-\tau_5 \cdot s} \cdot F(s) - e^{-(\tau_5+\tau_{pw}) \cdot s} \cdot F(s)\right]$$

whose inverse Laplace transform is:

$$\boxed{y1_{vp}(t) := V_{pp} \cdot A_5 \cdot \omega_5^2 \cdot \left[f1\left(t - \tau_5\right) \cdot \Phi\left(t - \tau_5\right) - f1\left[t - \left(\tau_5 + \tau_{pw}\right)\right] \cdot \Phi\left[t - \left(\tau_5 + \tau_{pw}\right)\right]\right]} \qquad (5.2.2.17)$$

2) case: $\zeta = \omega_5$

Rewrite the Laplace transform of the response in this form:

$$Y_{vp}(s) = V_{pp}\cdot\frac{e^{-\tau_5\cdot s}}{s}\cdot\frac{A_5\cdot\omega_5^{\ 2}}{(s+\omega_5)^2} - V_{pp}\cdot\frac{e^{-(\tau_5+\tau_{pw})\cdot s}}{s}\cdot\frac{A_5\cdot\omega_5^{\ 2}}{(s+\omega_5)^2} \qquad (5.2.2.18)$$

namely:

$$Y_{vp}(s) = V_{pp}\cdot A_5\cdot\omega_5^{\ 2}\cdot\left[\frac{e^{-\tau_5\cdot s}}{s\cdot(s+\omega_5)^2} - \frac{e^{-(\tau_5+\tau_{pw})\cdot s}}{s\cdot(s+\omega_5)^2}\right] \qquad (5.2.2.19)$$

$$\text{call: } G(s) = \frac{1}{s\cdot(s+\omega_5)^2} \qquad (5.2.2.20)$$

so that the first term of the second member can be also written:

$$\frac{e^{-\tau_5\cdot s}}{s\cdot(s+\omega_5)^2} = e^{-\tau_5\cdot s}\cdot G(s) = \mathcal{L}\left(f(t-\tau_5)\right)$$

While the second term is:

$$\frac{e^{-(\tau_5+\tau_{pw})\cdot s}}{s\cdot(s+\omega_5)^2} = e^{-(\tau_5+\tau_{pw})\cdot s}\cdot G(s) = \mathcal{L}\left[f\left[t-(\tau_5+\tau_{pw})\right]\right]$$

The Laplace transform of the output can be written:

$$Y_{vp}(s) = V_{pp}\cdot A_5\cdot\omega_5^{\ 2}\cdot\left[e^{-\tau_5\cdot s}\cdot G(s) - e^{-(\tau_5+\tau_{pw})\cdot s}\cdot G(s)\right] \qquad (5.2.2.21)$$

To calculate the response, it is sufficient to know the inverse Laplace transform of G(s):

place :

$$g_1(t) = \mathcal{L}^{-1}(G(s)) = \mathcal{L}^{-1}\left[\frac{1}{s\cdot(s+\omega_5)^2}\right]$$

Calculation of the inverse Laplace transform of G(s):

$s := s \qquad \zeta := \zeta$

$$\frac{1}{s\cdot(s+\zeta)^2} \left|\begin{array}{l} \text{invlaplace}, s \\ \text{simplify}, \text{max} \\ \text{factor} \\ \text{collect}, e^{-\zeta\cdot t} \\ \text{collect}, \frac{1}{\zeta^2} \end{array}\right. \rightarrow$$

so, the result is:

$$g_1(t) := \frac{1-e^{-\zeta\cdot t}\cdot(\zeta\cdot t+1)}{\zeta^2} \qquad (5.2.2.22)$$

Finally the inverse Laplace transform of the filter's output:

$$Y2_{vp}(s) = V_{pp}\cdot A_5\cdot\omega_5^{\ 2}\cdot\left[e^{-\tau_5\cdot s}\cdot G(s) - e^{-(\tau_5+\tau_{pw})\cdot s}\cdot G(s)\right]$$

is: $y2_{vp}(t) := V_{pp} \cdot A_5 \cdot \omega_5^{\,2} \cdot \left[g_1(t - \tau_5) \cdot \Phi(t - \tau_5) - g_1[t - (\tau_5 + \tau_{pw})] \cdot \Phi[t - (\tau_5 + \tau_{pw})] \right]$

As one can see, the two waveforms, according to which $\zeta = \omega_{5dB}$ or not, are slightly different.

$$\zeta = 0.018 \cdot \frac{\text{Grads}}{\text{sec}} \qquad \omega_5 = 0.192 \cdot \frac{\text{Grads}}{\text{sec}}$$

Hence, the time response to the Short Voltage Pulse is:

$$y_{pulse}(t) := \begin{cases} y1_{vp}(t) & \text{if } \zeta \neq \omega_5 \\ y2_{vp}(t) & \text{otherwise} \end{cases} \qquad (5.2.2.23)$$

Taking into account the op. amp. saturation voltage, the output is truncated:

$$y_{wsat}(t) := \text{if}\left(-V_{sat} \le y_{pulse}(t) \le V_{sat}, y_{pulse}(t), \text{if}\left(y_{pulse}(t) \le 0.0 \cdot \text{volt}, -V_{sat}, V_{sat}\right)\right)$$

▶ Graph controls

$y_{wsat}(\tau_{pw} + \tau_5) = -15\text{V}$ $\qquad t := -4 \cdot \tau_{pw}, -4 \cdot \tau_{pw} + \frac{6 \cdot \tau_{pw}}{10000} .. 2 \cdot \tau_{pw}$ $\qquad A_5 \cdot V_{pp} = -50\text{V}$

$$\zeta = 0.018 \cdot \frac{\text{Grads}}{\text{sec}} \qquad \omega_5 = 0.192 \cdot \frac{\text{Grads}}{\text{sec}}$$

Output Waveforms With and Without OPAMP Saturation

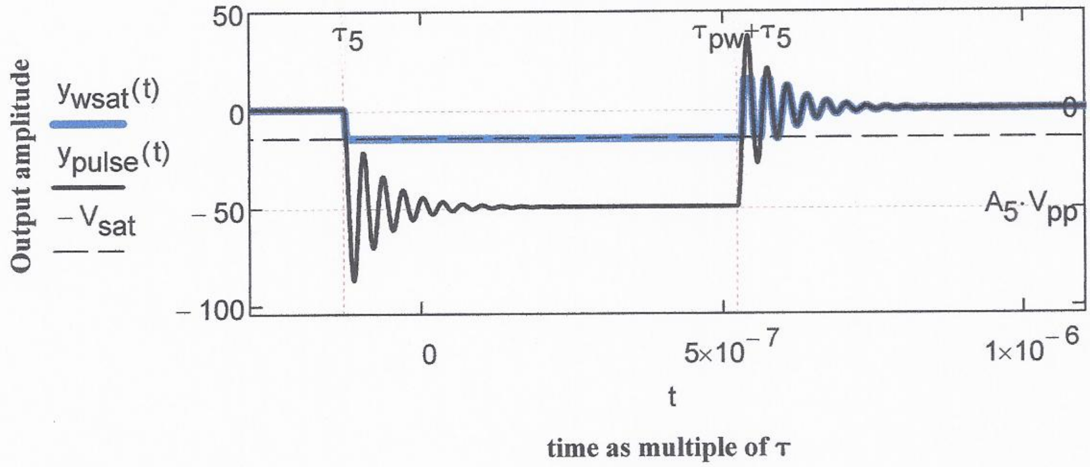


fig.:5.2.2.3

Dimensionless Output $\qquad vp(t) := \frac{y_{wsat}(t)}{V}$

Analog filter Output sampling.
Consider now the same signal repeated periodically, with period $T_{vp} := 2 \cdot (\tau_{pw} + \tau_5)$, in such a way that it is possible to calculate the bandwidth using BCSA:

Description of the program's parameters:
BCSA(*Dimensionless signal name, relative error, polynomial degree, start time, signal period*)
BCSA stands for "Bandwidth Calculation and Signal Analysis"

$rt_{gd} = 1 \cdot \%$ $\qquad Sb_{vp} := \text{BCSA}(vp, rt_{gd}, 50, 0.0 \cdot \text{sec}, T_{vp})$ $\qquad (5.2.2.24)$

▶ Bandwidth Calculation

Signal bandwidth:	$B_{vp} = 0.046 \cdot \text{GHz}$

$f_{test} = 0.061 \cdot \text{GHz}$

$\boxed{Parseval_{vp} = 251.043V^2}$ $\boxed{Average_{vp} = -7.51V}$ $\boxed{RMS_{vp} = 11.24V}$

Sampling frequency: $f_{samp} = \frac{1}{T_{samp}} \geq 2 \cdot f_1$

Chosen sampling frequency $f_{svp} := 2 \cdot B_{vp}$ $\qquad f_{svp} = 0.091 \cdot GHz$

$$\frac{N0}{f_{svp}} \cdot \frac{1}{T_{test}} = 170.667 \qquad T_{svp} := \frac{1}{f_{svp}} \qquad n_{svp_k} := \frac{k}{f_{svp}} + \tau_5 \tag{5.2.2.25}$$

Output sampling considering Op Amp saturation:

$$Vp_k := vp\left(n_{svp_k}\right) \tag{5.2.2.26}$$

Output sampling without considering Op Amp saturation:

$$ywin_k := \frac{y_{pulse}\left(n_{svp_k}\right)}{volt} \tag{5.2.2.27}$$

$N0 = 256 \qquad Q_5 = 5.4 \qquad A_5 = -10$

Dimensionless output sampling considering Op Amp saturation:

$$A_5 \cdot V_{pp} = -50V \qquad ywins_k := \frac{y_{wsat}\left(n_{svp_k}\right)}{volt} \tag{5.2.2.28}$$

Sampling Of The Output Waveform With And Without OPAMP Saturation

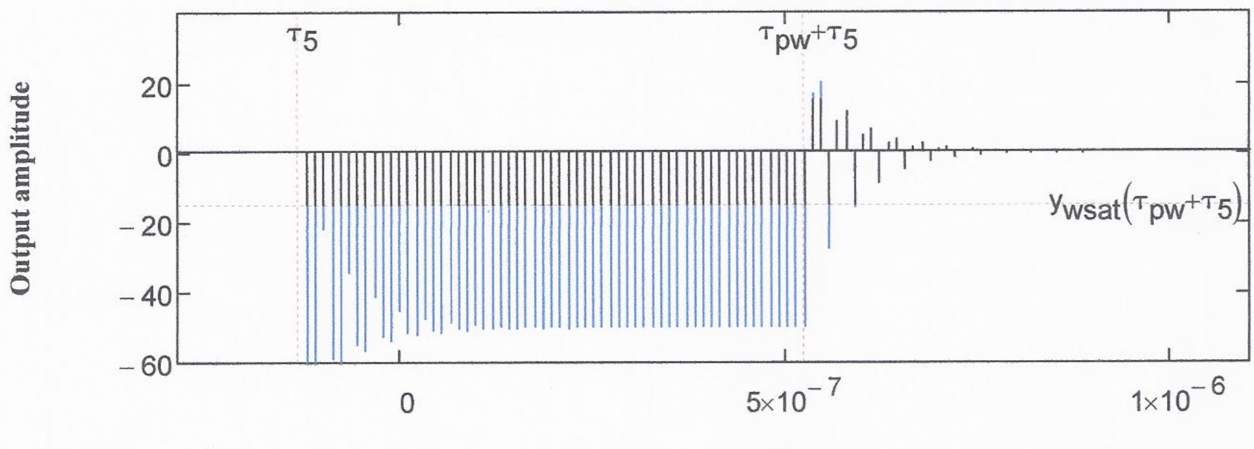


fig.:5.2.2.4

5.2 ANALOG FILTER OUTPUT ANALYSIS

5.2.3 Sawtooth response.

Input signal defined in "Test Signal.xmcd":

$$V_{sw}(t) := \frac{v1_{sw}\left(t, T_{test}, V_{pp}\right)}{volt} \tag{5.2.3.1}$$

$V_{pp} = 5V$

Bipolar Sawtooth Waveform with positive slope

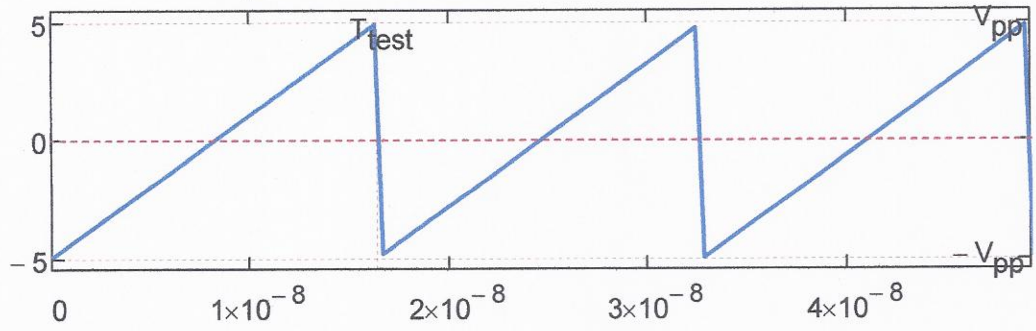


fig.:5.2.3.1

For a correct sampling one must know the signal bandwidth.
Numerical search of the signal bandwidth. All harmonics with amplitude less than $rt_{gd} = 1\cdot\%$ of the fundamental one, are neglected . To do that it is used the function BCSA(...) defined in "Fourier Series.xmcd".

Description of the program's parameters:
BCSA(*Dimensionless signal name, relative error, polynomial degree, start time, signal period*)
BCSA stands for "Bandwidth Calculation and Signal Analysis"

$N = 50$ $\qquad Sb_{sw} := BCSA\left(V_{sw}, rt_{gd}, N, 0.0\cdot sec, T_{test}\right)$ (5.2.3.2)

Bandwidth Calculation

Signal frequency: $f_{sw} = 60.997\cdot MHz$ $\qquad \omega_{sw} := 2\cdot\pi\cdot f_{sw}$ $\qquad$ Signal bandwidth: $B_{sw} = 2.928\cdot GHz$

$Parseval_{sw} = 16.466$ $\qquad Average1\cdot volt = 0V$ $\qquad RMS1\cdot volt = 2.887V$

$j70 := 0 .. rows\left(Sb_{sw}^{\langle 1\rangle}\right) - 1$ $\qquad T_{test} = 16.394\cdot ns$

$$Xsw_{fs} := max\left[\sqrt{\left(Sb_{sw}^{\langle 1\rangle}\right)^2 + \left(Sb_{sw}^{\langle 2\rangle}\right)^2}\right]$$

$$mxsw_{fs} := \frac{\sqrt{\left[\left(Sb_{sw}^{\langle 1\rangle}\right)_{j_{sw}}\right]^2 + \left[\left(Sb_{sw}^{\langle 2\rangle}\right)_{j_{sw}}\right]^2}}{Xsw_{fs}}$$

$$\omega_{sw} = 383.254\cdot\frac{Mrads}{sec}$$

Sawtooth Frequency Spectrum

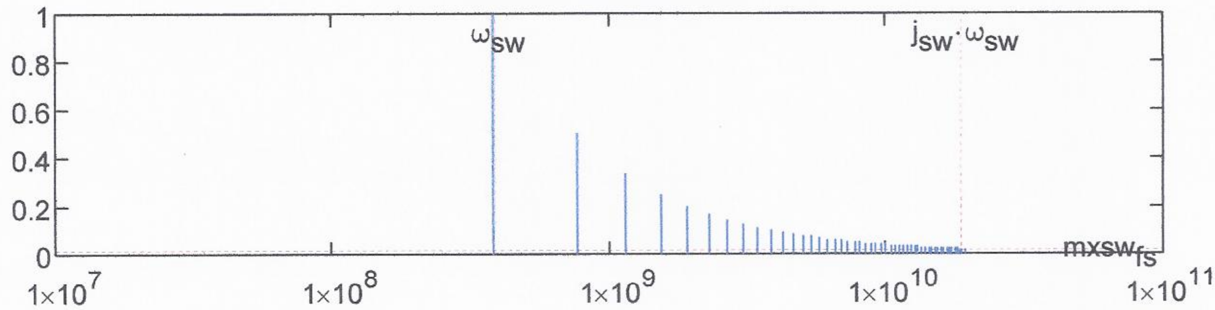


fig.:5.2.3.2

(Min) sampling frequency (Nyquist rate) $f_{ssw} := 2 \cdot B_{sw}$ $\quad f_{ssw} = 5.856 \cdot \text{GHz} \quad T_{ssw} := \frac{1}{f_{ssw}}$

sampling time step: $\quad n_{sw_k} := \frac{k}{f_{ssw}}$ (5.2.3.3)

$\text{rows}(n_{sw}) = 256 \qquad \frac{1}{T_{test}} = 60.997 \cdot \text{MHz} \qquad \frac{N0}{f_{ssw}} \cdot \frac{1}{T_{test}} = 2.667$

$RMS1 = 2.887$ Sampled signal: $u10_k := v1_{sw}\left(n_{sw_k}, T_{test}, V_{pp}\right)$ (5.2.3.4)

Sampling of the Input Waveform

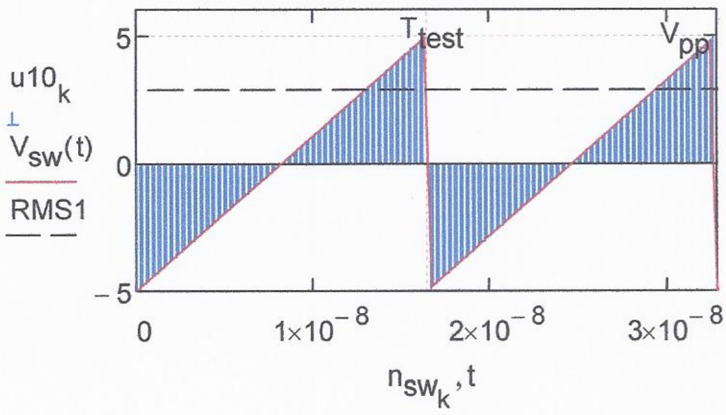


fig.:5.2.3.3

Approximate signal reconstruction according to the Shannon sampling theorem:

$$\omega_{s2} := 2 \cdot \pi \cdot B_{sw} \qquad sh3(t) := \left[\sum_{n=0}^{N0-1} \left(u10_n \cdot \text{sinc}\left(\omega_{s2} \cdot t - n \cdot \pi\right)\right)\right] \qquad (5.2.3.5)$$

$N0 - 1 = 255$

Reconstructed Input Waveform Compared with the Original One

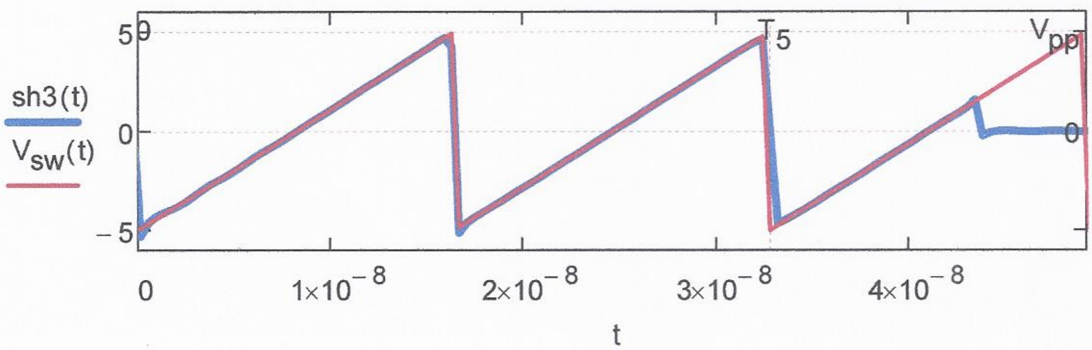


fig.:5.2.3.4

$rt_{gd} = 1 \cdot \%$

Search of the filter's transient response

Given the signal:

$$v1_{sw}\left(t, T_{test}, V_{pp}\right) := 2 \cdot \frac{V_{pp}}{T_{test}} \cdot \sum_{k=0}^{\infty} \left[\left(t - k \cdot T_{test}\right) \cdot \text{rect1}\left(t - k \cdot T_{test}, 0.0 \cdot T_{test}, T_{test}\right)\right] - V_{pp}$$

(5.2.3.6)

or:

$$v1_{sw0}(t, T_{test}, V_{pp}) := 2 \cdot \frac{V_{pp}}{T_{test}} \cdot \sum_{k=0}^{20} \left[(t - k \cdot T_{test}) \cdot \Phi(t - k \cdot T_{test}) \cdots + (-1) \cdot \left[(t - k \cdot T_{test}) \cdot \Phi[t - T_{test} \cdot (k+1)] \right] \right] - V_{pp}$$

$RMS1 = 2.887$ (5.2.3.7)

RMS of the Sawtooth Waveform

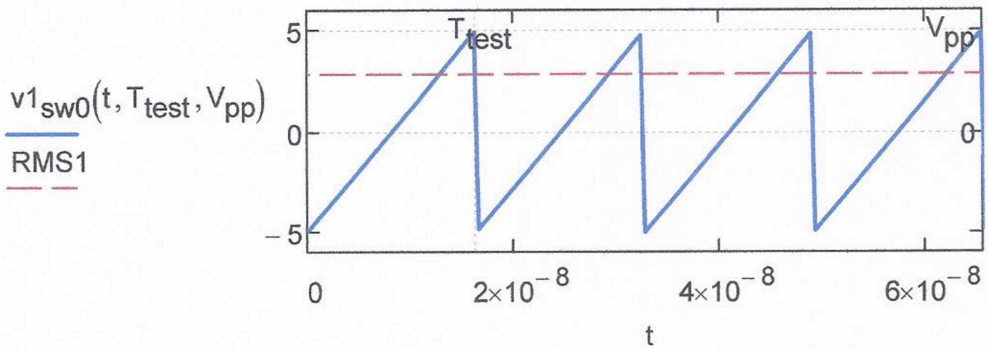


fig.:5.2.3.5

▶ Laplace Transform calculation of the Output Signal .

Hence, the compact laplace transform of the (21) is:

$$\mathcal{L}(v1_{sw}(t, T_{test}, V_{pp})) = 2 \cdot \frac{V_{pp}}{T_{test}} \cdot \frac{1}{s} \cdot \left[\frac{1}{s} - \left(T_{test} + \frac{1}{s} \right) \cdot e^{-s \cdot T_{test}} \right] \cdot \sum_{k=0}^{\infty} \left(e^{-T_{test} \cdot k \cdot s} \right) - \frac{V_{pp}}{s},$$

or

$$\boxed{\mathcal{L}(v1_{sw}(t, T_{test}, V_{pp})) = \frac{V_{pp}}{s} \cdot \left(\frac{2}{T_{test} \cdot s} - \coth\left(\frac{T_{test} \cdot s}{2} \right) \right)} \quad (5.2.3.10)$$

Search of the corresponding output waveform of the filter.

$$\text{Given the transfer function: } W_{lp}(s) := \begin{cases} \dfrac{A_5 \cdot \omega_5^2}{s^2 + 2 \cdot \zeta \cdot s + \omega_5^2} & \text{if } \zeta \neq \omega_5 \\ A_5 \cdot \dfrac{\omega_5^2}{(s + \omega_5)^2} & \text{otherwise} \end{cases} \quad (5.2.3.11)$$

the Laplace transform of the filter's output is: $V_{osw}(s) = W_{lp}(s) \cdot \mathcal{L}(v1_{sw}(t, T_{test}, V_{pp}))$

First case $\boxed{\zeta \neq \omega_5}$

$$V_{osw}(s) = \frac{V_{pp} \cdot A_5 \cdot \omega_5^2}{s^2 + 2 \cdot \zeta \cdot s + \omega_5^2} \cdot \frac{1}{s} \cdot \left(\frac{2}{T_{test} \cdot s} - \coth\left(\frac{T_{test} \cdot s}{2} \right) \right) \quad (5.2.3.12)$$

Second case $\boxed{\zeta = \omega_5}$

$$V_{osw}(s) = \frac{V_{pp} \cdot A_5 \cdot \omega_5^2}{(s + \omega_5)^2} \cdot \frac{1}{s} \cdot \left(\frac{2}{T_{test} \cdot s} - \coth\left(\frac{T_{test} \cdot s}{2} \right) \right) \quad (5.2.3.13)$$

Now apply the following theorems:

Initial value theorem: $\lim_{t \to 0} f(t) = \lim_{s \to \infty} (s \cdot F(s))$,

Final value theorem: $\lim_{t \to \infty} f(t) = \lim_{s \to 0} (s \cdot F(s))$

$T_{test} := T_{test}$ $\omega_5 := \omega_5$ $\zeta := \zeta$ $A_5 := A_5$ $V_{pp} := V_{pp}$ $s := s$

Final value of the output voltage:

$\zeta \neq \omega_5$

$$V_{ofin} := \lim_{s \to 0} \left[s \cdot \left[\frac{V_{pp} \cdot A_5 \cdot \omega_5^2}{T_{test}} \cdot \frac{\left(2 - T_{test} \cdot s \cdot \coth\left(\frac{T_{test} \cdot s}{2}\right)\right)}{s^2 \cdot \left(s^2 + 2 \cdot \zeta \cdot s + \omega_5^2\right)} \right] \right] \Bigg| \begin{array}{l} \text{assume}, (T_{test} > 0.0) \\ \text{simplify} \end{array} \to 0$$

$\zeta = \omega_5$

$$V_{ofin1} := \lim_{s \to 0} \left[s \cdot \left[\frac{V_{pp} \cdot A_5 \cdot \omega_5^2}{T_{test}} \cdot \frac{\left(2 - T_{test} \cdot s \cdot \coth\left(\frac{T_{test} \cdot s}{2}\right)\right)}{s^2 \cdot (s + \omega_5)^2} \right] \right] \Bigg| \begin{array}{l} \text{assume}, (T_{test} > 0.0) \\ \text{simplify} \end{array} \to 0$$

Initial value of the output voltage:

$T_{test} := T_{test}$ $\omega_5 := \omega_5$ $A_5 := A_5$ $V_{pp} := V_{pp}$

$\zeta \neq \omega_5$

$$\lim_{s \to \infty} \left[s \cdot \left[\frac{V_{pp} \cdot A_5 \cdot \omega_5^2}{T_{test}} \cdot \frac{\left(2 - T_{test} \cdot s \cdot \coth\left(\frac{T_{test} \cdot s}{2}\right)\right)}{s^2 \cdot \left(s^2 + 2 \cdot \zeta \cdot s + \omega_5^2\right)} \right] \right] \Bigg| \begin{array}{l} \text{assume}, (T_{test} > 0.0) \\ \text{simplify} \end{array} \to$$

$\zeta = \omega_5$

$$\lim_{s \to \infty} \left[s \cdot \left[\frac{V_{pp} \cdot A_5 \cdot \omega_5^2}{T_{test}} \cdot \frac{\left(2 - T_{test} \cdot s \cdot \coth\left(\frac{T_{test} \cdot s}{2}\right)\right)}{s^2 \cdot (s + \omega_5)^2} \right] \right] \Bigg| \begin{array}{l} \text{assume}, (T_{test} > 0.0) \\ \text{simplify} \end{array} \to$$

it results that: $V_{ofin} = 0 \cdot V$

Transient response calculation:

As seen, the filter output is:

First case $\zeta \neq \omega_5$:

$$\boxed{v_{osw}(t) = \frac{V_{pp} \cdot A_5 \cdot \omega_5^2}{T_{test}} \cdot \mathcal{L}^{-1} \left[\frac{2 - T_{test} \cdot s \cdot \coth\left(\frac{T_{test} \cdot s}{2}\right)}{s^2 \cdot \left(s^2 + 2 \cdot \zeta \cdot s + \omega_5^2\right)} \right]} \quad (5.2.3.14)$$

Second case $\zeta = \omega_5$:

$$\boxed{v_{osw}(t) = \frac{V_{pp} \cdot A_5 \cdot \omega_5^2}{T_{test}} \cdot \mathcal{L}^{-1} \left[\frac{2 - T_{test} \cdot s \cdot \coth\left(\frac{T_{test} \cdot s}{2}\right)}{s^2 \cdot (s + \omega_5)^2} \right]} \quad (5.2.3.15)$$

$$\frac{V_{pp} \cdot A_5 \cdot \omega_5^{\,2}}{T_{test}} = -0.112 \cdot \frac{V}{ns^3}$$

Calculation of the Sawtooth time response

$N = 50$ $\quad t := 0 \cdot T_{test}, 0 \cdot T_{test} + \frac{20 \cdot T_{test} - 0 \cdot T_{test}}{1000} .. 20 \cdot T_{test}$ $\quad A_5 = -10$

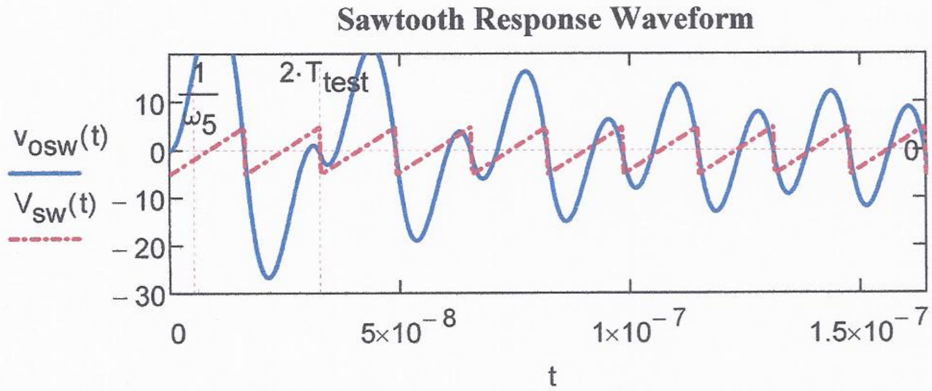


fig.:5.2.3.6

Convolution:

The second method is to calculate ***the time domain convolution product*** between the signal and the impulse response.

Sawtooth function. The number of sawtooth is limited to forty one. The function rect1 is defined in "Signal list.xmcd".

$$fs1(t) := \frac{2 \cdot V_{pp}}{T_{test}} \cdot \sum_{k=0}^{40} \left[\left(t - k \cdot T_{test}\right) \cdot rect1\left(t, k \cdot T_{test}, T_{test}\right) \right] - V_{pp} \qquad (5.2.3.16)$$

Convolution: $$v_{osw}(t) = \int_0^t V_{sw}(\tau) \cdot w(t-\tau)\, d\tau = \int_0^t V_{sw}(t-\tau) \cdot w(\tau)\, d\tau$$

Output: $$v_{oswconv}(t) := \int_0^t w(t-\sigma) \cdot fs1(\sigma)\, d\sigma \qquad (5.2.3.17)$$

$$t := -1 \cdot T_{test}, -1 \cdot T_{test} + \frac{5 \cdot T_5 + 1 \cdot T_{test}}{100} .. 5 \cdot T_5$$

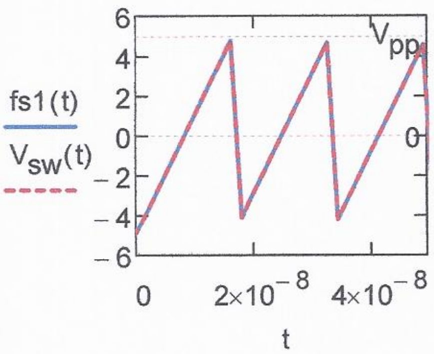


fig.:5.2.3.7

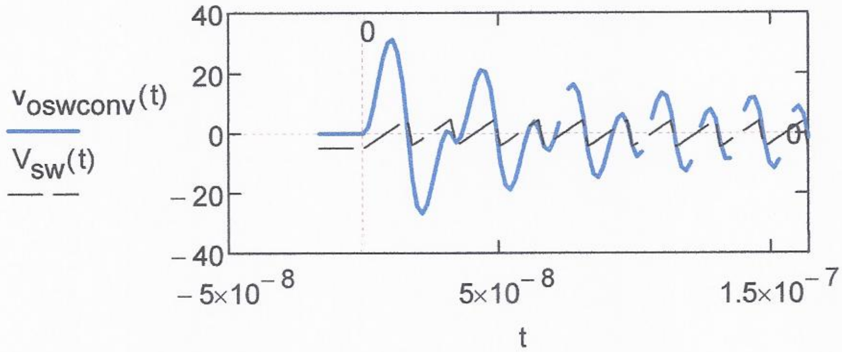


fig.:5.2.3.8

Now the output signal will be sampled. To do that correctly, one must know the signal bandwidth. The follo program (BCSA) will do that calculation.

Dimensionless output signal: $$V_{osw}(t) := \frac{v_{osw}(t)}{V} \qquad (5.2.3.18)$$

> Description of the program's parameters:
> BCSA(*Dimensionless signal name, relative error, polynomial degree, start time, signal period*)
> ***BCSA stands for "Bandwidth Calculation and Signal Analysis"***

$N = 50$ $$Sb_{osw} := BCSA\left(V_{osw}, rt_{gd}, N, 0.0 \cdot sec, T_{test}\right) \qquad (5.2.3.19)$$

▶ Bandwidth Calculation

Output signal frequency: $f_{osw} = 60.997 \cdot MHz$

Output signal pulsation: $\omega_{osw} := 2 \cdot \pi \cdot f_{osw}$ Output signal bandwidth: $B_{osw} = 2.928 \cdot GHz$

$Parseval_{osw} = 861.113$	$Average1o \cdot volt = 17.656V$	$RMS1o \cdot volt = 20.75V$

$j70 := 0 .. rows\left(Sb_{osw}{}^{\langle 1 \rangle}\right) - 1$ $T_{test} = 16.394 \cdot ns$

$$Xsw_{fso} := \max\left[\sqrt{\left(Sb_{osw}{}^{\langle 1\rangle}\right)^2 + \left(Sb_{osw}{}^{\langle 2\rangle}\right)^2}\right]$$

$$mxsw_{fso} := \frac{\sqrt{\left[\left(Sb_{osw}{}^{\langle 1\rangle}\right)_{j_{sw}}\right]^2 + \left[\left(Sb_{osw}{}^{\langle 2\rangle}\right)_{j_{sw}}\right]^2}}{Xsw_{fso}}$$

$$\omega_{osw} = 383.254 \cdot \frac{Mrads}{sec}$$

Output Frequency Spectrum (in blue) Compared with the Input (red)

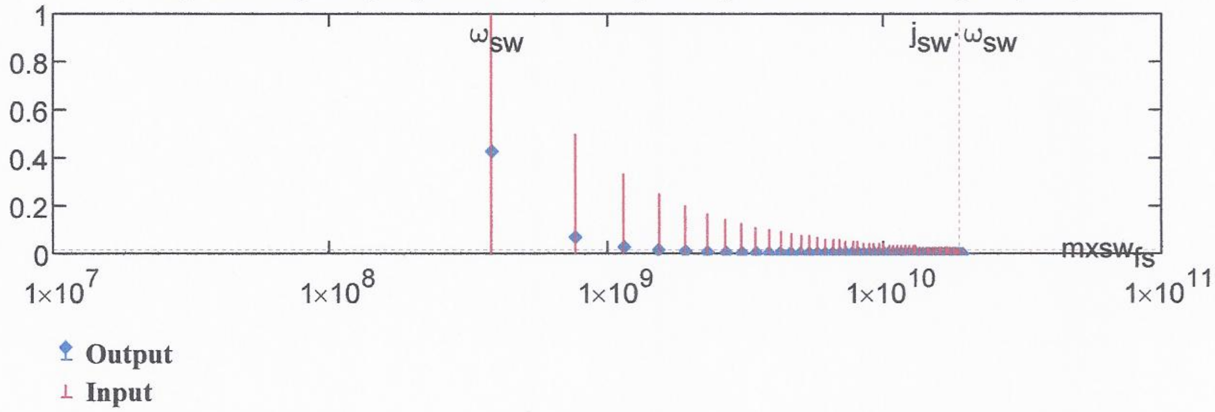


fig.:5.2.3.9

(Min) sampling frequency (Nyquist rate) $f_{sswo} := 2 \cdot B_{osw}$ $\quad f_{sswo} = 5.856 \cdot GHz \quad T_{sswo} := \frac{1}{f_{sswo}}$

sampling time step: $n_{swo_k} := \frac{k}{f_{sswo}}$ (5.2.3.20)

$T_{test} = 16.394 \cdot ns \quad rows\left(n_{swo}\right) = 256 \quad \left|A_5 \cdot V_{pp}\right| = 50V$

$\frac{1}{T_{test}} = 60.997 \cdot MHz \quad \frac{N0}{f_{sswo}} \cdot \frac{1}{T_{test}} = 2.667$

$$vosw_k := v_{osw}\left(n_{swo_k}\right) \quad (5.2.3.21)$$

Output Voltage Waveform

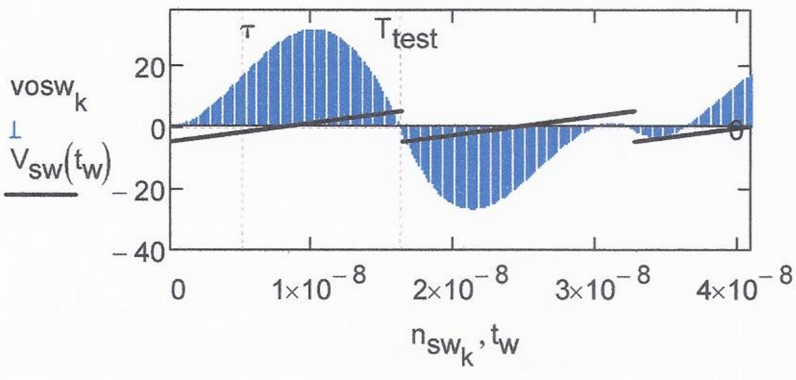


fig.:5.2.3.10

Approximate output signal reconstruction according to the Shannon sampling theorem:

$\omega_{s2o} := 2 \cdot \pi \cdot B_{osw}$ $\qquad sh3o(t) := \left[\sum_{n=0}^{N0-1} \left(vosw_n \cdot sinc\left(\omega_{s2o} \cdot t - n \cdot \pi \right) \right) \right]$ $\qquad$ (5.2.3.22)

$N0 - 1 = 255$

$rt_{gd} = 1 \cdot \%$

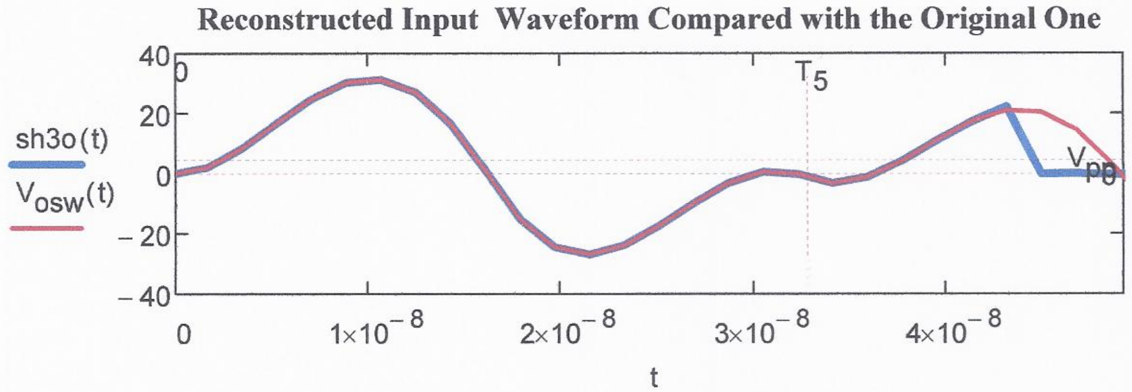


fig.:5.2.3.11

Fourier Transform of the Test signal

$f_{test} = 0.061 \cdot GHz$ $\qquad \frac{f_{sswo}}{f_{test}} = 96$ $\qquad \frac{N0}{f_{sswo}} \cdot \frac{1}{T_{test}} = 2.667$

$Osw := fft(vosw)$ $\qquad$ (5.2.3.23)

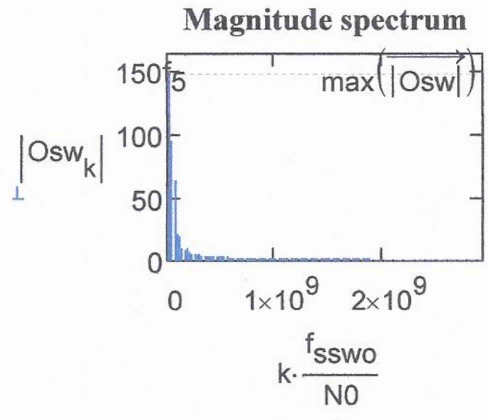


fig.:48

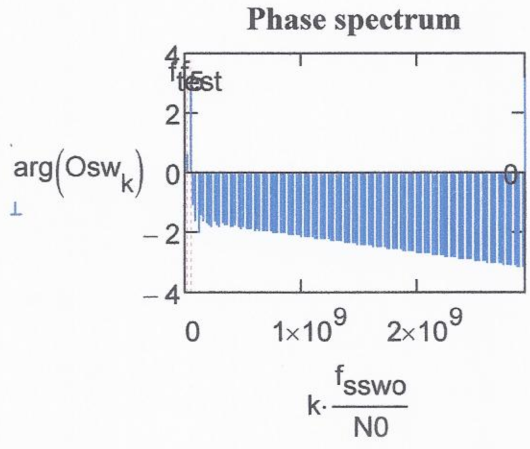


fig.:49

5.2 ANALOG FILTER OUTPUT ANALYSIS

5.2.4 Bipolar Square Wave response.

Signal Bandwidth calculation using the Fourier series' harmonics. Are excluded all harmonics with amplitude less than $rt_{gd} = 1\cdot\%$ of the fundamental.

$$V_{pp} = 5\times 10^{3}\cdot mV \qquad V_{sqwb}(t) := \frac{v_{sqw}(t, T_{test}, V_{pp})}{volt} \qquad (5.2.4.1)$$

$$T_{test} = 1.639\times 10^{-8}\,s$$

Signal bandwidth:

Description of the program's parameters:

BCSA(*Dimensionless signal name, relative error, polynomial degree, start time, signal period*)

BCSA stands for "Bandwidth Calculation and Signal Analysis"

$$rt_{gd} = 1\cdot\% \qquad Sb_{sqw} := BCSA(V_{sqwb}, rt_{gd}, 50, 0.0\cdot sec, T_{test}) \qquad (5.2.4.2)$$

▶ Bandwidth Calculation

The function returns a three columns matrix.
The first column contains:
pos. 0: relative error,
pos. 1: bandwidth (Dimensionless),
pos. 2: the nth. harmonic number corresponding to the give relative error,
pos. 3: temporary variable,
pos. 4: Parseval,
pos. 5: signal average,
pos. 6: signal rms.

The second column contains the coefficients a_k of the Fourier series,
the third column contains the coefficients b_k of the Fourier series.

Signal bandwidth: $B_{sqw} = 2.928\cdot GHz$

$Parseval_{sqwb} = 49.595V^2$ $\qquad$ $Average2 = 0V$ $\qquad$ $RMS2 = 2.887V$

sampling frequency (Nyquist rate) $f_{ssqw} := 2\cdot B_{sqw} \qquad f_{ssqw} = 5.856\cdot\frac{Grads}{sec} \qquad (5.2.4.3)$

$$n_{sqw_k} := \frac{k}{f_{ssqw}} \qquad T_{ssqw} := \frac{1}{f_{ssqw}} \qquad (5.2.4.4)$$

$$sqw_k := V_{sqwb}(n_{sqw_k}) \qquad \frac{N0}{f_{ssqw}}\cdot\frac{1}{T_{test}} = 2.667 \qquad (5.2.4.5)$$

Sampled Input

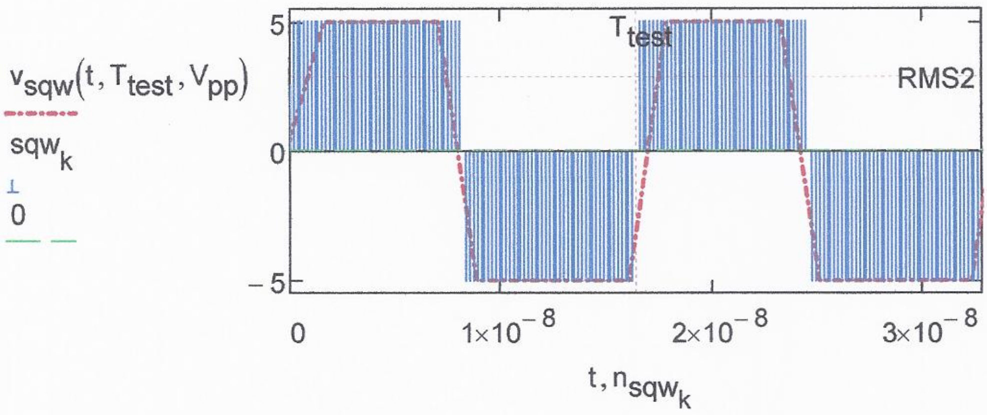


fig.:5.2.4.1

Approximate signal reconstruction according to the Shannon sampling theorem:

$$\omega_{s4} := 2\cdot\pi\cdot B_{sqw} \quad sh4(t) := \left[\sum_{n=0}^{N0-1} \left(sqw_n \cdot \mathrm{sinc}\left(\omega_{s4}\cdot t - n\cdot\pi\right)\right)\right] \tag{5.2.4.6}$$

$$t := 0\cdot T_{test}, 0\cdot T_{test} + \frac{5\cdot T_{test} - 0\cdot T_{test}}{1000} .. 5\cdot T_{test} \qquad rt_{gd} = 1\cdot\%$$

Reconstructed Waveform Compared to the Original One

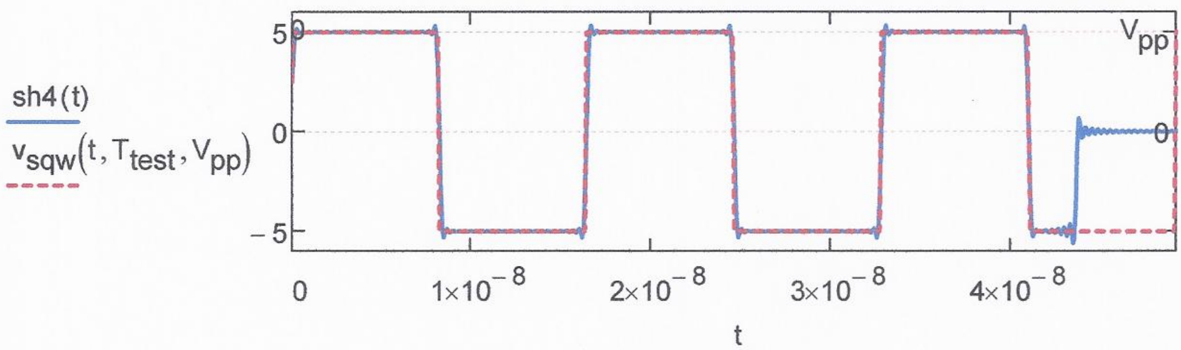


fig.:5.2.4.2

Output calculation using the Laplace transform of the input and the filter's transfer function.

Input L. t. $$\mathcal{L}\left(v_i\left(t, T_{test}, V_{pp}\right)\right) = \frac{V_{pp}}{s}\cdot\tanh\left(\frac{T_{test}\cdot s}{4}\right) \tag{5.2.4.7}$$

First case $\zeta \neq \omega_5$:

Output inverse L. t..: $$V_{opt}(t) = \mathcal{L}^{-1}\left(\frac{V_{pp}}{s}\cdot\tanh\left(\frac{T_{test}\cdot s}{4}\right)\cdot\frac{A_5\cdot\omega_5^2}{s^2 + 2\cdot\zeta\cdot s + \omega_5^2}\right) \tag{5.2.4.8}$$

Second case $\zeta = \omega_5$:

Output inverse L. t..: $$V_{opt}(t) = \mathcal{L}^{-1}\left[\frac{V_{pp}}{s}\cdot\tanh\left(\frac{T_{test}\cdot s}{4}\right)\cdot\frac{A_5\cdot\omega_5^2}{\left(s + \omega_5\right)^2}\right] \tag{5.2.4.9}$$

Output final value:

First case $\zeta \neq \omega_5$: $$\lim_{s \to 0} \left(\frac{V_{pp} \cdot s}{s} \cdot \tanh\left(\frac{T_{test} \cdot s}{4}\right) \cdot \frac{A_5 \cdot \omega_5^2}{s^2 + 2 \cdot \zeta \cdot s + \omega_5^2} \right) \to 0$$

Second case $\zeta = \omega_5$: $$\lim_{s \to 0} \left[\frac{V_{pp} \cdot s}{s} \cdot \tanh\left(\frac{T_{test} \cdot s}{4}\right) \cdot \frac{A_5 \cdot \omega_5^2}{(s + \omega_5)^2} \right] \to 0$$

How to calculate the Laplace transform of the input signal:

Shift theorem $\mathcal{L}(f(t-a)) = e^{-a \cdot s} \cdot F(s)$

$$V_i(s) = \frac{V_{pp}}{s} \cdot \sum_{k=0}^{N} \left[e^{-k \cdot T_{test} \cdot s} - 2 \cdot e^{-\frac{2 \cdot k+1}{2} \cdot T_{test} \cdot s} + e^{-(k+1) \cdot T_{test} \cdot s} \right] \qquad (5.2.4.10)$$

$$\boxed{\mathcal{L}\left(\sum_{k=0}^{\infty} f(t - k \cdot T)\right) = \frac{F(s)}{1 - e^{-T \cdot s}}}$$

Filter's output signal calculation:

First case $\zeta \neq \omega_5$: $$V_o(s) = V_i(s) \cdot \frac{A_5 \cdot \omega_5^2}{s^2 + 2 \cdot \zeta \cdot s + \omega_5^2} \qquad (5.2.4.11)$$

Second case $\zeta = \omega_5$: $$V_o(s) = V_i(s) \cdot \frac{A_5 \cdot \omega_5^2}{(s + \omega_5)^2} \qquad (5.2.4.12)$$

First case $\zeta \neq \omega_5$.

▶ First case: Output calculation

Once defined the function:

$$f1_{sq}(t) := A_5 \cdot \left[1 - e^{-\zeta \cdot t} \cdot \left(\cosh\left(t \cdot \sqrt{\zeta^2 - \omega_5^2}\right) + \frac{\zeta \cdot \sinh\left(t \cdot \sqrt{\zeta^2 - \omega_5^2}\right)}{\sqrt{\zeta^2 - \omega_5^2}} \right) \right] \cdot \Phi(t) \qquad (5.2.4.13)$$

the output is given by the sum:

$$V_{o1}(t) := V_{pp} \cdot \left[\sum_{k=0}^{N} \left[f1_{sq}(t - k \cdot T_{test}) - 2 \cdot f1_{sq}\left(t - \frac{2 \cdot k + 1}{2} \cdot T_{test}\right) + f1_{sq}[t - (k+1) \cdot T_{test}] \right] \right]$$

(5.2.4.13')

Second case $\zeta = \omega_5$:

▶ Second case:Output calculation

$$f2_{sq}(t) := A_5 \cdot \left[1 - e^{-t \cdot \omega_5} \cdot (1 + t \cdot \omega_5) \right] \cdot \Phi(t) \qquad (5.2.4.14)$$

$$V_{o2}(t) := V_{pp} \cdot \left[\sum_{k=0}^{N} \left[f2_{sq}(t - k \cdot T_{test}) - 2 \cdot f2_{sq}\left(t - \frac{2 \cdot k + 1}{2} \cdot T_{test}\right) + f2_{sq}\left[t - (k+1) \cdot T_{test}\right] \right] \right] \quad (5.2.4.15)$$

$$\zeta = 0.018 \cdot \frac{Grads}{sec} \qquad \frac{1}{T_{test}} = 0.061 \cdot GHz$$

$N = 50$

Output signal

$$V_{sqw}(t) := \begin{vmatrix} V_{o1}(t) & \text{if } \zeta \neq \omega_5 \\ V_{o2}(t) & \text{otherwise} \end{vmatrix} \qquad (5.2.4.16)$$

$$V_{sqw}\left(\frac{T_{test}}{1.3}\right) = -45.135 V$$

Graph of the bipolar Square Wave response considering the Op Amp saturation:

$$V_{osqw}(t) := if\left(-V_{sat} \le V_{sqw}(t) \le V_{sat}, V_{sqw}(t), if\left(V_{sqw}(t) \le 0.0 \cdot volt, -V_{sat}, V_{sat}\right)\right) \qquad (5.2.4.17)$$

$A_5 = -10 \qquad T_{test} = 16.394 \cdot ns$

Bipolar Square Wave response. $\qquad f_{test} = 0.061 \cdot GHz$

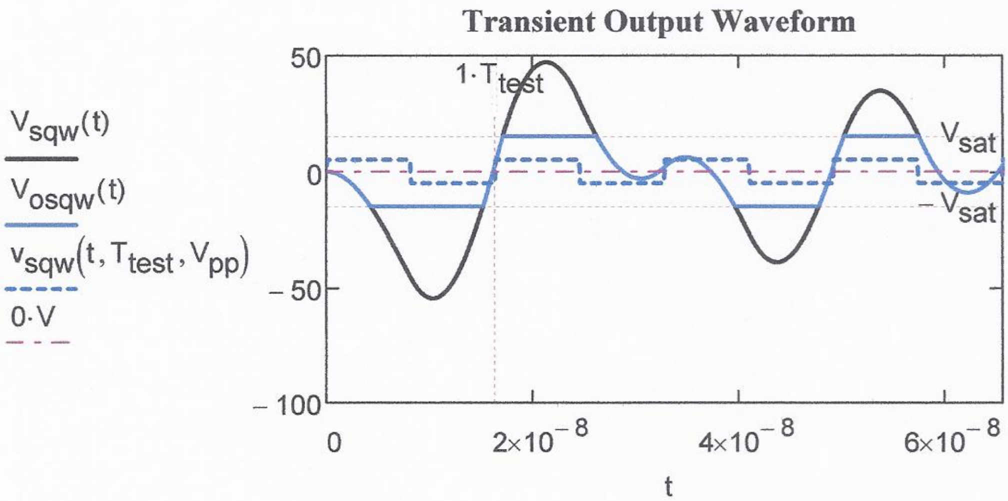


fig.:5.2.4.3

$$vosqw_k := V_{sqw}\left(n_{sqw_k}\right) \qquad (5.2.4.18)$$

Approximate output signal reconstruction according to the Shannon sampling theorem:

$$\omega_{s5} := 2 \cdot \pi \cdot B_{sqw} \qquad sh5(t) := \left[\sum_{n=0}^{N0-1} \left(vosqw_n \cdot sinc\left(\omega_{s5} \cdot t - n \cdot \pi\right)\right) \right] \qquad (5.2.4.19)$$

$rt_{gd} = 1 \cdot \%$

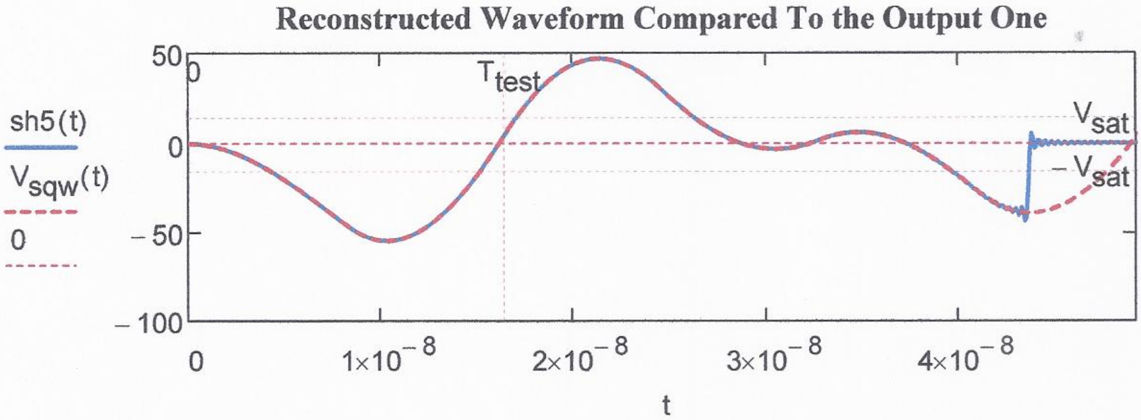


fig.:5.2.4.5

$$vopt_k := V_{osqw}\left(n_{sqw_k}\right) \qquad (5.2.4.20)$$

Approximate reconstruction of the output signal (with Op Amp saturation) according to the Shannon sampling theorem:

$$\omega_{s5} := 2 \cdot \pi \cdot B_{sqw} \quad sh5(t) := \left[\sum_{n=0}^{N0-1} \left(vopt_n \cdot \mathrm{sinc}\left(\omega_{s5} \cdot t - n \cdot \pi\right)\right)\right] \qquad (5.2.4.21)$$

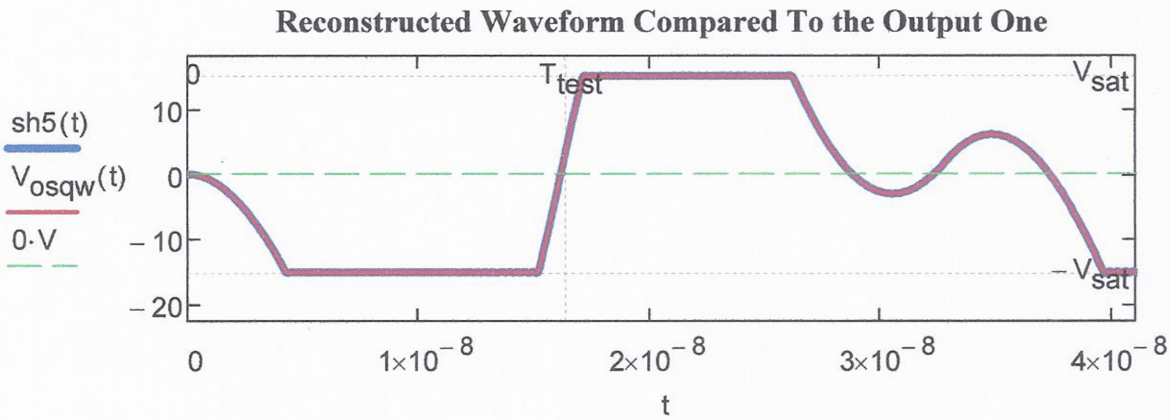


fig.:5.2.4.6

$$\frac{N0}{f_{ssqw}} \cdot \frac{1}{T_{test}} = 2.667 \qquad Opt := \mathrm{fft}(vopt) \qquad (5.2.4.22)$$

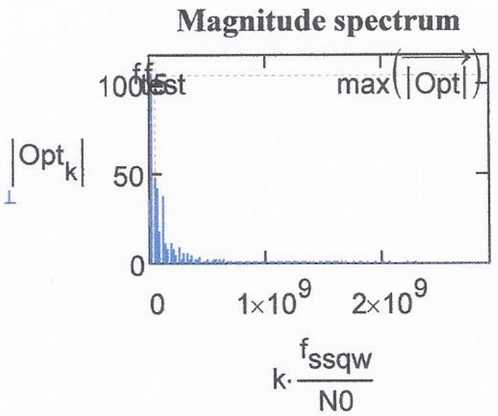


fig.:5.2.4.7

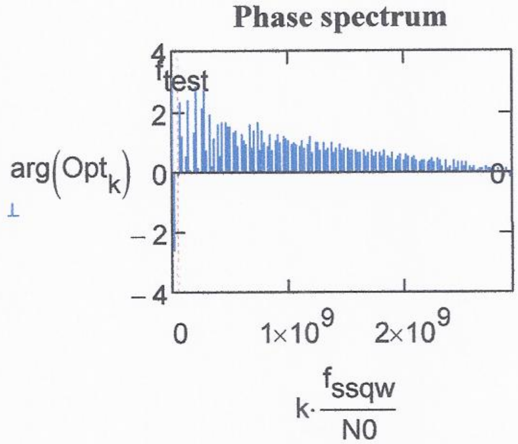


fig.:5.2.4.8

Convolution:

The second method is to calculate ***the time domain convolution product*** between the signal and the impulse response:

$$t := -1 \cdot T_{test}, -1 \cdot T_{test} + \frac{4 \cdot T_{test} + 1 \cdot T_{test}}{100} .. 4 \cdot T_{test}$$

$$v_{osqconv}(t) := \int_0^t w(t-\sigma) \cdot v_{sqw}(\sigma, T_{test}, V_{pp}) \, d\sigma \qquad (5.2.4.23)$$

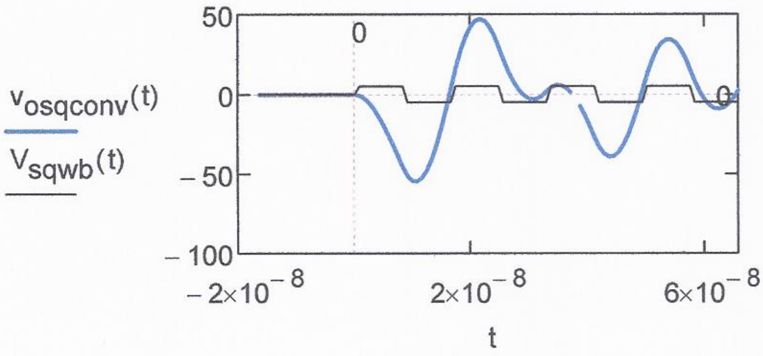


fig.:5.2.4.9

5.2 ANALOG FILTER OUTPUT ANALYSIS

5.2.5 (single tone) AM Signal response.

$A1 = 10V$ $\omega_c := 1\cdot\omega_{test}$ $f_c := \frac{\omega_c}{2\cdot\pi}$ $T_c := \frac{1}{f_c}$ $\omega_{mam} := \frac{1}{5}\cdot\omega_5$ $f_{mam} := \frac{\omega_{mam}}{2\cdot\pi}$

$B1 = 5.5V$ $f_c = 60.997\cdot MHz$

$T_{mam} := \frac{1}{f_{mam}}$ $f_{mam} = 6.1\cdot MHz$ modulation index: $m_{am} = 55\cdot\%$

$\omega_{mam} = 0.038\cdot\frac{Grads}{sec}$ $\omega_c = 0.383\cdot\frac{Grads}{sec}$ $Bw_{am} := f_c + 2\cdot f_{mam}$

sampling frequency (Nyquist rate): $f_{sam} := 2\cdot Bw_{am}$, $f_{sam} = 146.392\cdot MHz$,

sampling angular frequency: $\omega_{sam} := 2\cdot\pi\cdot f_{sam}$. $\omega_{sam} = 0.92\cdot\frac{Grads}{sec}$,

sampling period: $T_{sam} := \frac{1}{f_{sam}}$, $T_{sam} = 6.831\times10^{-3}\cdot\mu s$,

sampling time step: $nam_k := \frac{k}{f_{sam}}$,

$NO\cdot\frac{T_{sam}}{T_{test}} = 106.667$

$nam^T =$

	0	1	2	3	4
0	0	$6.831\cdot10^{-3}$	0.014	0.02	...

$\cdot\mu s$

$\frac{\omega_c}{\omega_{mam}} = 10$

$$V2_i(t) := v2_i(t, \omega_{mam}, \omega_c, A1, B1) \quad (5.2.5.1)$$

$$u\_7_k := \frac{V2_i(nam_k)}{volt} \quad (5.2.5.2)$$

$t := 0\cdot T_c, 0\cdot T_c + \frac{20\cdot T_c}{1000} .. 20\cdot T_c$ $\frac{NO}{f_{sam}}\cdot\frac{1}{T_{test}} = 106.667$

(Single Tone) AM Signal Sampled input Waveform

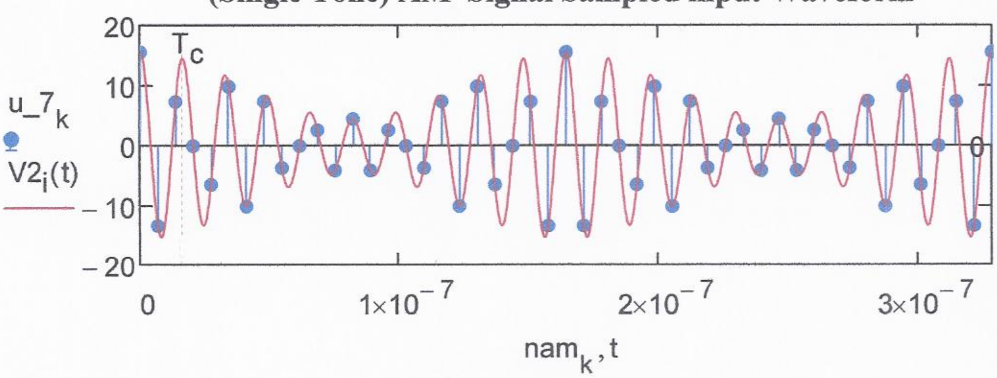


fig.:5.2.5.1

Approximate signal reconstruction according to the Shannon sampling theorem:

$\omega_{sh6} := 2\cdot\pi\cdot Bw_{am}$ $$sh6(t) := \left[\sum_{n=0}^{N0-1}\left(u\_7_n\cdot sinc\left(\omega_{sh6}\cdot t - n\cdot\pi\right)\right)\right] \quad (5.2.5.3)$$

$rt_{gd} = 1\cdot\%$

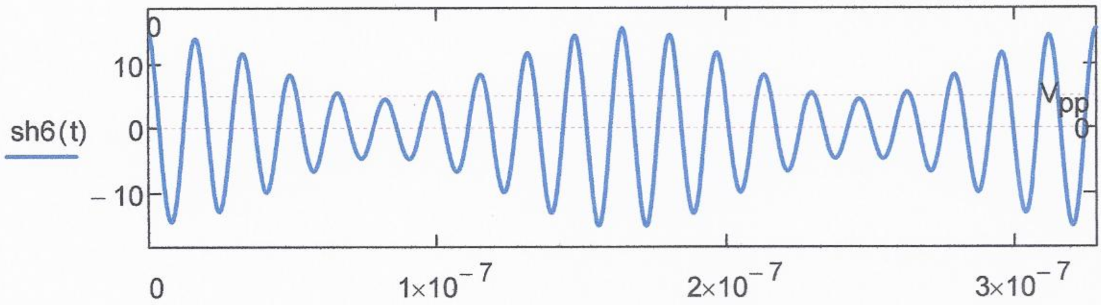


fig.:5.2.5.2

$$Spec\_7 := fft(u\_7)$$

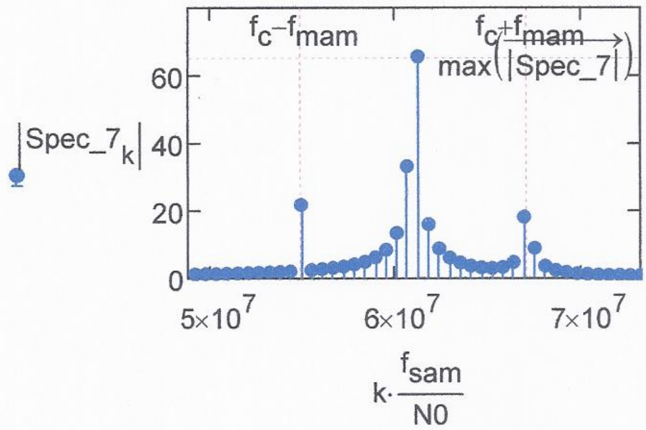


fig.:5.2.5.3

$V_{pp} = 5V$

Exact output: $$v_{oam}(t) := \int_0^t w(t-\sigma)\cdot V2_i(\sigma)\,d\sigma \quad (5.2.5.4)$$

$m_{am} = 55\cdot\%$ $$t := 0\cdot T_c, 0\cdot T_c + \frac{40\cdot T_c}{1000} .. 40\cdot T_c$$ $T_c = 0.016\cdot\mu s$

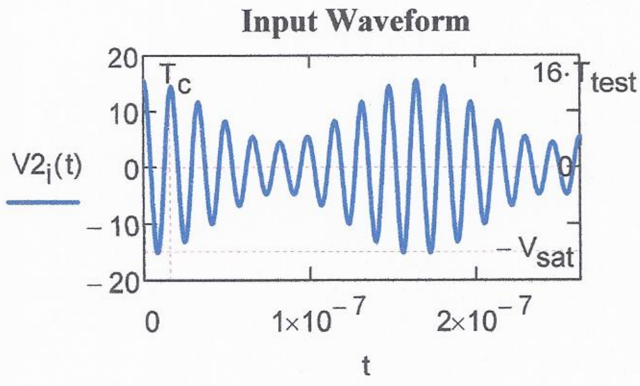


fig.:5.2.5.4

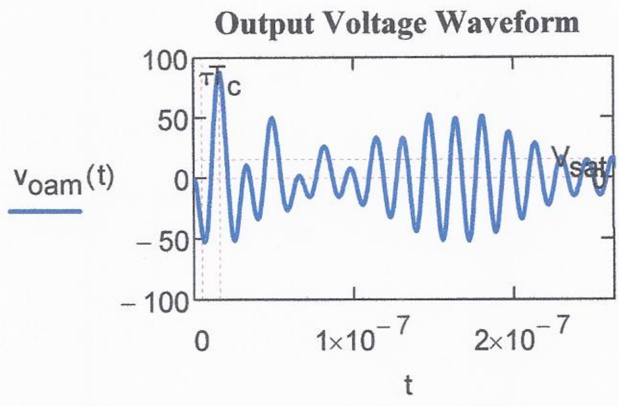


fig.:5.2.5.5

Output sampling: $Vam_k := v_{oam}(nam_k)$

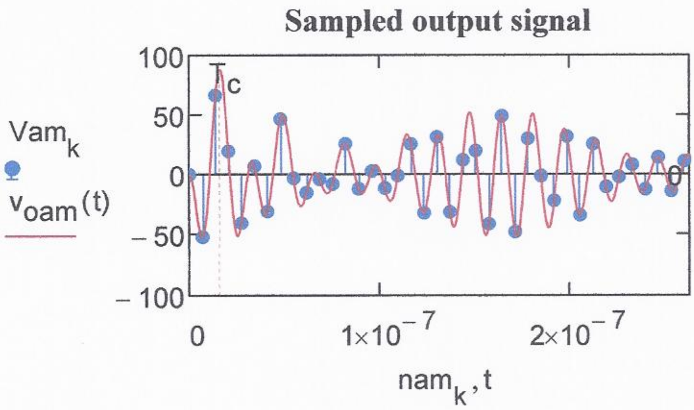


fig.:5.2.5.6

Fourier Transform of the output signal

$f_c = 0.061 \cdot GHz$ $\qquad \frac{f_{sam}}{f_c} = 2.4$

$Spec_{am} := fft(Vam)$

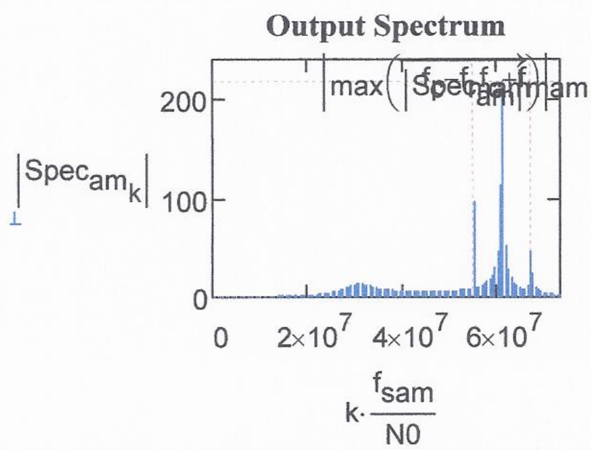


fig.:5.2.5.7

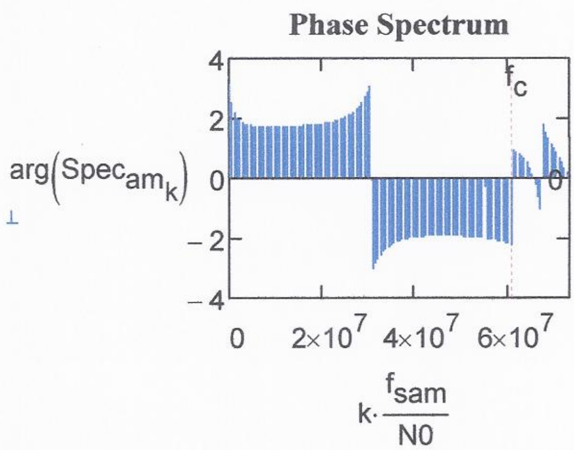


fig.:5.2.5.8

Magnitude of W(jω) in dB

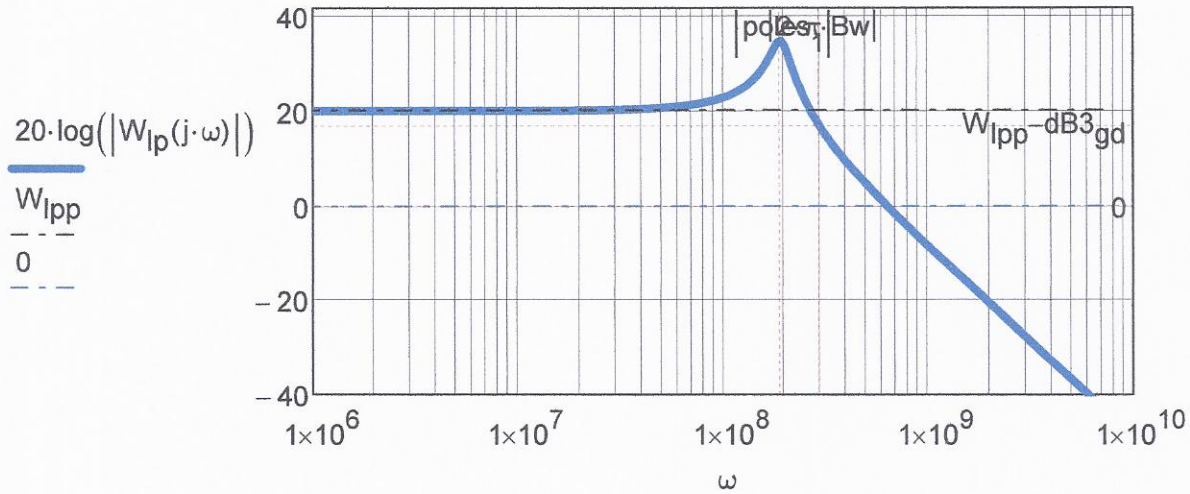


fig.:5.2.5.9

Phase of W(jω)

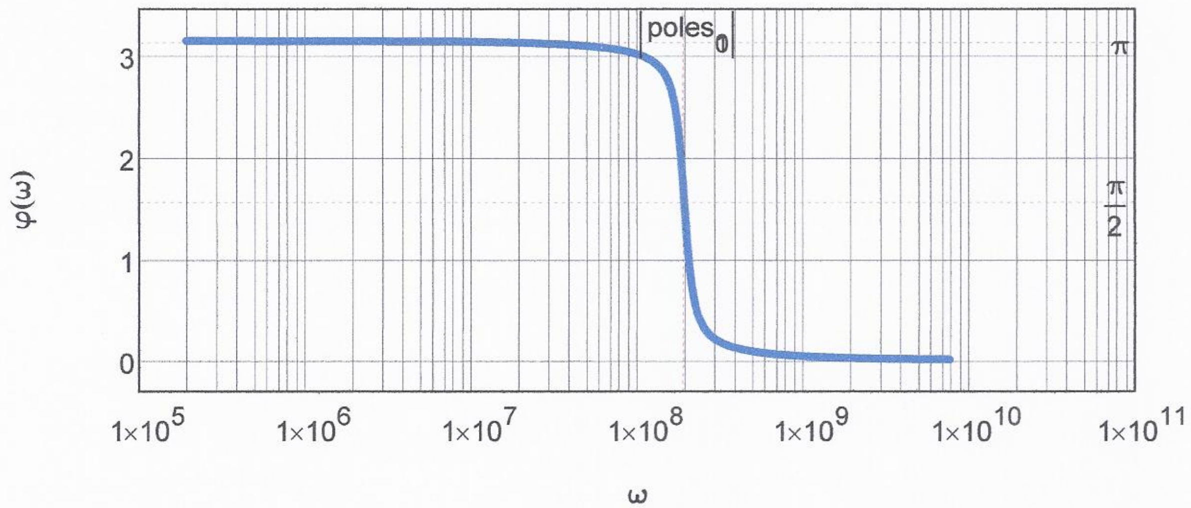


fig.:5.2.5.10

5.2.6 (single tone) Frequency Modulated carrier response.

$$\omega_{cfm} := 1\cdot\omega_{test} \qquad f_{cfm} := \frac{\omega_{cfm}}{2\cdot\pi}$$

$$T_{cfm} := \frac{1}{f_{cfm}} \qquad \omega_{mfm} := \frac{\omega_c}{20}$$

$$f_{mfm} := \frac{\omega_{mfm}}{2\cdot\pi} \qquad T_{mfm} := \frac{1}{f_{mfm}} \qquad T_{mfm} = 0.328\cdot\mu s$$

$$T_{cfm} = 0.016\cdot\mu s$$

frequency modulation index: $m_f := \frac{2\cdot Kst\cdot\pi\cdot B}{\omega_m}$ $\quad m_f := 8$ (5.2.6.1)

Carson bandwidth: $Cars1 := 2\cdot\omega_{mfm}\cdot(m_f + 1)$ $\quad Cars1 = 0.345\cdot\frac{Grads}{sec}$ (5.2.6.2)

sampling frequency (Nyquist rate): $f_{sfm} := 2\cdot Cars1$, (5.2.6.3)

$f_{sfm} = 0.69\cdot GHz$,

sampling angular frequency: $\omega_{sfm} := 2\cdot\pi\cdot f_{sfm}\cdot$ $\quad \omega_{sfm} = 4.335\cdot\frac{Grads}{sec}$,

sampling period: $T_{sfm} := \frac{1}{f_{sfm}}$, $\quad T_{sfm} = 1.45\times 10^{-3}\cdot\mu s$,

sampling time step: $nfm_k := \frac{k}{f_{sfm}}$, (5.2.6.4)

$$\frac{N0}{f_{sfm}}\cdot\frac{1}{T_{test}} = 22.635$$

$nfm^T =$ ·µs

	0	1	2	3	4	5	6
0	0	$1.45\cdot10^{-3}$	$2.899\cdot10^{-3}$	$4.349\cdot10^{-3}$	$5.798\cdot10^{-3}$	$7.248\cdot10^{-3}$	...

$\frac{\omega_{cfm}}{\omega_{mfm}} = 20$ $\qquad V_{fm}(t) := v_{fmsl}(t, \omega_{cfm}, \omega_{mfm}, A_{fm}, m_f)$ (5.2.6.5)

$A_{fm} = 0.02V$ $\qquad u8_k := \frac{V_{fm}(nfm_k)}{volt}$ $\qquad V_{pp} = 5V$ (5.2.6.6)

Approximate signal reconstruction according to the Shannon sampling theorem:

$\omega_{sh7} := 2\cdot\pi\cdot Cars1$ $\qquad sh7(t) := \left[\sum_{n=0}^{N0-1}\left(u8_n\cdot sinc(\omega_{sh7}\cdot t - n\cdot\pi)\right)\right]$ (5.2.6.7)

$rt_{gd} = 1 \cdot \%$

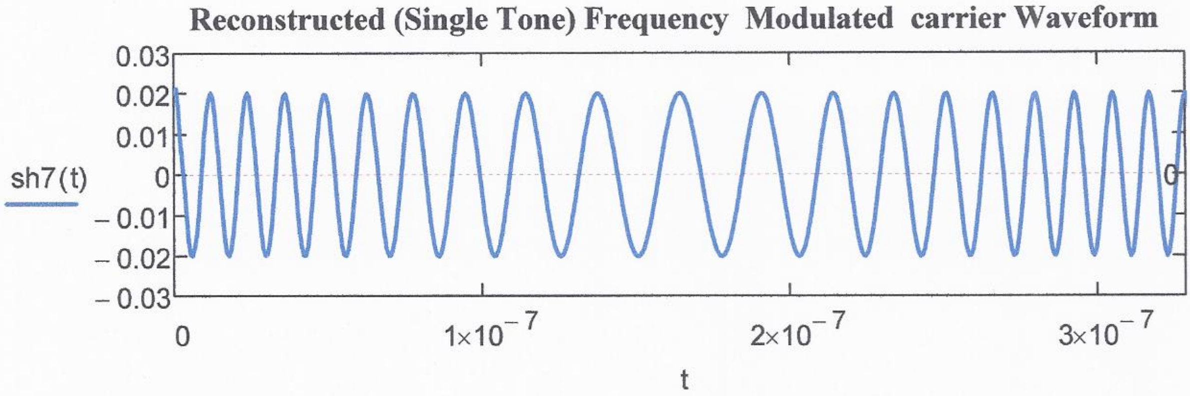


fig.:5.2.6.1

Exact output:

$$f2fm(t,\sigma,\omega_5,\zeta,\omega_{cfm},\omega_{mfm},m_f) := \sin\left[(t-\sigma)\cdot\sqrt{\omega_5{}^2-\zeta^2}\right]\cdot e^{\zeta\cdot\sigma}\cdot\cos\left(\omega_{cfm}\cdot\sigma + m_f\cdot\sin(\omega_{mfm}\cdot\sigma)\right)$$

$$v_{ofm}(t) := A_{fm}\cdot A_5\cdot\omega_5{}^2\cdot \begin{cases} \left(\dfrac{e^{-\zeta\cdot t}}{\sqrt{\omega_5{}^2-\zeta^2}}\cdot\displaystyle\int_0^t f2fm(t,\sigma,\omega_5,\zeta,\omega_{cfm},\omega_{mfm},m_f)\,d\sigma\right) & \text{if } \zeta \neq \omega_5 \\ \displaystyle\int_0^t (t-\sigma)\cdot e^{-(t-\sigma)\cdot\omega_5}\cdot\cos\left(\omega_{cfm}\cdot\sigma + m_f\cdot\sin(\omega_{mfm}\cdot\sigma)\right)d\sigma & \text{otherwise} \end{cases}$$

(5.2.6.8)

$$v_{ofm}(t) := \int_0^t w(t-\sigma)\cdot V_{fm}(\sigma)\,d\sigma \qquad (5.2.6.9)$$

$T_{test} = 0.016\cdot\mu s$

$$t_{fm} := \frac{T_{mfm}\cdot 0}{100}, \frac{T_{mfm}\cdot 0}{100} + \frac{1\cdot T_{mfm} - \frac{T_{mfm}\cdot 0}{100}}{100} \,..\, 1\cdot T_{mfm}$$

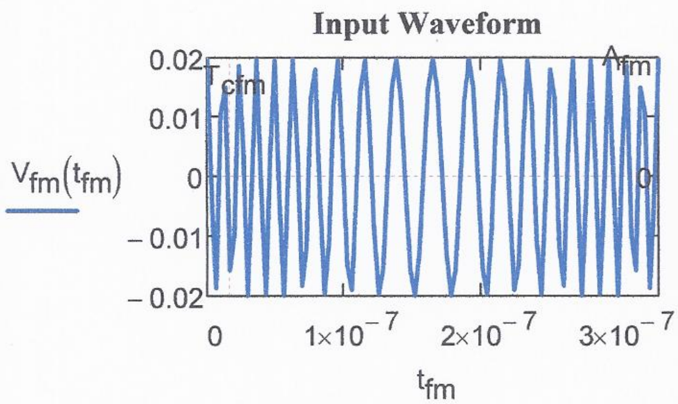


fig.:5.2.6.2

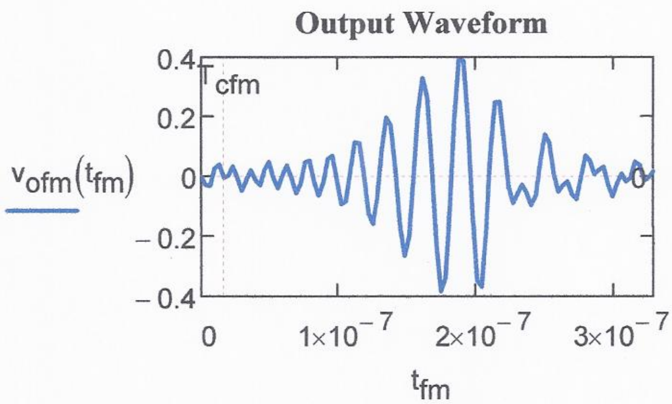


fig.:5.2.6.3

$v_{ofm}(nfm_{100}) = -0.199\,V$ Output sampling: $Ofm_k := v_{ofm}(nfm_k)$ (5.2.6.10)

Sampled Output Signal

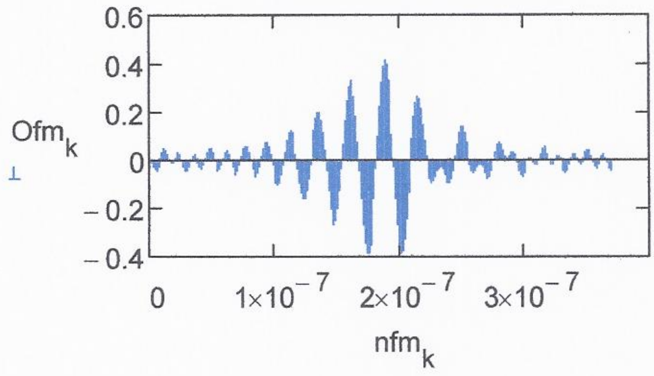


fig.:5.2.6.4

Fourier Transform of the Test signal

$m_f = 8$ $\qquad f_c = 0.061 \cdot GHz$ $\qquad \frac{f_{sfm}}{f_c} = 11.31$ $\qquad \frac{N0}{f_{sfm}} \cdot \frac{1}{T_{test}} = 22.635$

$\omega_{mfm} = 0.019 \cdot \frac{Grads}{sec}$ $\qquad OSpecfm := fft(Ofm)$ $\qquad$ (5.2.6.11)

FM Test Signal spectrum

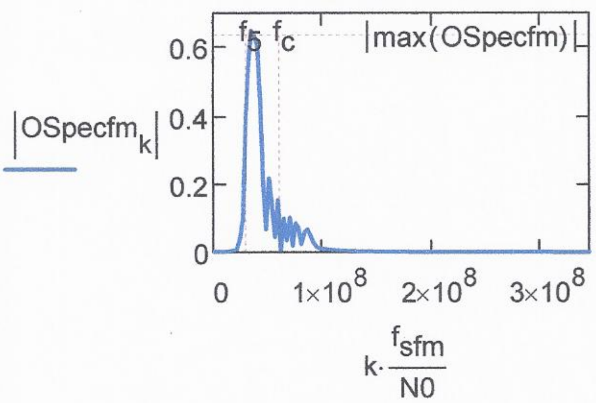


fig.:5.2.6.5

Phase spectrum

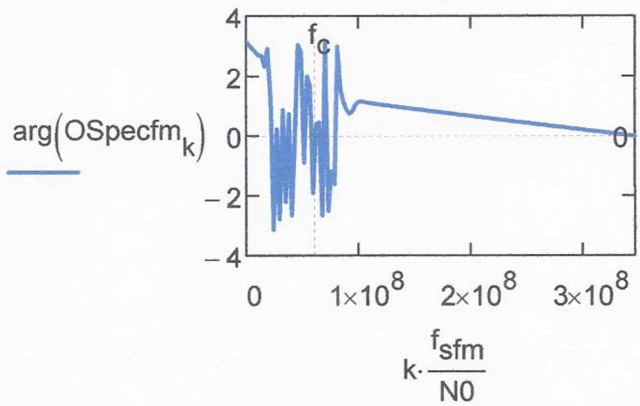


fig.:5.2.6.6

On the other hand if the carrier frequency is located in the passing band, the filter response is :

$$\omega 3_c := \frac{\omega_c}{100} \qquad \omega 3_m := \frac{\omega_{mfm}}{100} \qquad T3_c := \frac{2 \cdot \pi}{\omega 3_c} \qquad T3_m := \frac{2 \cdot \pi}{\omega 3_m}$$

Carson bandwidth: $\quad Cars3 := 2 \cdot \omega 3_m \cdot (m_f + 1) \qquad Cars3 = 3.449 \cdot \frac{Mrads}{sec}$ $\quad$ (5.2.6.12)

Carrier frequency: $\quad f3_c := \frac{\omega 3_c}{2 \cdot \pi}, \qquad f3_c = 6.1 \times 10^{-4} \cdot GHz$

sampling frequency (Nyquist rate): $\quad f3_{sfm} := 2 \cdot Cars3 \ , \qquad f3_{sfm} = 6.899 \times 10^{-3} \cdot GHz,$

sampling angular frequency: $\quad \omega 3_{sfm} := 2 \cdot \pi \cdot f3_{sfm} \cdot \qquad \omega 3_{sfm} = 0.043 \cdot \frac{Grads}{sec},$

sampling period: $\quad T3_{sfm} := \frac{1}{f3_{sfm}}, \qquad T3_{sfm} = 0.145 \cdot \mu s,$

sampling time step: $n3fm_k := \frac{k}{f3_{sfm}}$,

$$\frac{N0}{f3_{sfm}} \cdot \frac{1}{T_{test}} = 2.264 \times 10^3$$

$A_{fm} = 0.02V$ $\qquad V1_{fm}(t) := v_{fmsl}(t, \omega 3_c, \omega 3_m, A_{fm}, m_f)$ (5.2.6.13)

$$u8b_k := \frac{V1_{fm}(n3fm_k)}{volt} \qquad (5.2.6.14)$$

Approximate signal reconstruction according to the Shannon sampling theorem:

$\omega_{s7} := 2 \cdot \pi \cdot Cars1$ $\qquad sh7b(t) := \left[\sum_{n=0}^{N0-1} \left(u8b_n \cdot sinc(\omega_{s7} \cdot t - n \cdot \pi)\right)\right]$ (5.2.6.15)

$rt_{gd} = 1 \cdot \%$

Reconstructed Signal Waveform

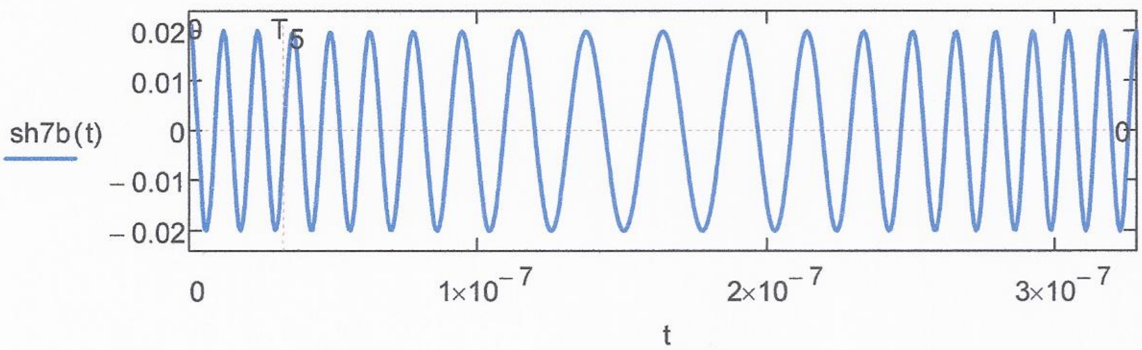


fig.:5.2.6.7

Exact output:

$$v2_{ofm}(t) := A_{fm} \cdot A_5 \cdot \omega_5^2 \cdot \begin{cases} \frac{e^{-\zeta \cdot t}}{\sqrt{\omega_5^2 - \zeta^2}} \cdot \int_0^t f2fm(t, \sigma, \omega_5, \zeta, \omega 3_c, \omega 3_m, m_f)\, d\sigma & \text{if } \zeta \neq \omega_5 \\ \int_0^t (t - \sigma) \cdot e^{-(t-\sigma) \cdot \omega_5} \cdot \cos(\omega 3_c \cdot \sigma + m_f \cdot \sin(\omega 3_m \cdot \sigma))\, d\sigma & \text{otherwise} \end{cases}$$

(5.2.6.16)

$$v2_{ofm}(t) := \int_0^t w(t - \sigma) \cdot V1_{fm}(\sigma)\, d\sigma \qquad (5.2.6.17)$$

$T_{test} = 0.016 \cdot \mu s$ $\qquad T1_c = 2 \times 10^3 \cdot \mu s$ $\qquad \frac{1}{f3_{sfm}} = 144.957 \cdot ns$ $\qquad V_{pp} = 5V$

$$t1_{fm} := T_{mfm}\cdot 0, T_{mfm}\cdot 0 + \frac{100\cdot T_{mfm} - T_{mfm}\cdot 0}{400} .. 100\cdot T_{mfm}$$

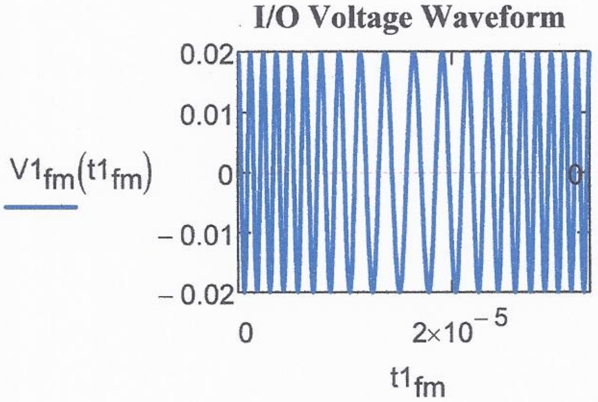


fig.:5.2.6.8

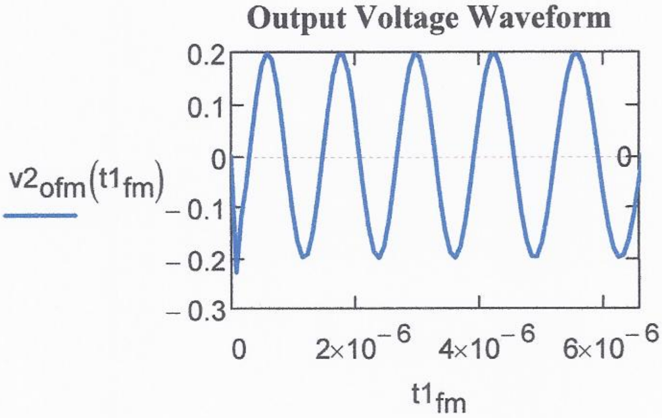


fig.:5.2.6.9

$n3fm_k := \frac{k}{N0}\cdot T3_c$ Output sampling: $Ofm_k := v2_{ofm}(n3fm_k)$ (5.2.6.18)

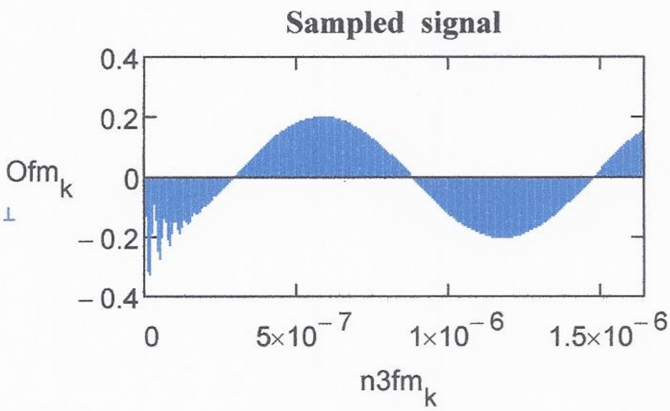


fig.:5.2.6.10

Fourier Transform of the Test signal $f3_c = 6.1\times 10^{-4}\cdot GHz$ $\frac{f3_{sfm}}{f3_c} = 11.31$ $m_f = 8$

$\omega_{mfm} = 0.019\cdot\frac{Grads}{sec}$ $OSpecfm3 := fft(Ofm)$ (5.2.6.19)

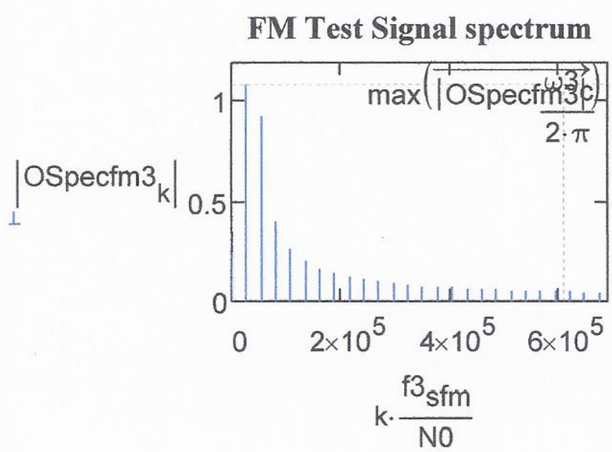


fig.:5.2.6.11

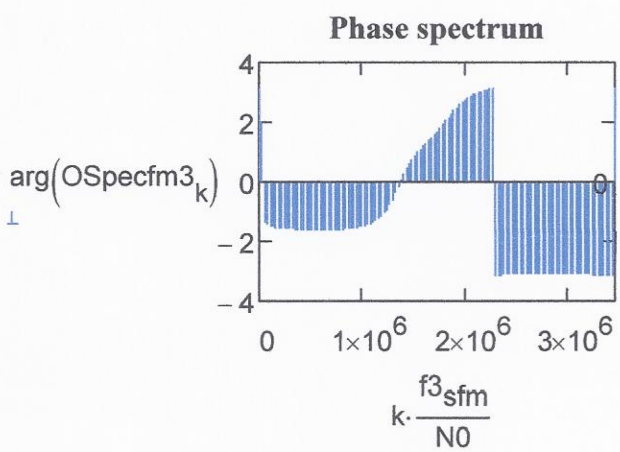


fig.:5.2.6.12

$$\omega := \frac{\omega_5}{20 \cdot U}, \frac{\omega_5}{20 \cdot U} + \frac{\omega_5 \cdot U - \frac{\omega_5}{20 \cdot U}}{U^2} .. 10 \cdot U \cdot \omega_5$$

$$Wdb\omega 1_c := 20 \cdot \log\left(\left|W_{lp}\left(j \cdot \omega 1_c\right)\right|\right) \qquad (5.2.6.20)$$

$Wdb\omega 1_c = 20 \cdot dB$

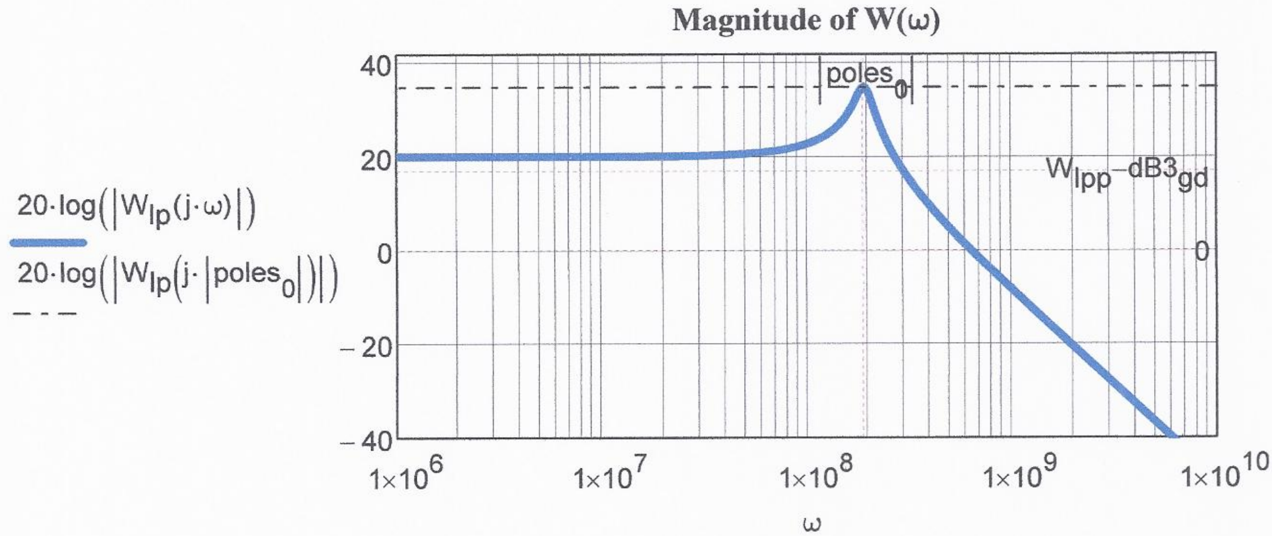


fig.:5.2.6.13

5.2.7 (single tone) Phase Modulated carrier response.

$$\omega_{cpm} := 4\cdot\omega_{test} \quad f_{cpm} := \frac{\omega_{cpm}}{2\cdot\pi} \quad T_{cpm} := \frac{1}{f_{cpm}} \qquad (5.2.7.1)$$

$$\omega_{mpm} := \frac{\omega_{cpm}}{20} \quad f_{mpm} := \frac{\omega_{mpm}}{2\cdot\pi} \quad T_{mpm} := \frac{1}{f_{mpm}} \qquad (5.2.7.2)$$

$$T_{cpm} = 4.099\times10^{-3}\cdot\mu s \qquad T_{mpm} = 0.082\cdot\mu s$$

$$m_p := 8$$

Carson bandwidth: $Cars4 := 2\cdot\omega_{mpm}\cdot(m_p + 1)$ (5.2.7.3)

sampling frequency (Nyquist rate): $f_{spm} := 2\cdot Cars4$, $f_{spm} = 2.759\cdot GHz$ (5.2.7.4)

sampling angular frequency: $\omega_{spm} := 2\cdot\pi\cdot f_{spm}\cdot$ $\omega_{spm} = 17.338\cdot\frac{Grads}{sec}$,

sampling period: $T_{spm} := \frac{1}{f_{spm}}$, $T_{spm} = 3.624\times10^{-4}\cdot\mu s$,

sampling time step: $npm_k := \frac{k}{f_{spm}}$, (5.2.7.5)

$$\frac{N0}{f_{spm}}\cdot\frac{1}{T_{test}} = 5.659 \qquad (5.2.7$$

$npm^T =$

	0	1	2	3	4
0	0	3.624·10-4	7.248·10-4	1.087·10-3	...

$\cdot\mu s$

$A_{pm} = 0.02V$ $\qquad V_{pm}(t) := v_{pm}(t, \omega_{cpm}, \omega_{mpm}, A_{pm}, m_p)$ (5.2.7.7)

$$u9_k := \frac{V_{pm}(npm_k)}{volt} \qquad (5.2.7.8)$$

Sampled PM Waveform

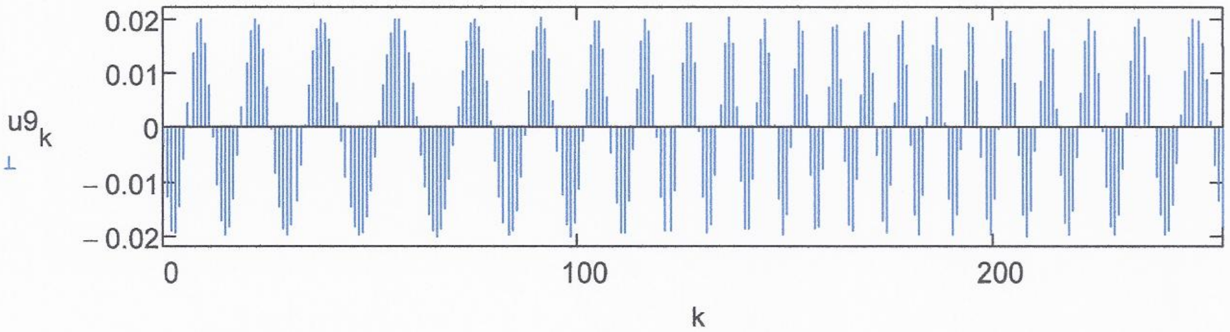


fig.:5.2.7.1

Approximate signal reconstruction according to the Shannon sampling theorem:

$\omega_{ts8} := 2\cdot\pi\cdot Cars4$ $$sh8(t) := \left[\sum_{n=0}^{N0-1} \left(u9_n \cdot \mathrm{sinc}\left(\omega_{ts8}\cdot t - n\cdot\pi\right)\right)\right] \qquad (5.2.7.9)$$

$$t_{pm} := 0\cdot T_{cpm}, 0\cdot T_{cpm} + \frac{40\cdot T_{cpm}}{10000} .. 40\cdot T_{cpm} \qquad rt_{gd} = 1\cdot\%$$

Waveform of The Reconstructed Signal

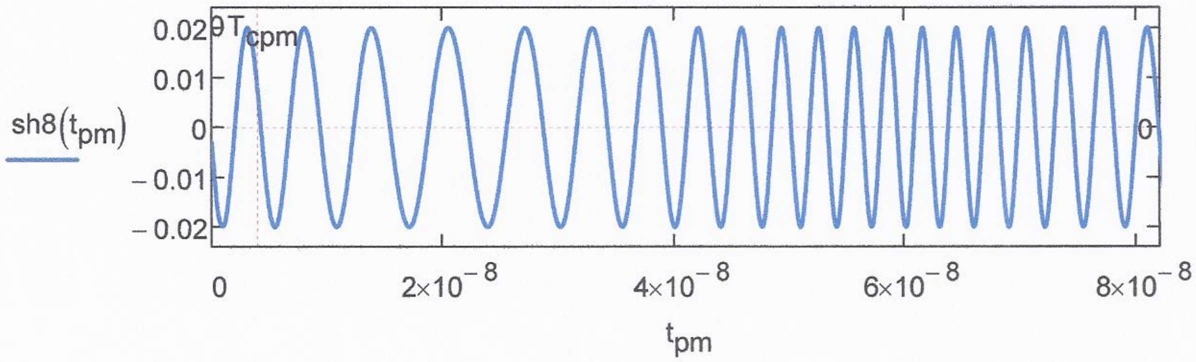


fig.:5.2.7.2

$V_{pp} = 5V$ Exact output: $$v_{opm}(t) := \int_0^t w(t-\sigma)\cdot V_{pm}(\sigma)\, d\sigma \qquad (5.2.7.10)$$

$$T_{cpm} = 4.099\times10^{-3}\cdot\mu s \quad T_{mpm} = 0.082\cdot\mu s$$

Output Voltage Waveform

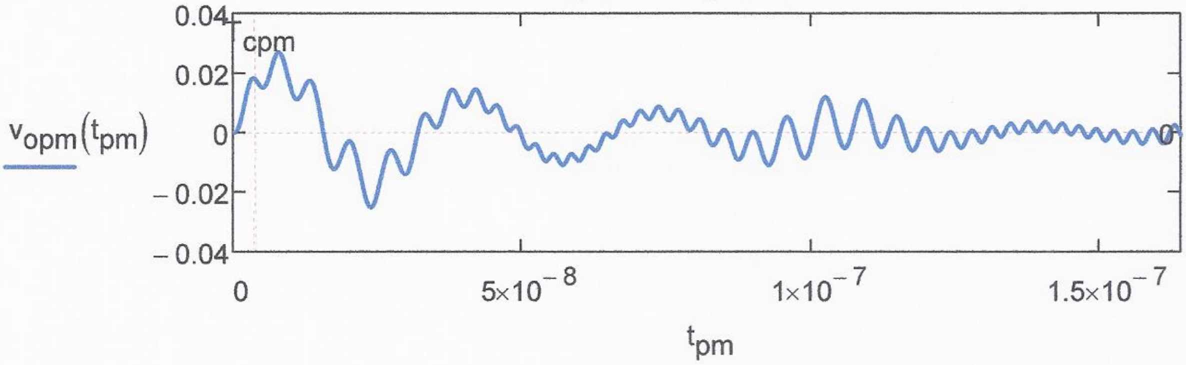


fig.:5.2.7.3

$$Opm_k := \frac{v_{opm}(npm_k)}{volt} \qquad (5.2.7.11)$$

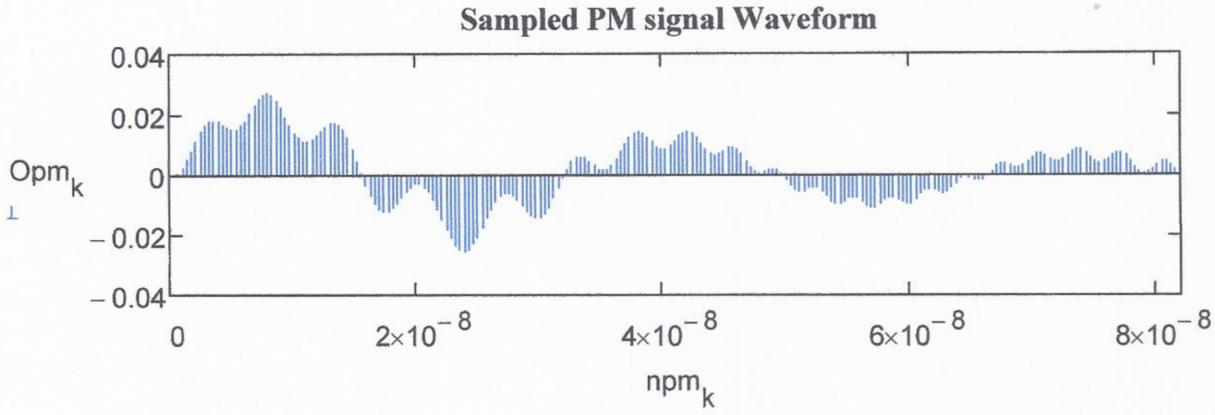


fig.:5.2.7.4

Fourier Transform of the Test signal

$$f_{cpm} = 0.244\cdot GHz \qquad \frac{f_{spm}}{f_{cpm}} = 11.31$$

$$m_p = 8 \qquad \omega_{mpm} = 0.077\cdot\frac{Grads}{sec} \qquad OSpecpm := fft(Opm) \qquad (5.2.7.12)$$

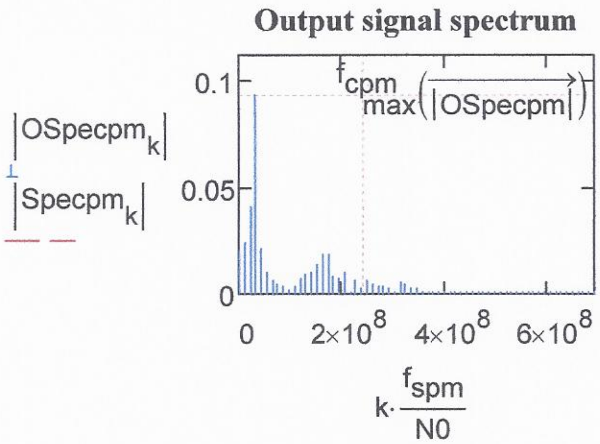


fig.:5.2.7.5

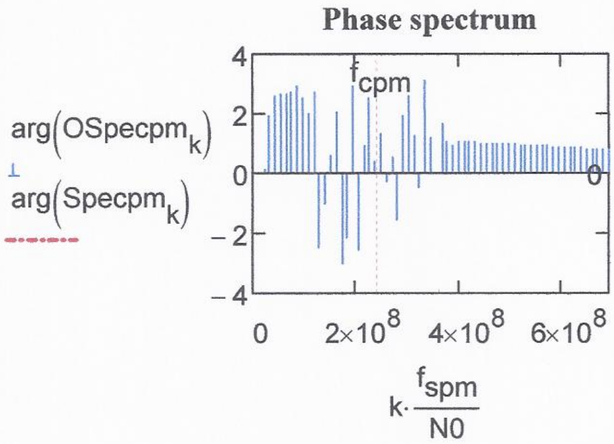


fig.:5.2.7.6

$$Wdb\omega_c := 20\cdot\log\left(\left|W_{lp}(j\cdot\omega_c)\right|\right) \qquad Wdb\omega_c = 10.392\cdot dB$$

ω_c is the angular frequency of the carrier

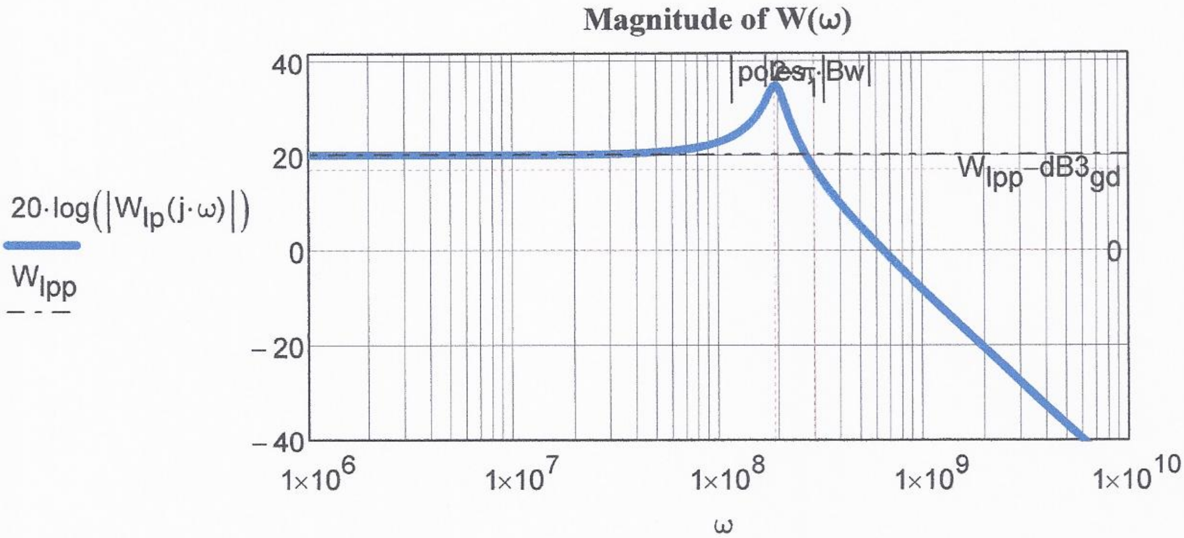

Magnitude of W(ω)
20·log(|Wlp(j·ω)|)
Wlpp
|poles1|
2π·Bw
Wlpp−dB3gd
0
40
20
0
−20
−40
1×10^6
1×10^7
1×10^8
1×10^9
1×10^10
ω

fig.:5.2.7.7

5.3
Analog-Equivalent Digital Low Pass II°order Filter

5.3.1 Z-transfer function of the II° Order Low Pass Digital Filter.

Now the previous analog results will be compared with the digital one

Consider the first order approximation with the change of variable: $s = \frac{1 - z^{-1}}{T_s}$ (5.3.1.1)

(see file" 1)DIGITAL FILTERS EQUIVALENT TO LINEAR CLASSICS - BASICS.xmcd, § 1.1.1)
From Laplace transform to Z transform),
and the following substitution into the transfer function

$A_5 := A_5 \quad \omega_5$

If $\zeta \neq \omega_5$ $\quad H_{lp}(z) := \frac{A_5 \cdot \omega_5^2}{s^2 + 2 \cdot \zeta \cdot s + \omega_5^2} \left| \begin{array}{l} \text{substitute}, s = \frac{1 - z^{-1}}{T_s} \\ \text{collect}, z \end{array} \right. \rightarrow$

otherwise $x = z^{-1}$: $H_{lp}(x) := A_5 \cdot \frac{\omega_5^2}{(s + \omega_5)^2} \left| \begin{array}{l} \text{substitute}, s = \frac{1 - x}{T_s} \\ \text{collect}, x \\ \text{factor} \end{array} \right. \rightarrow \frac{A_5 \cdot T_s^2 \cdot \omega_5^2}{(T_s \cdot \omega_5 - x + 1)^2}$

the result is:

$$H_{lp}(z) = \left| \begin{array}{ll} \frac{A_5 \cdot T_s^2 \cdot \omega_5^2}{z^{-2} - z^{-1} \cdot 2 \cdot (T_s \cdot \zeta + 1) + 2 \cdot T_s \cdot \zeta + T_s^2 \cdot \omega_5^2 + 1} & \text{if } \zeta \neq \omega_5 \\ \frac{A_5 \cdot T_s^2 \cdot \omega_5^2}{(-z^{-1} + T_s \cdot \omega_5 + 1)^2} & \text{otherwise} \end{array} \right. \quad (5.3.1.2)$$

To simplify define the following parameters:

Place the sampling period $T_s := T_{sstp}$ which is the one defined for the step function. In addition define he constants:

$A0 := A_5 \cdot \omega_5^2 \cdot T_s^2 \qquad B0 := 2 \cdot (1 + \zeta \cdot T_s) \qquad C0 := T_s \cdot (\omega_5^2 \cdot T_s + 2 \cdot \zeta) + 1 \qquad D0 := T_s \cdot \omega_5 + 1$

$A0 = -0.98696044 \qquad B0 = 2.0581776417 \qquad C0 = 1.1568736857 \qquad D0 = 1.314$

$T_s = 1.639 \times 10^{-6} \cdot ms$

you get the following result for the t. f. as a function of z:

$$H_{lp}(z) := \begin{cases} \dfrac{A0}{z^{-2} - B0 \cdot z^{-1} + C0} & \text{if } \zeta \neq \omega_5 \\ \dfrac{A0}{\left(D0 - z^{-1}\right)^2} & \text{otherwise} \end{cases} \qquad (5.3.1.3)$$

$$W1_{lpp} := 20 \cdot \log\left(\left|\frac{A_5 \cdot \omega_5}{2 \cdot \zeta}\right|\right)$$

BODE PLOTS (Low Pass (II° order)):

Magnitude in dB of Wlp compared with Hlp

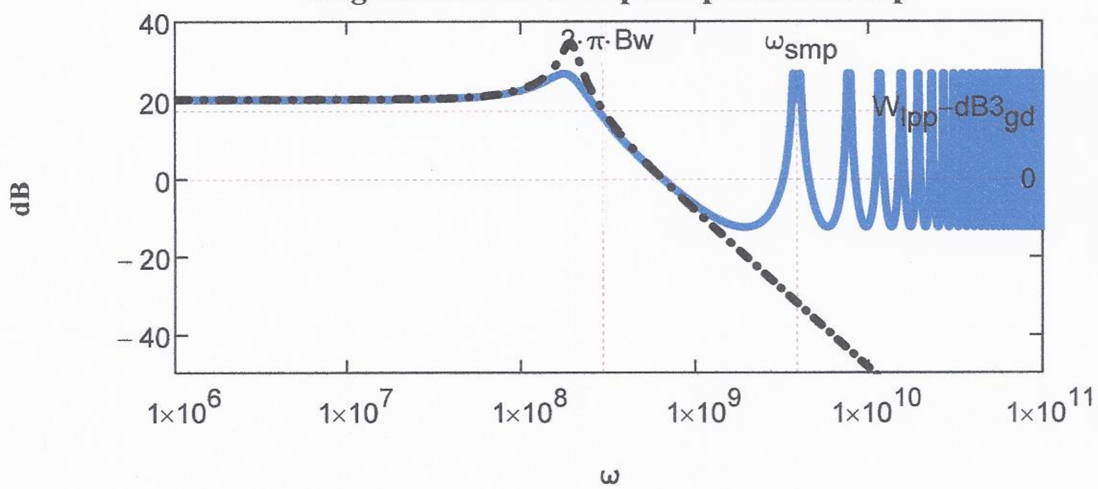


fig.:5.3.1.1

Phase

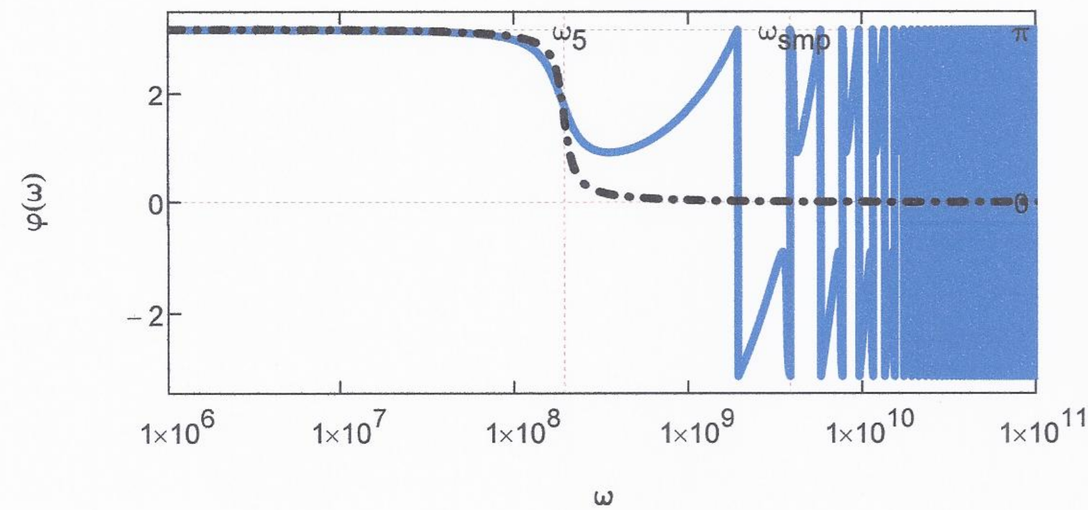


fig.:5.3.1.2

5.3.2 Difference equations Low Pass II°order filter. Canonical form.

Given the transfer function: $$H_{lp}(z) = \begin{cases} \dfrac{A0}{z^{-2} - B0\cdot z^{-1} + C0} & \text{if } \zeta \neq \omega_5 \\ \dfrac{A0}{\left(D0 - z^{-1}\right)^2} & \text{otherwise} \end{cases} \qquad (5.3.2.1)$$

Multiply and divide its definition for the same function G(z), so that

$$H_{lp}(z) = \frac{Y(z)}{X(z)} = \frac{Y(z)}{G(z)}\cdot\frac{G(z)}{X(z)} \qquad (5.3.2.2)$$

$$\frac{Y(z)}{G(z)} = A0$$

$$Y(z) = A0\cdot G(z)$$

$$\boxed{y(\nu) = A0\cdot g(\nu)} \qquad (5.3.2.3)$$

$$\frac{G(z)}{X(z)} = \begin{cases} \dfrac{1}{z^{-2} - B0\cdot z^{-1} + C0} & \text{if } \zeta \neq \omega_5 \\ \dfrac{1}{\left(D0 - z^{-1}\right)^2} & \text{otherwise} \end{cases} \qquad (5.3.2.4)$$

$$X(z) = \begin{cases} \left[\left(z^{-2} - B0\cdot z^{-1} + C0\right)\cdot G(z)\right] & \text{if } \zeta \neq \omega_5 \\ \left(D0 - z^{-1}\right)^2\cdot G(z) & \text{otherwise} \end{cases} \qquad (5.3.2.5)$$

$$X(z) = \begin{cases} \left(C0\cdot G(z) - B0\cdot z^{-1}\cdot G(z) + z^{-2}\cdot G(z)\right) & \text{if } \zeta \neq \omega_5 \\ G(z)\cdot D0^2 - 2\cdot G(z)\cdot D0\cdot z^{-1} + G(z)\cdot z^{-2} & \text{otherwise} \end{cases} \qquad (5.3.2.6)$$

$$\boxed{x(\nu) = \begin{cases} \left(C0\cdot g(\nu) - B0\cdot g(\nu-1) + g(\nu-2)\right) & \text{if } \zeta \neq \omega_5 \\ g(\nu)\cdot D0^2 - 2\cdot g(\nu-1)\cdot D0 + g(\nu-2) & \text{otherwise} \end{cases}} \qquad (5.3.2.7)$$

The corresponding set of difference equations is:

1) $$g(\nu) = \begin{cases} \dfrac{x(\nu) + B0\cdot g(\nu-1) - g(\nu-2)}{C0} & \text{if } \zeta \neq \omega_5 \\ \dfrac{x(\nu) + 2\cdot D0\cdot g(\nu-1) - g(\nu-2)}{D0^2} & \text{otherwise} \end{cases} \qquad (5.3.2.8)$$

2) $y(\nu) = A0\cdot g(\nu)$ (5.3.2.9)

$A0 := A0 \qquad B0 := B0 \qquad C0 := C0$

Z T. Initial value theorem: $\lim_{z \to \infty} \left(\frac{A0}{z^{-2} - B0\cdot z^{-1} + C0} \right) \to \frac{A0}{C0}$

Z T. Final value theorem: $\lim_{z \to 0} \left(\frac{A0}{z^{-2} - B0\cdot z^{-1} + C0} \right) \to 0$

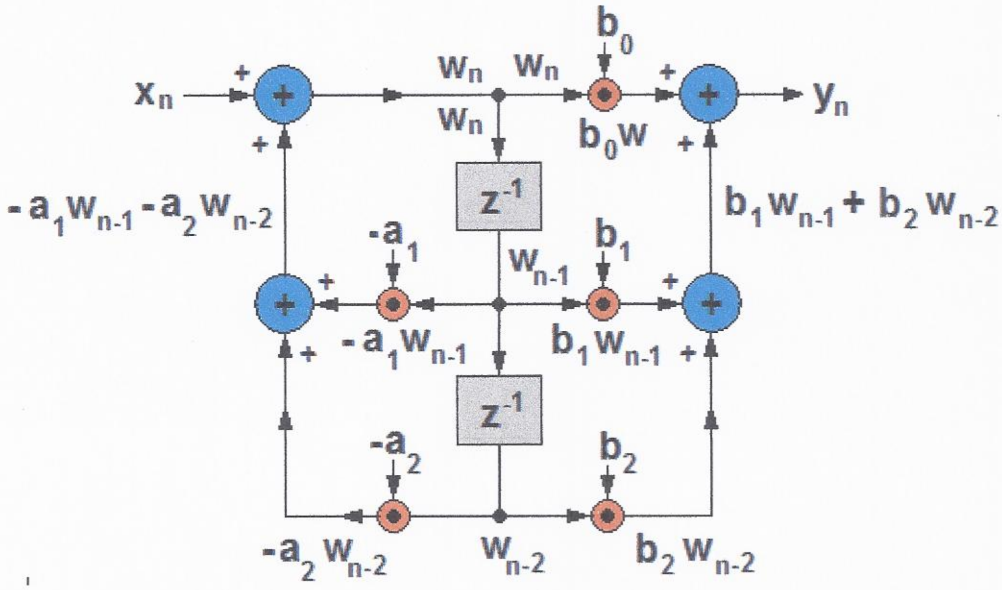


fig.:5.3.2.1

Recurrence relations:

1) $$g(\nu) := \begin{cases} \text{if } \nu > 1 \\ \quad \begin{cases} \dfrac{v_i(\nu) + B0\cdot g(\nu-1) - g(\nu-2)}{C0} & \text{if } \zeta \neq \omega_5 \\ \dfrac{v_i(\nu) + 2\cdot D0\cdot g(\nu-1) - g(\nu-2)}{D0^2} & \text{otherwise} \end{cases} \\ \text{if } \nu = 0 \\ \quad \begin{cases} \dfrac{v_i(0)}{C0} & \text{if } \zeta \neq \omega_5 \\ \dfrac{v_i(0)}{D0^2} & \text{otherwise} \end{cases} \\ \text{if } \nu = 1 \\ \quad \begin{cases} \dfrac{v_i(1) + B0\cdot g(0)}{C0} & \text{if } \zeta \neq \omega_5 \\ \dfrac{v_i(1) + 2\cdot D0\cdot g(0)}{D0^2} & \text{otherwise} \end{cases} \end{cases} \qquad (5.3.2.10)$$

2) $$y(\nu) := \begin{cases} A0\cdot g(\nu) & \text{if } \nu > 0 \\ 0 & \text{otherwise} \end{cases} \qquad (5.3.2.11)$$

Vectorized:

$Glp_k := 0$ $\qquad$ $rows(Glp) = 256$ $\qquad$ (5.3.2.12)

$$Glp_k := \begin{cases} \text{if } k > 1 & \begin{cases} \dfrac{v_i(k) + B0 \cdot Glp_{k-1} - Glp_{k-2}}{C0} & \text{if } \zeta \neq \omega_5 \\ \dfrac{v_i(k) + 2 \cdot D0 \cdot Glp_{k-1} - Glp_{k-2}}{D0^2} & \text{otherwise} \end{cases} \\ \text{if } k = 0 & \begin{cases} \dfrac{v_i(0)}{C0} & \text{if } \zeta \neq \omega_5 \\ \dfrac{v_i(0)}{D0^2} & \text{otherwise} \end{cases} \\ \text{if } k = 1 & \begin{cases} \dfrac{v_i(1) + B0 \cdot Glp_0}{C0} & \text{if } \zeta \neq \omega_5 \\ \dfrac{v_i(1) + 2 \cdot D0 \cdot Glp_0}{D0^2} & \text{otherwise} \end{cases} \end{cases} \qquad (5.3.2.13)$$

$$Y22_k := \begin{cases} A0 \cdot Glp_k & \text{if } k > 0 \\ 0 & \text{otherwise} \end{cases} \qquad \text{rows}(Y22) = \blacksquare \qquad (5.3.2.14)$$

Block diagram of the difference equation algorithm for a second order system

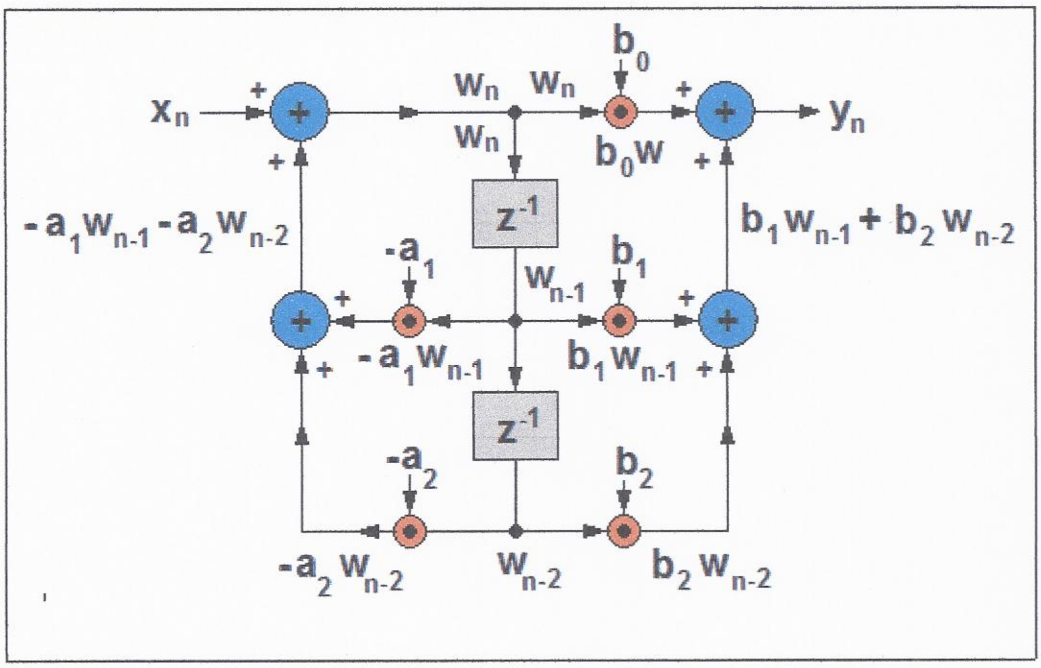


fig.:5.3.2.2

To save space when applying the previous algorithm, it is convenient to call the following program CANONIC2LP (it is the acronym of: Canonical Form Second Order Low Pass):

▶ CANONIC2LP

5.3 Equivalent Digital Low Pass Filter (II°order)

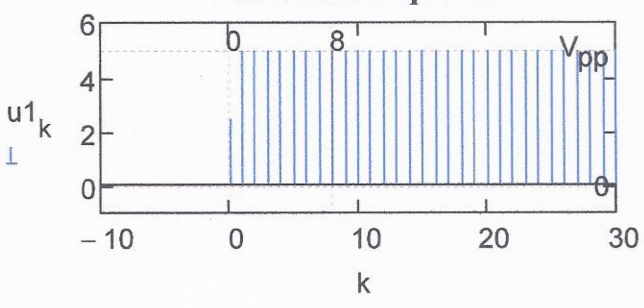
5.3.2.1 Sequence of the voltage Step response.

Given the filter's input signal: $u1_k := V_{stpsl}(nstp_k, V_{pp})$ (5.3.2.1.1)

and the filter's z transfer function, it will be calculated the output.

$V_{pp} = 5 \times 10^3 \cdot mV$

Unit Pulses Sequence.



rows(u1) = 256

fig.:5.3.2.1.1

Transfer function coefficients:

A0 := A0   B0 := B0   C0 := C0

Output's $\mathcal{Z}$ transform Initial value theorem: $\lim_{z \to \infty} \left[\frac{A0}{z^{-2} - B0 \cdot z^{-1} + C0} \cdot \frac{z}{(z-1)^2} \right] \to 0$

$v_i(\nu) := \frac{u1_\nu}{V}$ $v_i(0) = 2.5$ (5.3.2.1.2)

Numerical calculation of the filter's response to the input step function

$svsr := CANONIC2LP(v_i, A_5, \zeta, \omega_5, T_{sstp}, N0)$ (5.3.2.1.3)

svsr = (−0.987 2.058 1.157 {256,1} {256,1} 1.314)

Calculated transfer function coefficients:

$A01 := svsr_{0,0}$ $B01 := svsr_{0,1}$ $C01 := svsr_{0,2}$ $D01 := svsr_{0,5}$

A01 = −0.98696044   B01 = 2.0581776417   C01 = 1.1568736857

Sequences of the state function and of the output:

$Glp0 := svsr_{0,3}$ $Y00 := svsr_{0,4}$

Sequence of the voltage Step response.

$t := -1 \cdot T_{test}, -1 \cdot T_{test} + \frac{20 \cdot T_5 + 1 \cdot T_{test}}{10000} .. 20 \cdot T_5$ $Q_5 = 5.4$

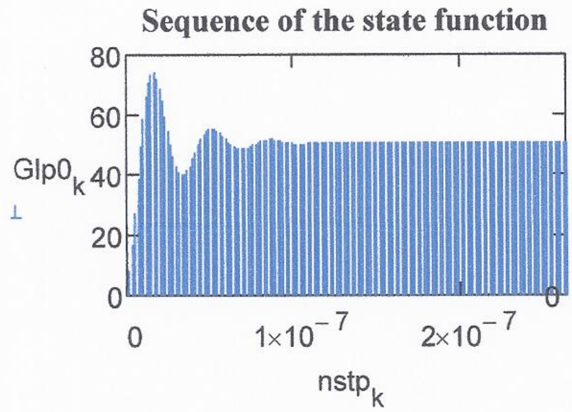


fig.:5.3.2.1.2

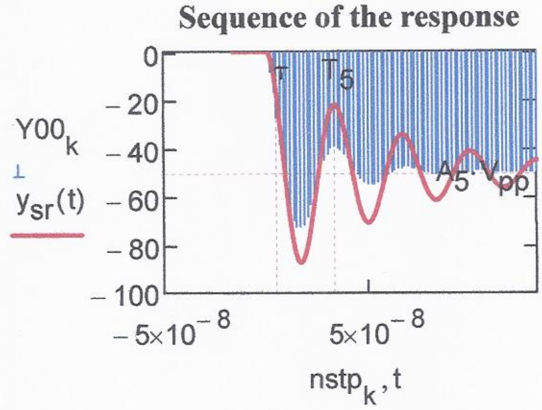


fig.:5.3.2.1.3

Spec0x := FFT(Y00)

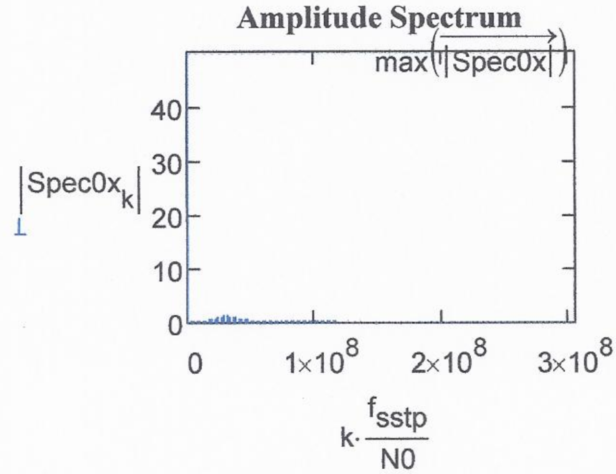


fig.:5.3.2.1.4

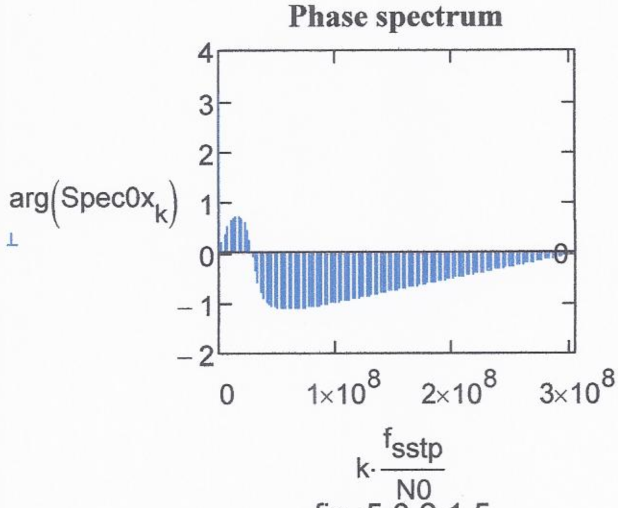


fig.:5.3.2.1.5

Bode plots of the $\mathcal{Z}$ transfer function:

$$H_{lp}(z) := \begin{cases} \dfrac{A01}{z^{-2} - B01 \cdot z^{-1} + C01} & \text{if } \zeta \neq \omega_5 \\ \dfrac{A01}{\left(D01 - z^{-1}\right)^2} & \text{otherwise} \end{cases} \qquad (5.3.2.1.4)$$

Frequency Responses for sampling period T_{sstp}

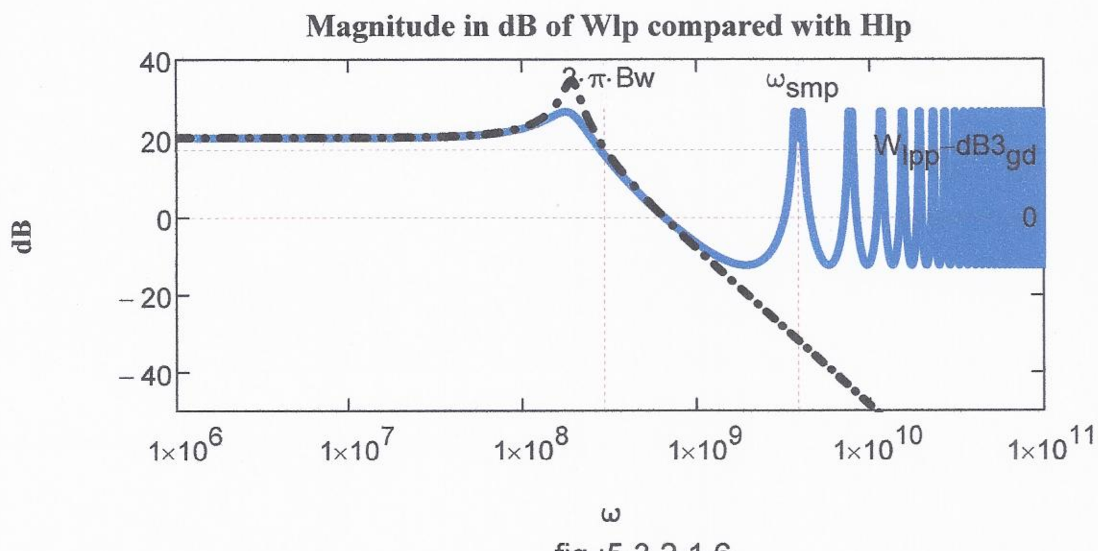


fig.:5.3.2.1.6

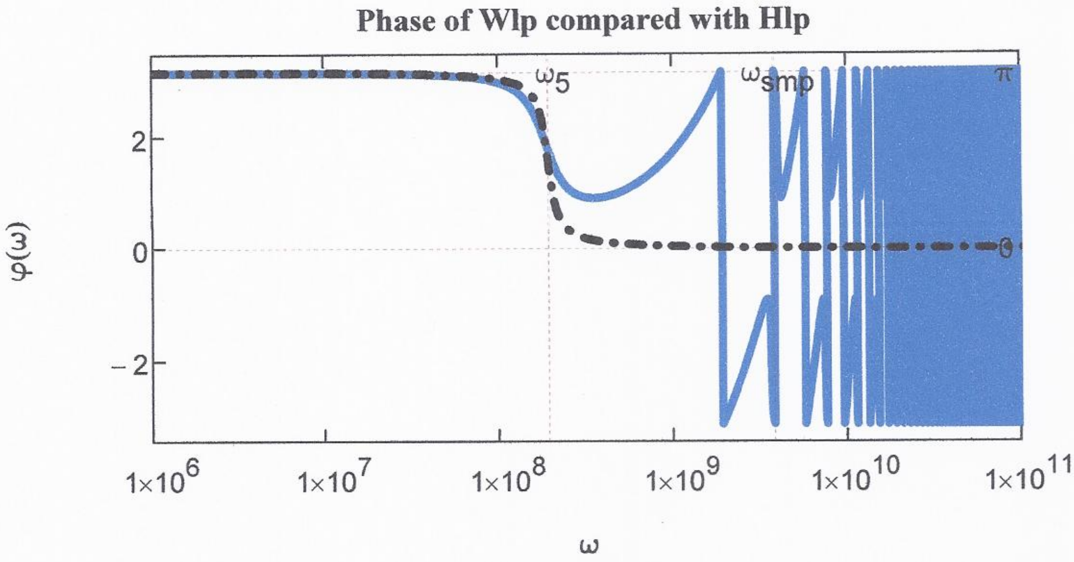

Phase of Wlp compared with Hlp
φ(ω)
2
0
-2
ω5
ωsmp
π
0
1×10^6
1×10^7
1×10^8
1×10^9
1×10^10
1×10^11
ω

fig.:5.3.2.1.7

5.3.2.2 Sequence of the Short Voltage Pulse response.

In this paragraph, given the second order low pass filter difference equations, it will be calculated numeric using the program "CANONIC2LP", the filter's response to a very short time duration voltage pulse:

$$V_{pp} = 5 \times 10^{3} \cdot mV \quad t := -(\tau_5 + 2 \cdot \tau_{pw}), -(\tau_5 + 2 \cdot \tau_{pw}) + \frac{2 \cdot (\tau_5 + 2 \cdot \tau_{pw})}{10000} .. \tau_5 + 2 \cdot \tau_{pw}$$

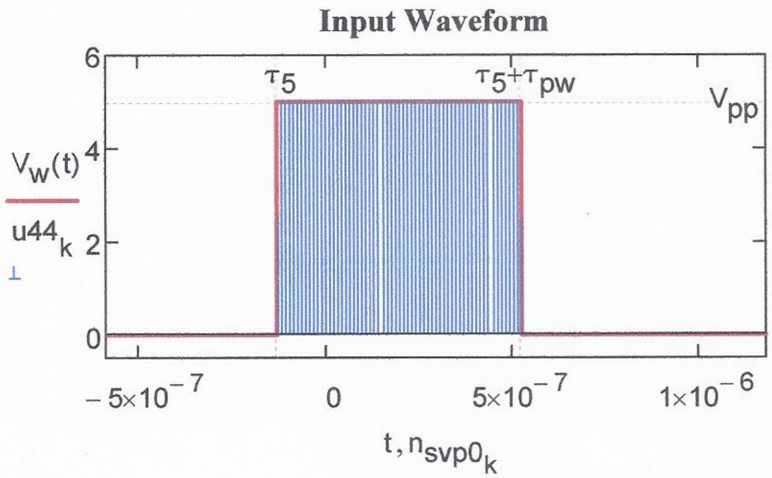


fig.:5.3.2.2.1

Sampling period of the short pulse defined earlier (5.2.2.25): $T_s := T_{svp}$

The first parameter of the program is the time discrete input function $v_i(\nu) := u44_\nu$

The calculation's results will be stored into the following vector:

$$svsr1 := CANONIC2LP(v_i, A_5, \zeta, \omega_5, T_s, N0) \qquad (5.3.2.2.1)$$

$$svsr1 = (-43.865 \quad 2.388 \quad 5.774 \quad \{256,1\} \quad \{256,1\} \quad 3.094)$$

Calculated transfer function coefficients:

$a1 := svsr1_{0,0}$ $\quad b1 := svsr1_{0,1}$ $\quad c1 := svsr1_{0,2}$ $\quad d01 := svsr1_{0,5}$

$a1 = -43.86490845$ $\quad b1 = 2.3878509449$ $\quad c1 = 5.7743417898$ $\quad d01 = 3.094$

Sequences of the state function and of the output:

$$Glp1 := svsr1_{0,3} \qquad Y01 := svsr1_{0,4}$$

Block diagram of the difference equation algorithm for a second order system

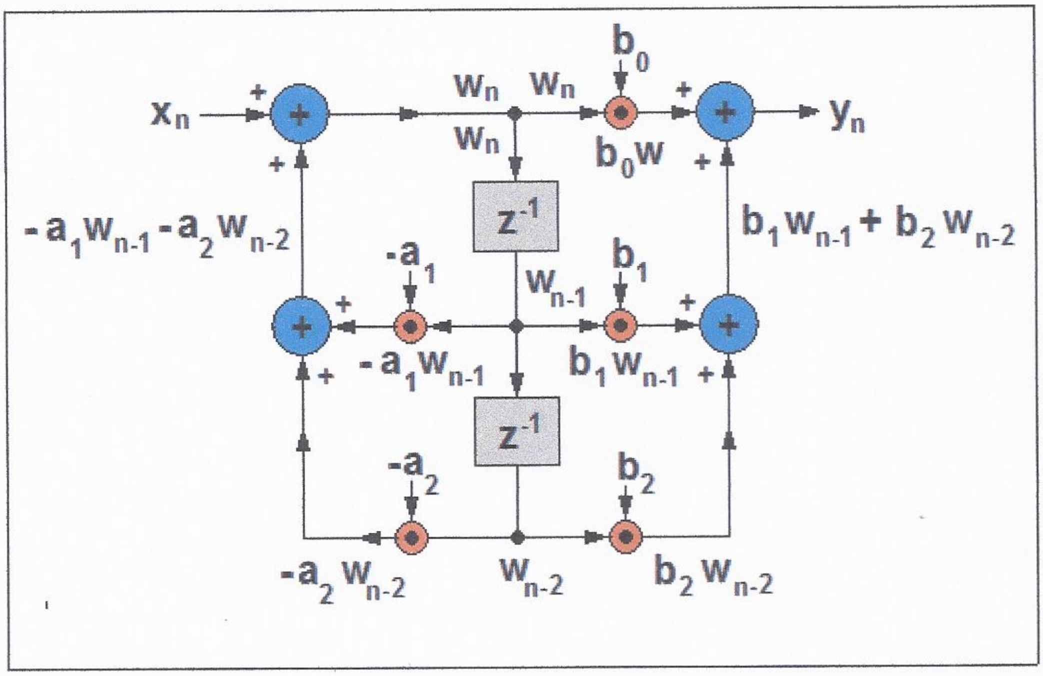


fig.:5.3.2.2.2

$$t := -0.1\cdot\tau_5, -0.1\cdot\tau_5 + \frac{\tau_5 + 8\cdot(\tau_{pw} + \tau_5) + 0.1\cdot\tau_5}{20000} .. \tau_5 + 8\cdot(\tau_{pw} + \tau_5)$$

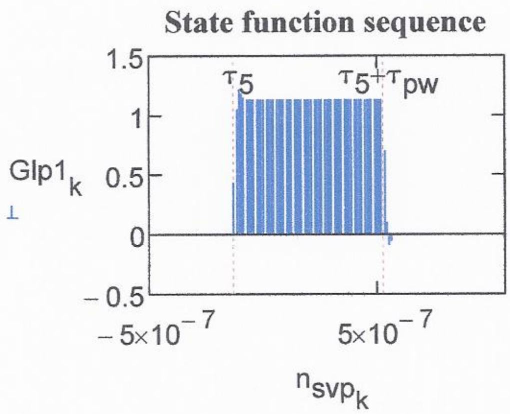


fig.:5.3.2.2.3

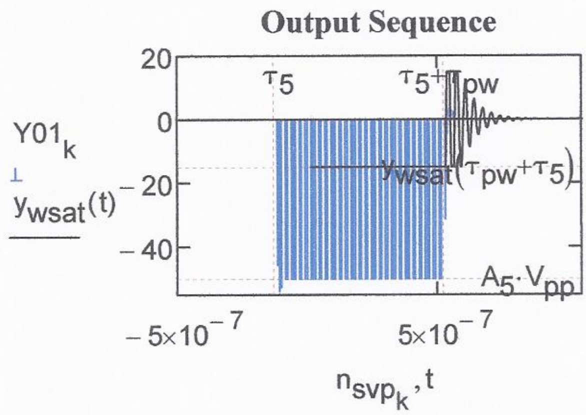


fig.:5.3.2.2.4

Sampled signal: $Spec1x := fft(Y01)$ (5.3.2.2.2)

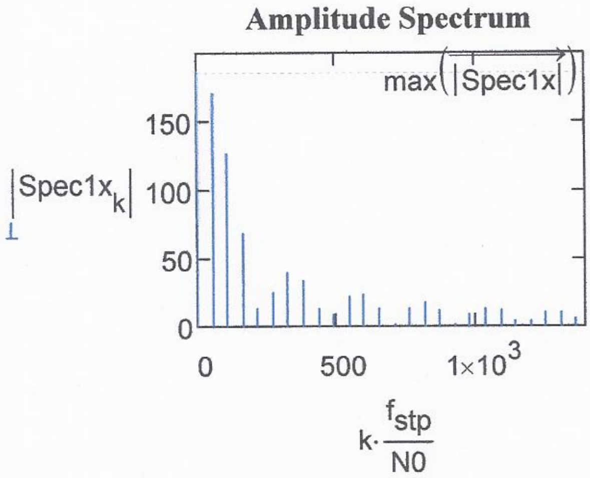


fig.:5.3.2.2.5

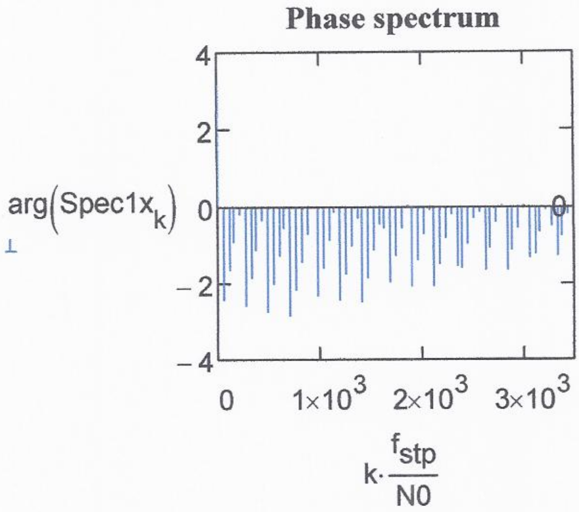


fig.:5.3.2.2.6

$$H_{lp}(z) := \begin{cases} \dfrac{a1}{z^{-2} - b1\cdot z^{-1} + c1} & \text{if } \zeta \neq \omega_5 \\ \dfrac{a1}{\left(d01 - z^{-1}\right)^2} & \text{otherwise} \end{cases} \tag{5.3.2.2.3}$$

Frequency Responses for sampling period T_{svp}

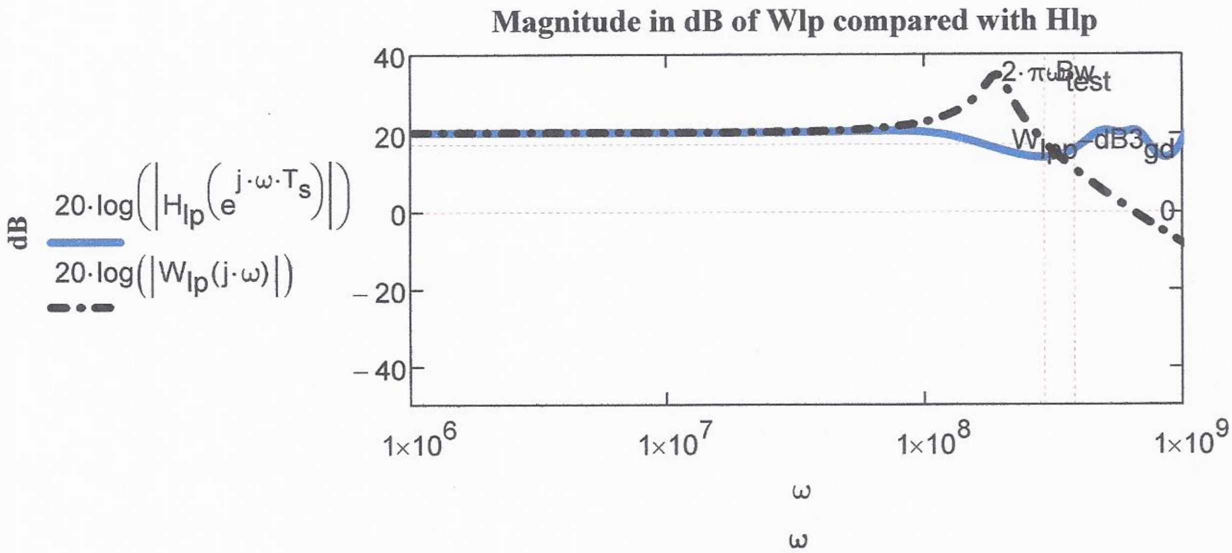


fig.:5.3.2.2.7

5.3.2.3 Sequence of the Sawtooth response

Place $T_s := T_{ssw}$ $\quad T_s = 1.708 \times 10^{-4} \cdot \mu s$ $\quad \omega_{smp} := \frac{2 \cdot \pi}{T_s}$

$$svsr2 := CANONIC2LP(v_i, A_5, \zeta, \omega_5, T_s, N0) \qquad (5.3.2.3.1)$$

$$svsr2 = (-0.011 \quad 2.006 \quad 1.007 \quad \{256,1\} \quad \{256,1\} \quad 1.033)$$

$A_0 := svsr2_{0,0}$ $\quad B_0 := svsr2_{0,1}$ $\quad C_0 := svsr2_{0,2}$ $\quad Glp2 := svsr2_{0,3}$ $\quad Y2 := svsr2_{0,4}$ $\quad D_0 := svsr2_{0,5}$

$A_0 = -0.01070921$ $\quad B_0 = 2.006060171$ $\quad C_0 = 1.0071310916$ $\quad D_0 = 1.033$

you get the following result for the t. f. as a function of z:

$T_s = 1.708 \times 10^{-4} \cdot \mu s$

$$H_{lp}(z) := \begin{vmatrix} \dfrac{A_0}{z^{-2} - B_0 \cdot z^{-1} + C_0} & \text{if } \zeta \neq \omega_5 \\ \dfrac{A_0}{\left(D_0 - z^{-1}\right)^2} & \text{otherwise} \end{vmatrix} \qquad (5.3.2.3.2)$$

$$W1_{lpp} := 20 \cdot \log\left(\left|\frac{A_5 \cdot \omega_5}{2 \cdot \zeta}\right|\right)$$

BODE PLOTS (Low Pass (II° order)):

$\omega_{smp} = 36.792 \cdot \frac{Grads}{sec}$

Frequency Responses for sampling period T_{ssw}

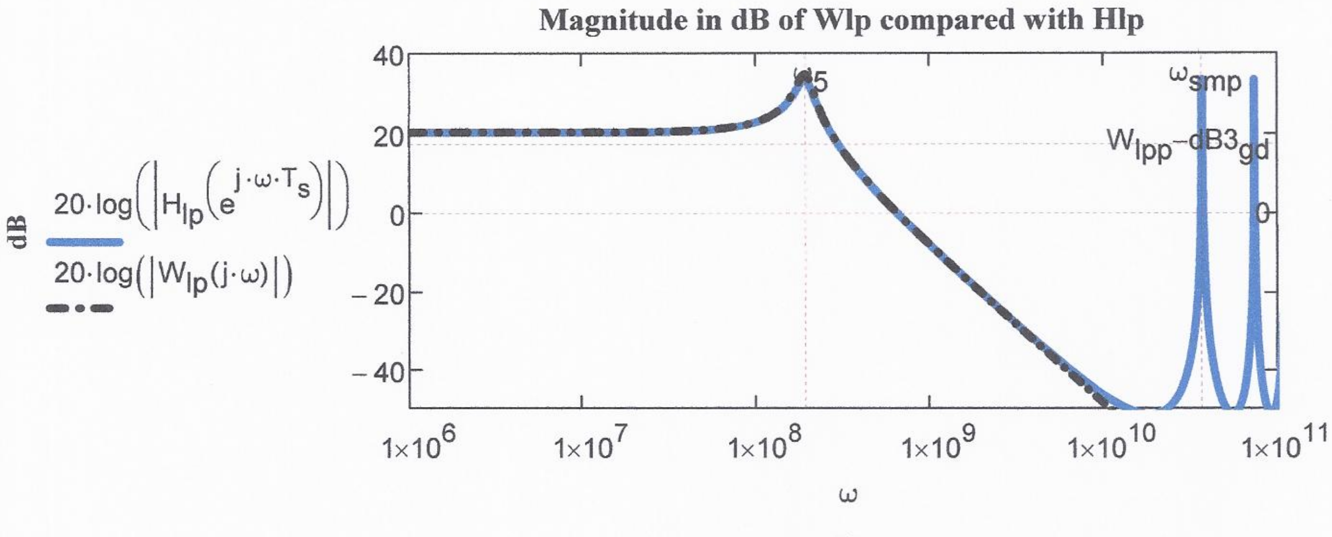


fig.:5.3.2.3.1

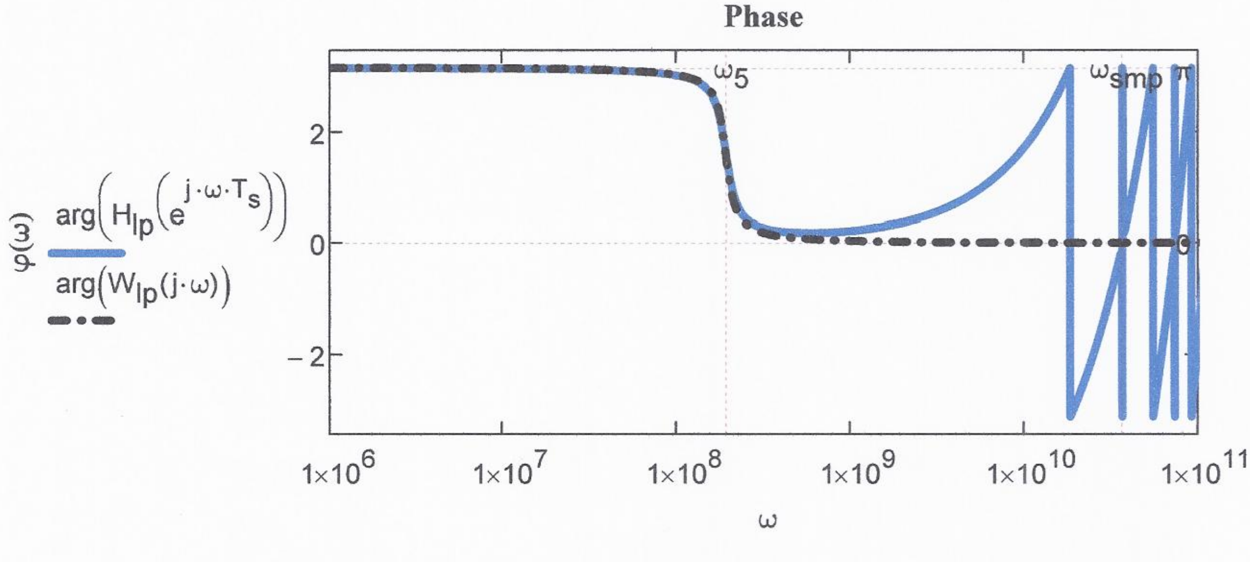


fig.:5.3.2.3.2

Digital first order Low pass filter difference equations:

$V_{pp} = 5 \times 10^3 \cdot mV$ $\qquad u55_k := v1_{sw}\left(n_{sw_k}, T_{test}, V_{pp}\right)$ $\qquad$ (5.3.2.3.3)

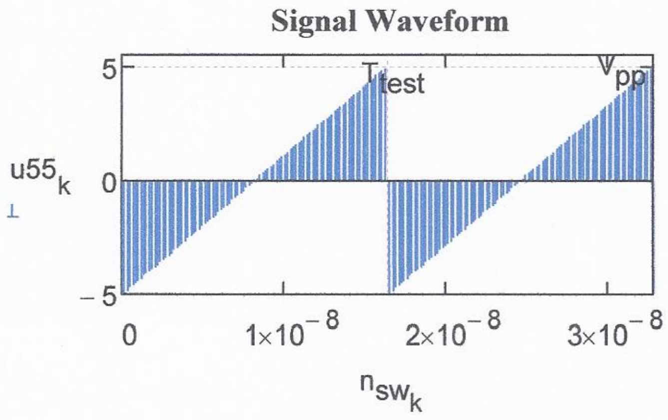


fig.:5.3.2.3.3

$$\omega_{smp} := \frac{2\cdot\pi}{T_s} \qquad v3_i(\nu) := \frac{u55_\nu}{V} \tag{5.3.2.3.4}$$

$$svsr3 := CANONIC2LP\left(v3_i, A_5, \zeta, \omega_5, T_s, N0\right) \tag{5.3.2.3.5}$$

$$svsr3 = (-0.011 \quad 2.006 \quad 1.007 \quad \{256,1\} \quad \{256,1\} \quad 1.033)$$

$a3 := svsr3_{0,0}$ $b3 := svsr3_{0,1}$ $c3 := svsr3_{0,2}$ $Glp3 := svsr3_{0,3}$ $Y3 := svsr3_{0,4}$ $d3 := svsr3_{0,5}$

$a3 = -0.01070921$ $b3 = 2.006060171$ $c3 = 1.0071310916$ $d3 = 1.033$

you get the following result for the t. f. as a function of z:

$$T_s = 1.708\times 10^{-4}\cdot\mu s \qquad H_{lp}(z) := \begin{cases} \dfrac{a3}{z^{-2} - b3\cdot z^{-1} + c3} & \text{if } \zeta \neq \omega_5 \\ \dfrac{a3}{\left(d3 - z^{-1}\right)^2} & \text{otherwise} \end{cases} \tag{5.3.2.3.6}$$

Block diagram of the difference equation algorithm for a second order system

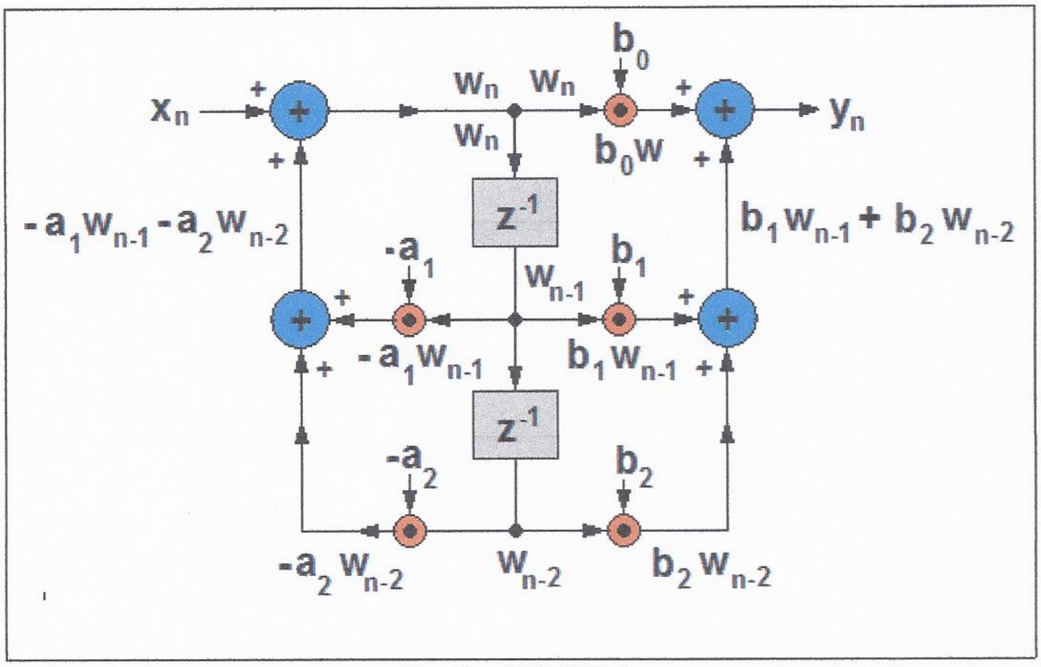


fig.:5.3.2.3.4

$$t := 0\cdot T_{test}, 0\cdot T_{test} + \frac{20\cdot T_{test} - 0\cdot T_{test}}{1000} .. 20\cdot T_{test}$$

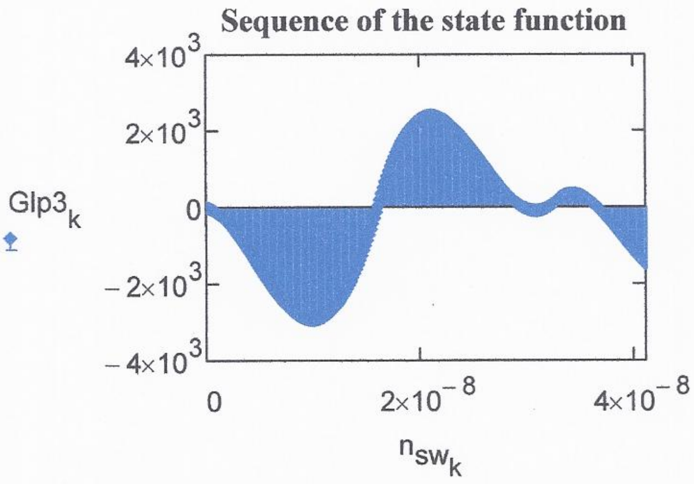


fig.:5.3.2.3.5

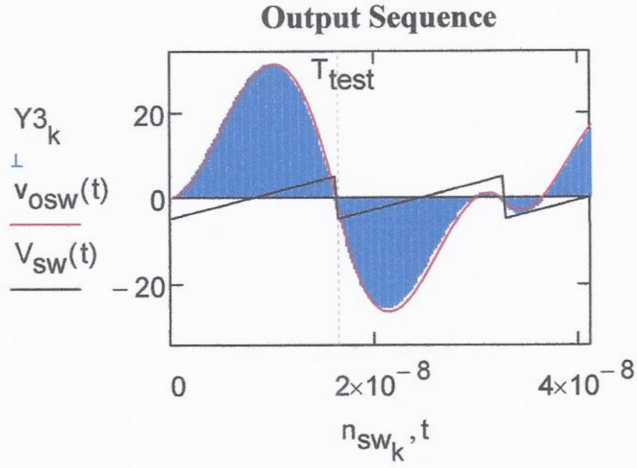


fig.:5.3.2.3.6

$$\text{Spec3x} := \text{FFT}(\text{Y3}) \qquad (5.3.2.3.7)$$

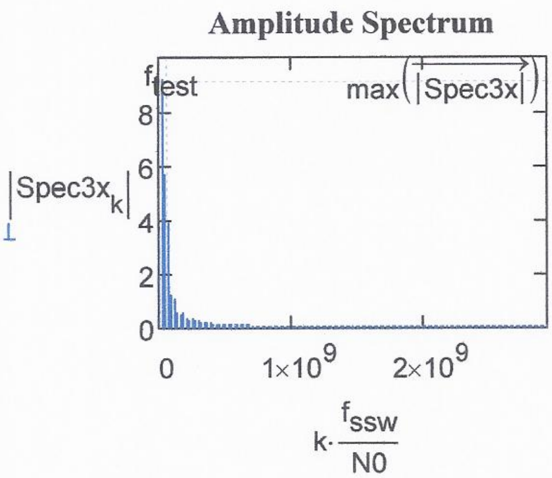


fig.:5.3.2.3.7

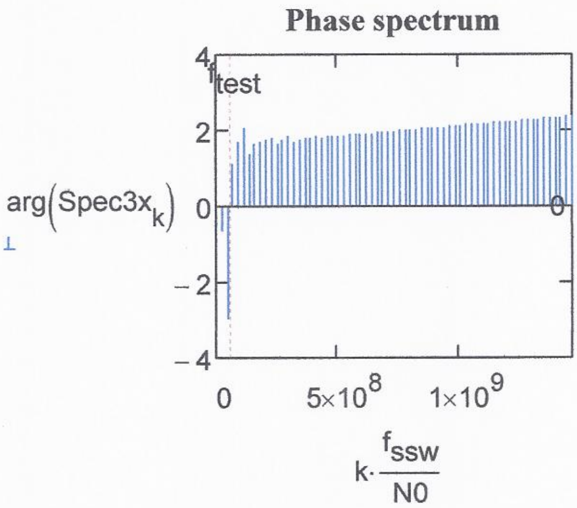


fig.:5.3.2.3.8

5.3 Equivalent Digital Low Pass Filter (II°order)

5.3.2.4 Sequence of the Bipolar Square Wave response.

$$u66_k := V_{sqwb}\left(n_{sqw_k}\right) \qquad (5.3.2.4.1)$$

$$t := 0\cdot T_{test}, 0\cdot T_{test} + \frac{4\cdot T_{test} - 0\cdot T_{test}}{1000} .. 4\cdot T_{test}$$

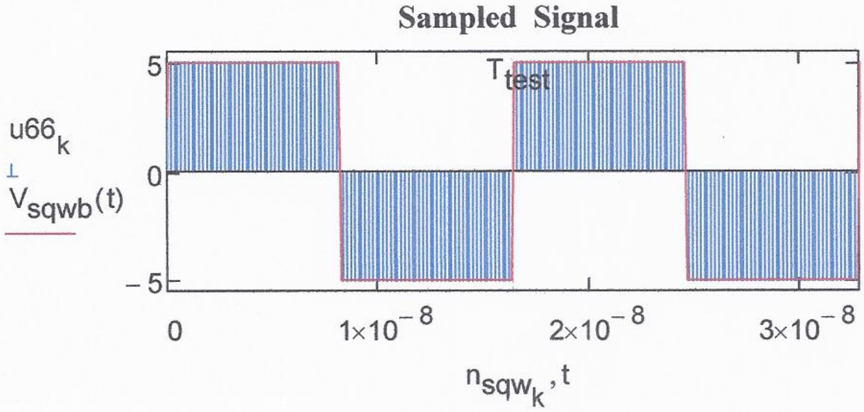


fig.:5.3.2.4.1

Place $T_s := T_{ssqw}$ $\qquad T_s = 0.171\cdot ns \qquad \omega_{smp} := \frac{2\cdot\pi}{T_s}$

$$v4_i(\nu) := u66_\nu \qquad (5.3.2.4.2)$$

$$svsr4 := CANONIC2LP\left(v4_i, A_5, \zeta, \omega_5, T_s, N0\right) \qquad (5.3.2.4.3)$$

$$svsr4 = (-0.011 \quad 2.006 \quad 1.007 \quad \{256,1\} \quad \{256,1\} \quad 1.033)$$

$a4 := svsr4_{0,0}$ $\quad b4 := svsr4_{0,1}$ $\quad c4 := svsr4_{0,2}$ $\quad Glp4 := svsr4_{0,3}$ $\quad Y4 := svsr4_{0,4}$ $\quad d4 := svsr4_{0,5}$

$a4 = -0.01070921 \qquad b4 = 2.006060171 \qquad c4 = 1.0071310916 \qquad d4 = 1.033$

you get the following result for the t. f. as a function of z:

$T_s = 1.708\times10^{-7}\cdot ms$

$$H_{lp}(z) := \begin{cases} \dfrac{a4}{z^{-2} - b4\cdot z^{-1} + c4} & \text{if } \zeta \neq \omega_5 \\ \dfrac{a4}{\left(d4 - z^{-1}\right)^2} & \text{otherwise} \end{cases} \qquad (5.3.2.4.4)$$

$$W1_{lpp} := 20\cdot\log\left(\left|\frac{A_5\cdot\omega_5}{2\cdot\zeta}\right|\right)$$

BODE PLOTS (Low Pass (II° order)):

Frequency Responses for sampling period T_{ssqw}

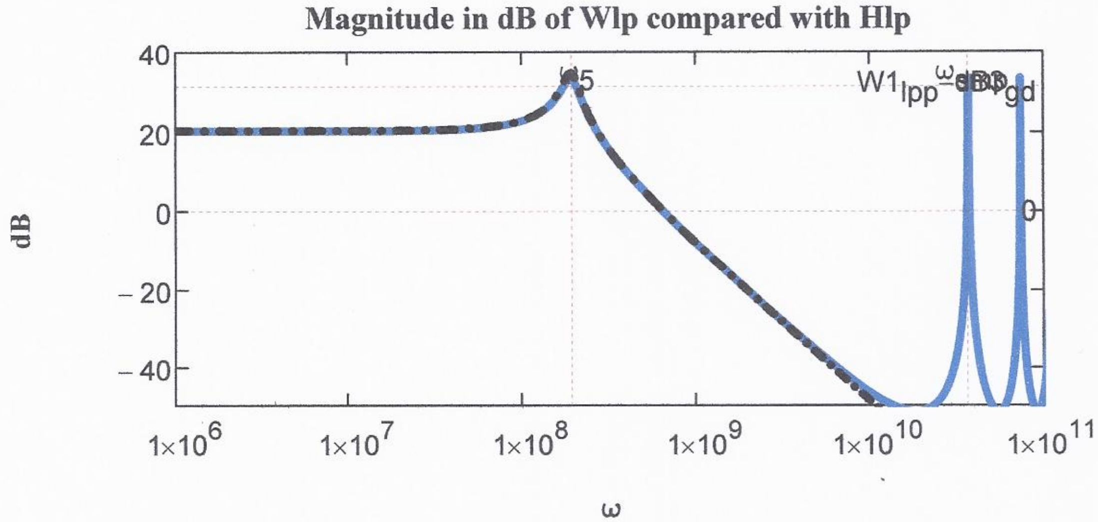


fig.:5.3.2.4.2

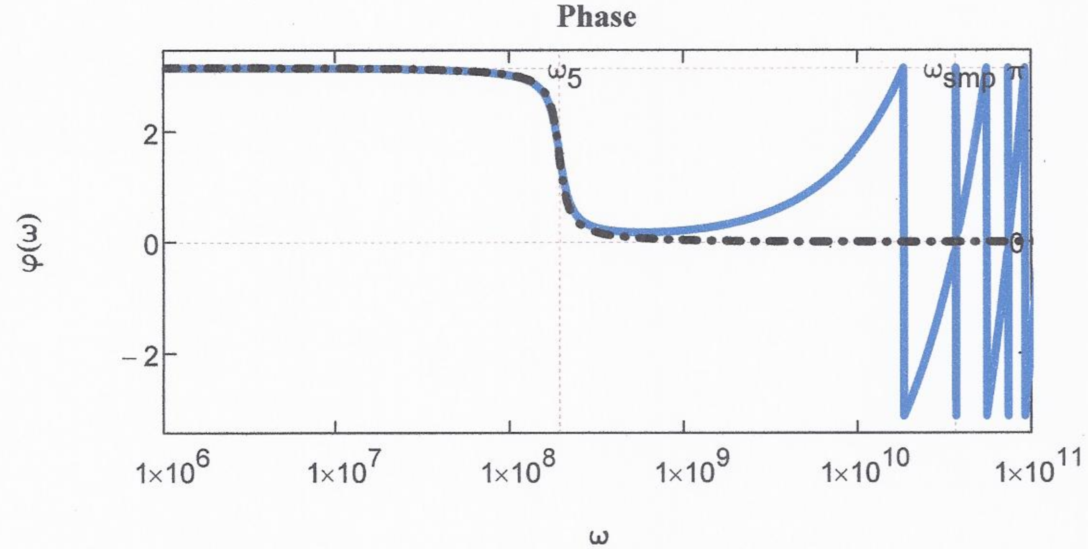


fig.:5.3.2.4.3

Block diagram of the difference equation algorithm for a second order system

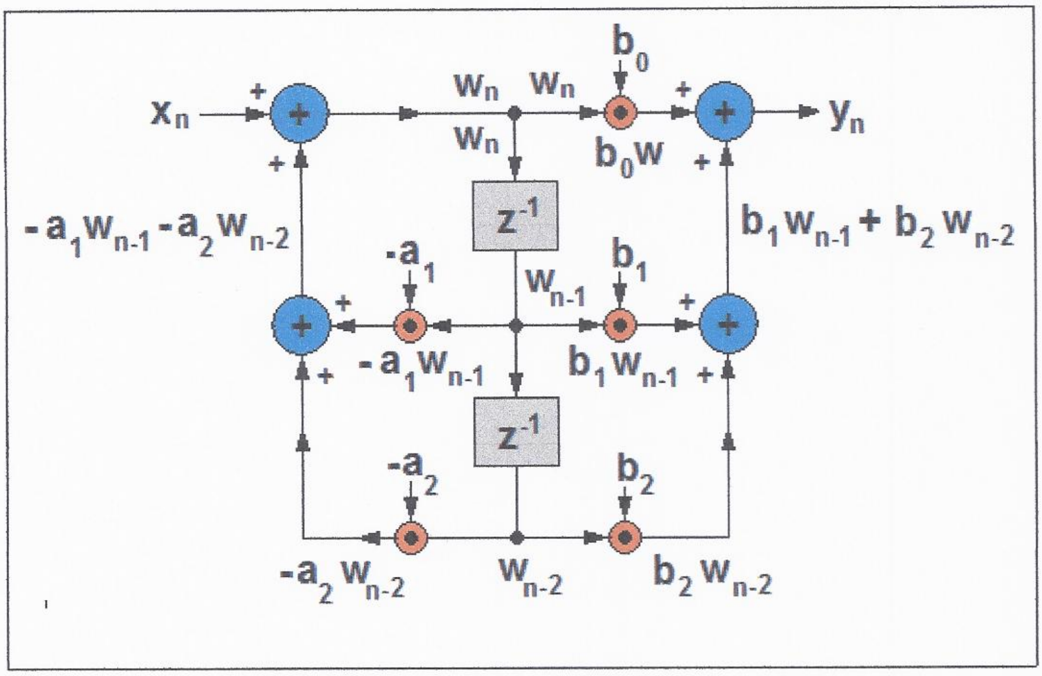


fig.:5.3.2.4.4

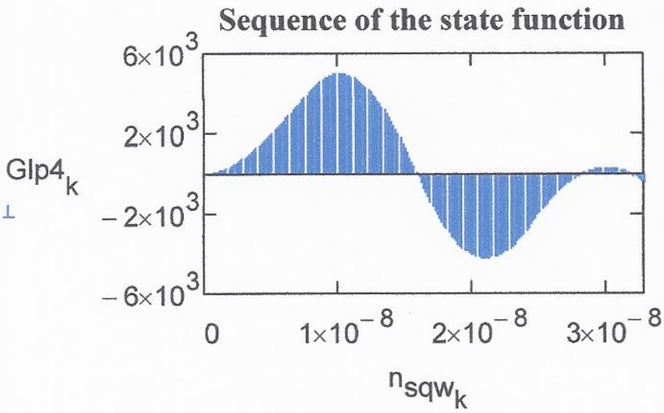


fig.:5.3.2.4.5

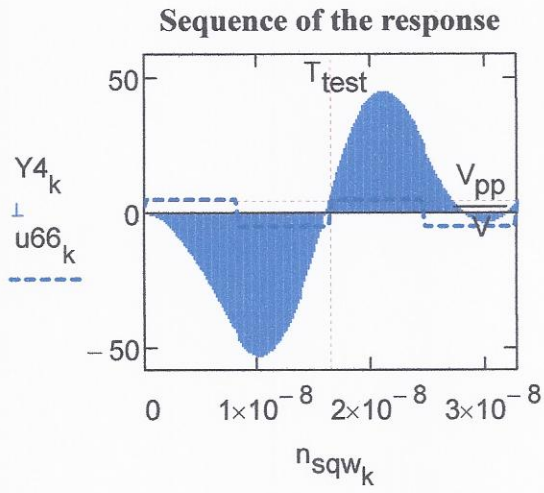


fig.:5.3.2.4.6

Spec4x := FFT(Y4)

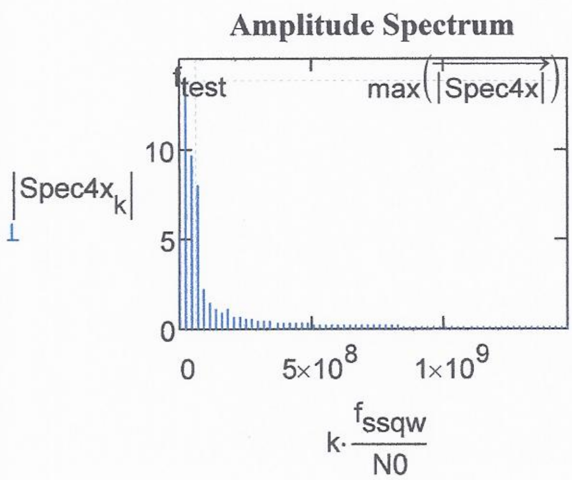


fig.:5.3.2.4.7

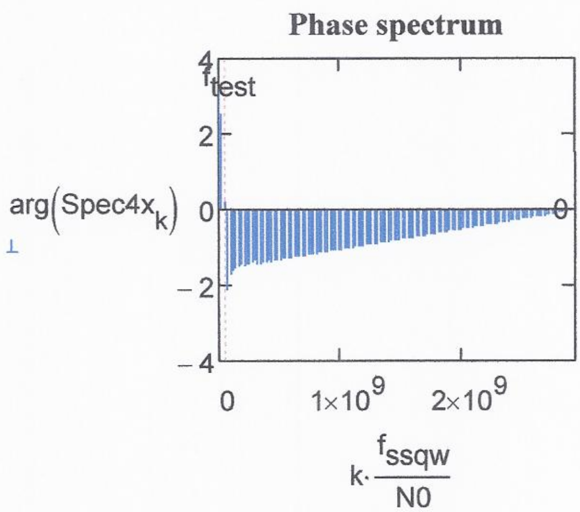


fig.:5.3.2.4.8

5.3 Equivalent Digital Low Pass Filter (II°order)

5.3.2.5 (single tone) Sequence of the AM Signal response.

$$u77_k := v2_i\left(nam_k, \omega_{mam}, \omega_c, A1, B1\right) \qquad (5.3.2.5.1)$$

$\omega_{mam} = 0.038 \cdot \frac{Grads}{sec}$ $\quad \omega_c = 0.383 \cdot \frac{Grads}{sec}$ $\quad \frac{\omega_c}{\omega_{mam}} = 10$ $\quad \frac{N0}{f_{sam}} \cdot \frac{1}{T_5} = 53.333$

$$Spec77 := fft(u77) \qquad (5.3.2.4.2)$$

$$t := 0 \cdot T_{mam}, 0 \cdot T_{mam} + \frac{3 \cdot T_{mam}}{1000} .. 3 \cdot T_{mam}$$

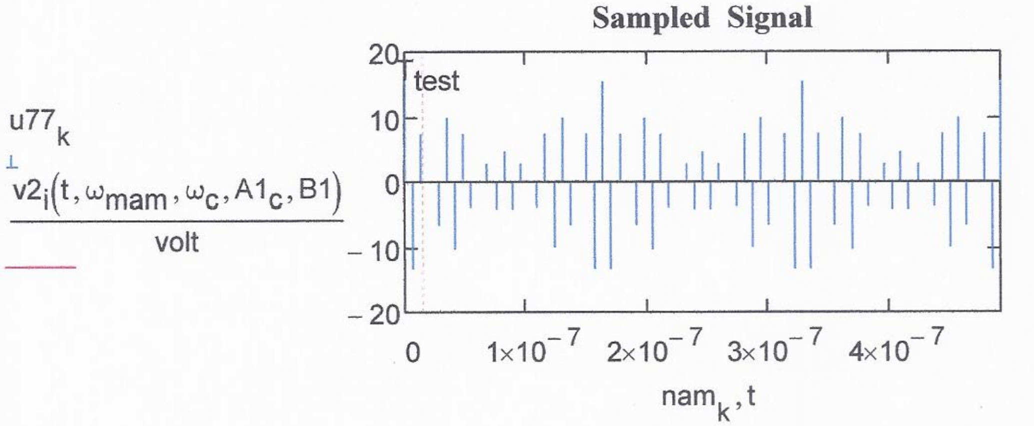


fig.:5.3.2.5.1

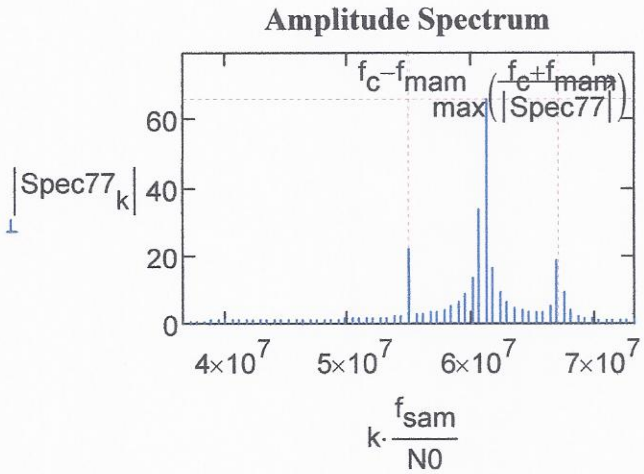


fig.:5.3.2.5.2

Place $T_s := T_{sam}$ $\qquad T_s = 6.831 \cdot ns$ $\qquad \omega_{smp} := \frac{2 \cdot \pi}{T_s}$

$$v5_i(\nu) := \frac{u77_\nu}{V} \qquad (5.3.2.4.3)$$

$$svsr5 := CANONIC2LP\left(v5_i, A_5, \zeta, \omega_5, T_s, N0\right) \qquad (5.3.2.4.4)$$

$$svsr5 = (-17.135 \quad 2.242 \quad 2.956 \quad \{256,1\} \quad \{256,1\} \quad 2.309)$$

$a5 := svsr5_{0,0}$ $b5 := svsr5_{0,1}$ $c5 := svsr5_{0,2}$ $Glp5 := svsr5_{0,3}$ $Y5 := svsr5_{0,4}$ $d5 := svsr5_{0,5}$

$a5 = -17.13472986$ $b5 = 2.2424068406$ $c5 = 2.9558798269$ $d5 = 2.309$

you get the following result for the t. f. as a function of z:

$$T_s = 6.831 \times 10^{-6} \cdot ms \quad H_{lp}(z) := \begin{cases} \dfrac{a5}{z^{-2} - b5 \cdot z^{-1} + c5} & \text{if } \zeta \neq \omega_5 \\ \dfrac{a5}{\left(d5 - z^{-1}\right)^2} & \text{otherwise} \end{cases} \tag{5.3.2.4.5}$$

$$W1_{lpp} := 20 \cdot \log\left(\left|\frac{A_5 \cdot \omega_5}{2 \cdot \zeta}\right|\right)$$

BODE PLOTS (Low Pass (II° order)):

Frequency Responses for sampling period T_{sam}

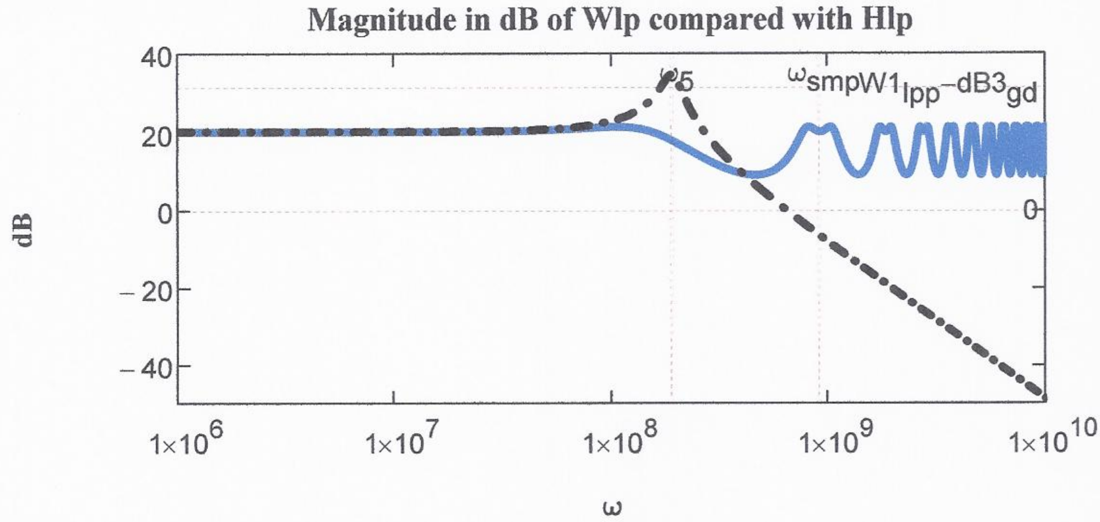


fig.:5.3.2.5.3

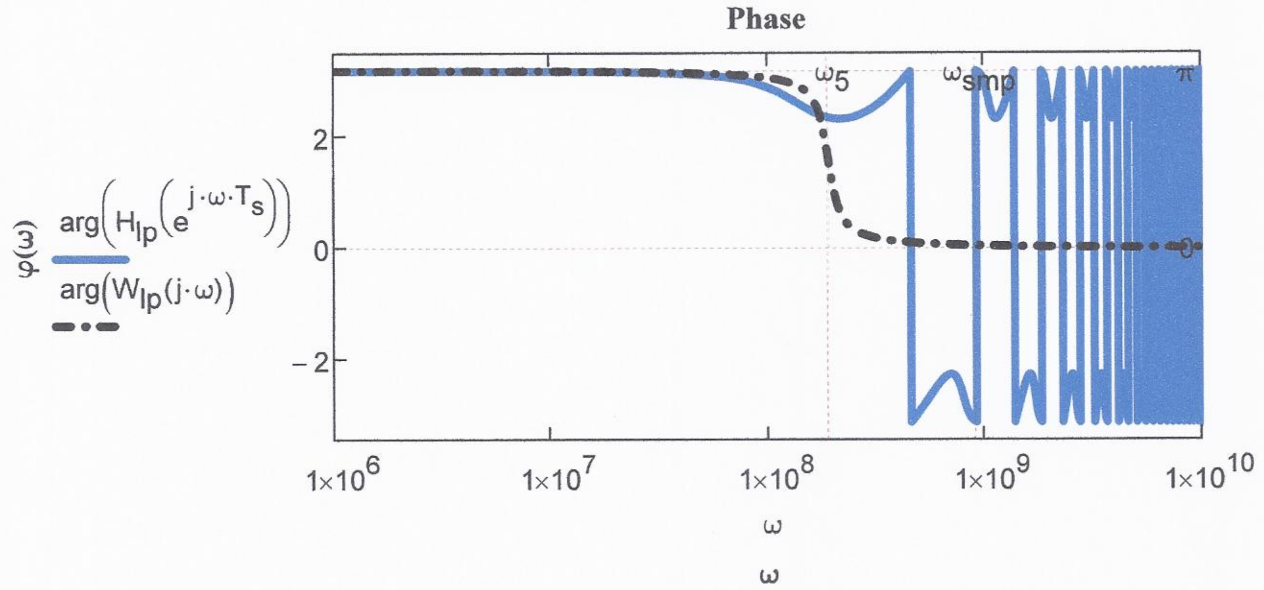


fig.:5.3.2.5.4

Block diagram of the difference equation algorithm for a second order system

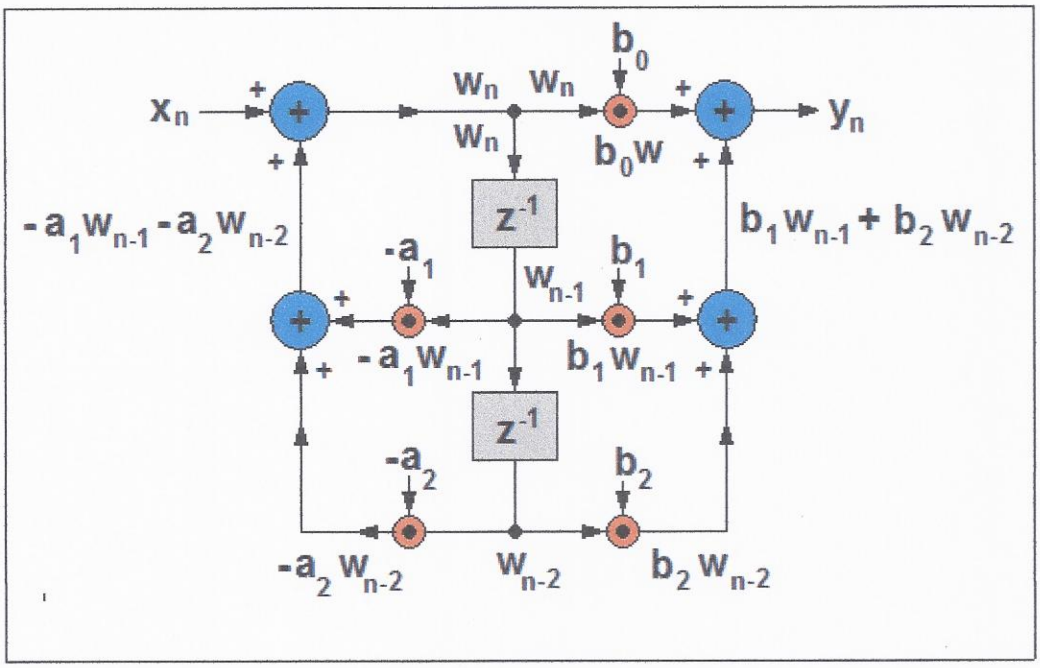


fig.:5.3.2.5.5

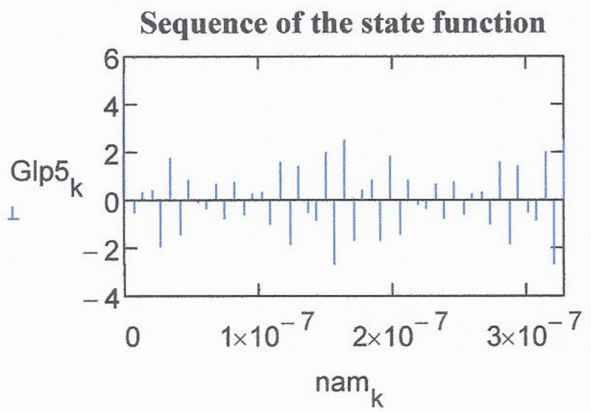


fig.:5.3.2.5.6

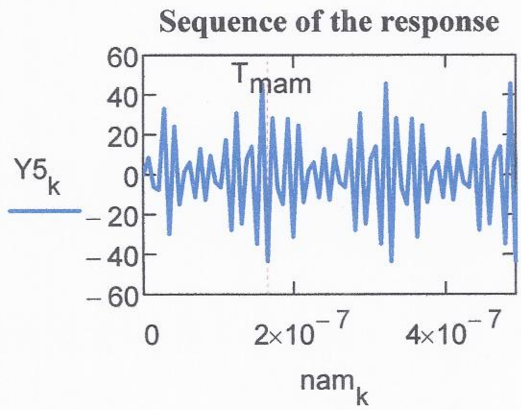


fig.:5.3.2.5.7

$Spec5x := FFT(Y5) \qquad max\left(\overrightarrow{|Spec5x|}\right) = 12.148$

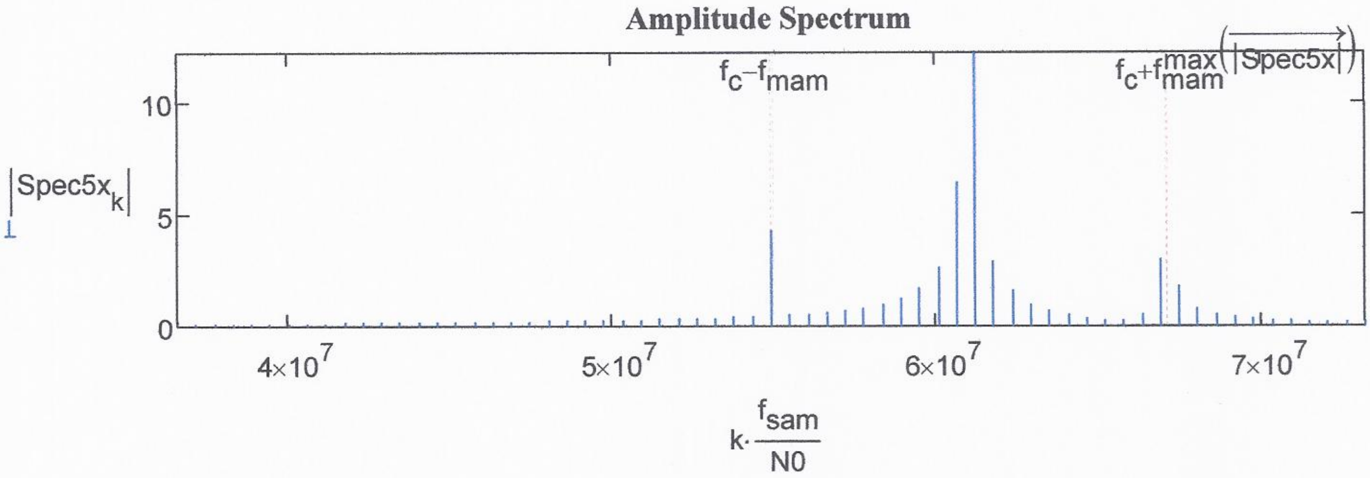


fig.:5.3.2.5.8

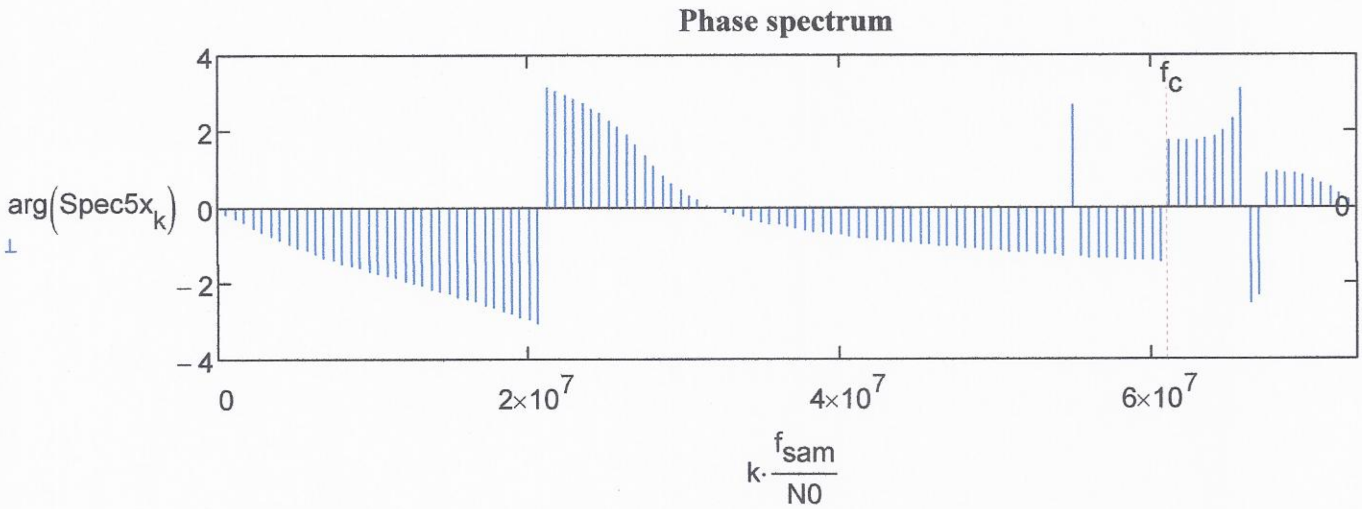


fig.:5.3.2.5.9

5.3 Equivalent Digital Low Pass Filter (II°order)

5.3.2.6 Sequence of the (single tone) Frequency Modulated carrier response.

$A_{fm} = 0.02V$ $\quad m_f = 8$ $\quad \omega_{mfm} = 1.916 \times 10^4 \cdot \frac{krads}{sec}$ $\quad \omega_c = 0.383 \cdot \frac{Grads}{sec}$

$$u\_9_k := \frac{v_{fmsl}(nfm_k, \omega_c, \omega_{mfm}, A_{fm}, m_f)}{volt} \qquad (5.3.2.6.1)$$

$$Spec\_9 := fft(u\_9) \qquad (5.3.2.6.2)$$

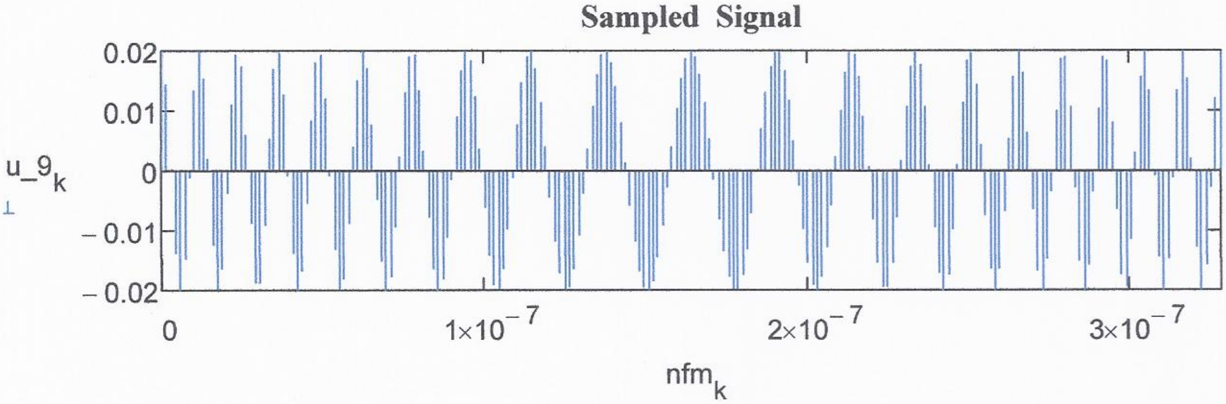


fig.:5.3.2.6.1

Place $T_s := T_{sfm}$ $\quad T_s = 1.45 \cdot ns$ $\quad \omega_{smp} := \frac{2 \cdot \pi}{T_s}$

$$v6_i(\nu) := u\_9_\nu \qquad (5.3.2.6.3)$$

$$svsr6 := CANONIC2LP(v6_i, A_5, \zeta, \omega_5, T_s, N0) \qquad (5.3.2.6.4)$$

$$svsr6 = (-0.772 \quad 2.051 \quad 1.129 \quad \{256,1\} \quad \{256,1\} \quad 1.278)$$

$a6 := svsr6_{0,0}$ $\quad b6 := svsr6_{0,1}$ $\quad c6 := svsr6_{0,2}$ $\quad Glp6 := svsr6_{0,3}$ $\quad Y6 := svsr6_{0,4}$ $\quad d6 := svsr6_{0,5}$

$a6 = -0.77160494$ $\quad b6 = 2.0514403292$ $\quad c6 = 1.128600823$ $\quad d6 = 1.278$

you get the following result for the t. f. as a function of z:

$T_s = 1.45 \times 10^{-6} \cdot ms$

$$H_{lp}(z) := \begin{cases} \frac{a6}{z^{-2} - b6 \cdot z^{-1} + c6} & \text{if } \zeta \neq \omega_5 \\ \frac{a6}{(d6 - z^{-1})^2} & \text{otherwise} \end{cases} \qquad (5.3.2.6.5)$$

$$W1_{lpp} := 20 \cdot \log\left(\left|\frac{A_5 \cdot \omega_5}{2 \cdot \zeta}\right|\right)$$

BODE PLOTS (Low Pass (II° order)):

Frequency Responses for sampling period T_{sfm}

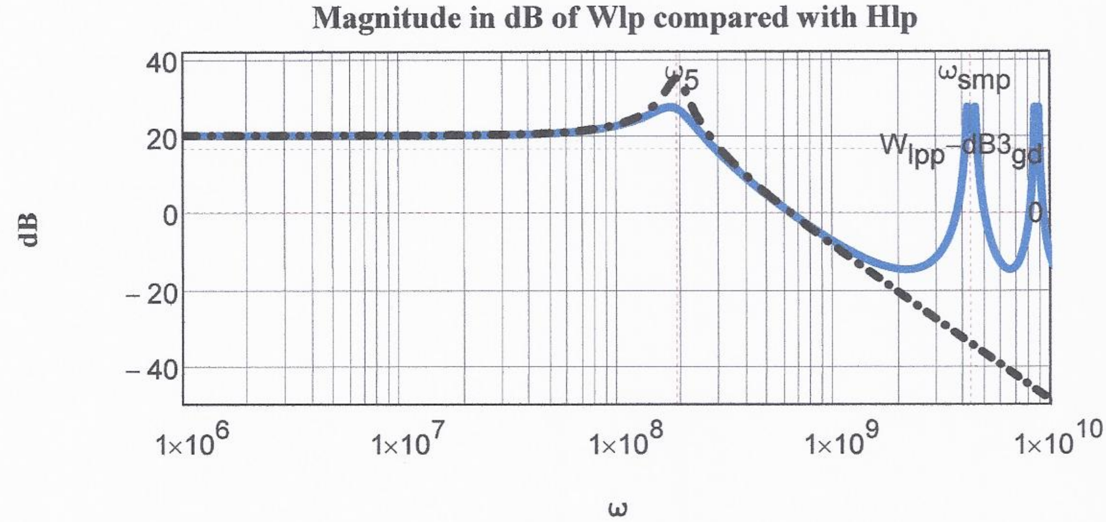


fig.:5.3.2.6.2

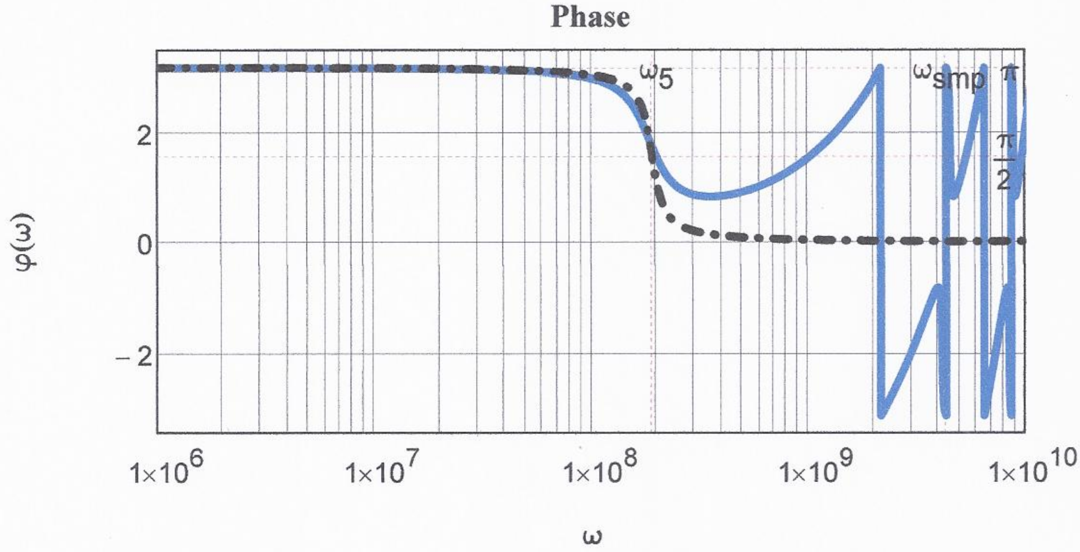


fig.:5.3.2.6.3

Block diagram of the difference equation algorithm for a second order system

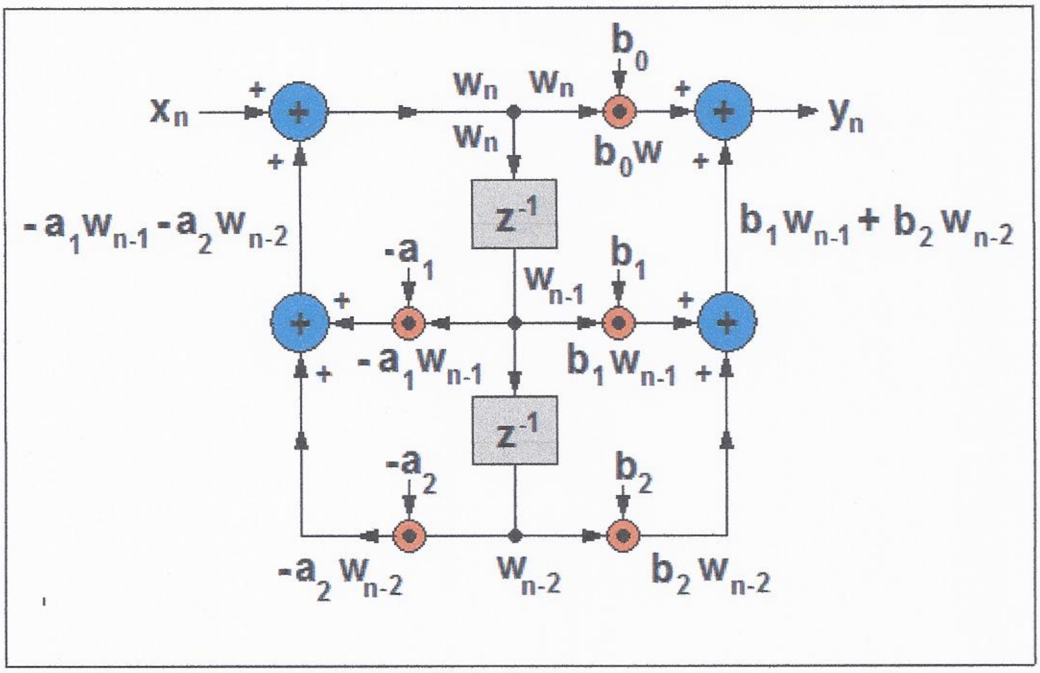


fig.:5.3.2.6.4

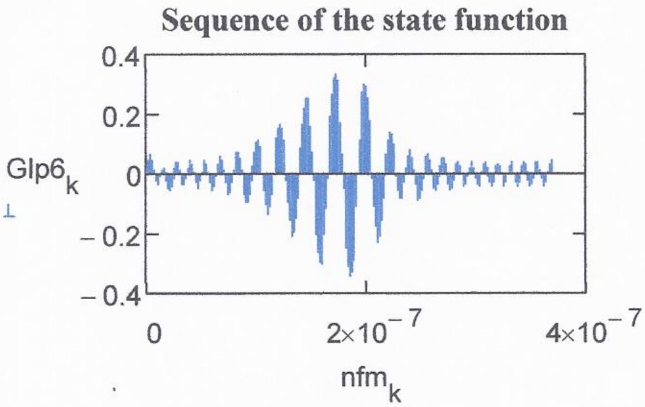


fig.:5.3.2.6.5

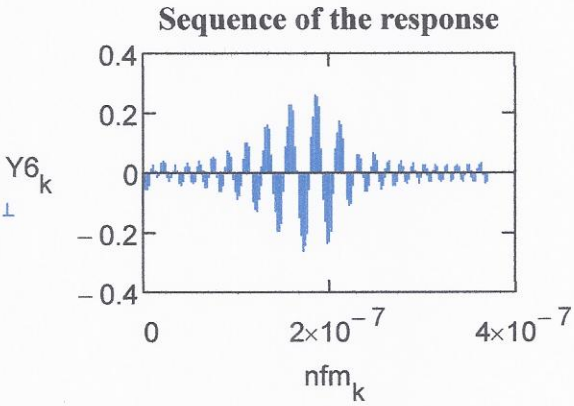


fig.:5.3.2.6.6

$$\text{Spec6x} := \text{FFT}(\text{Y6}) \qquad \max\left(\overrightarrow{|\text{Spec6x}|}\right) = 0.029 \qquad (5.3.2.6.6)$$

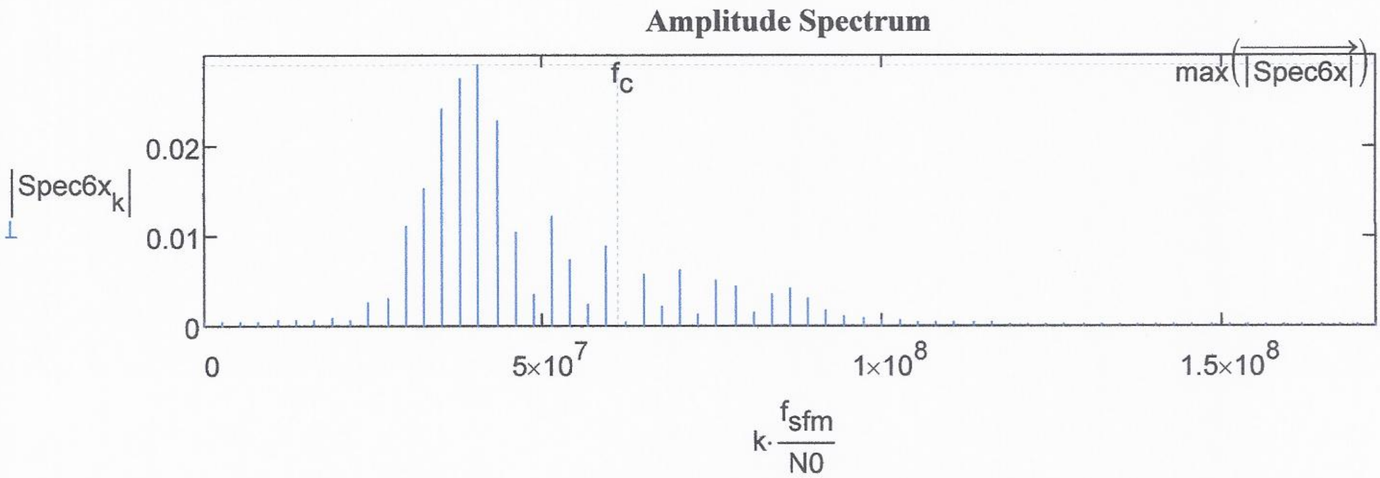


fig.:5.3.2.6.7

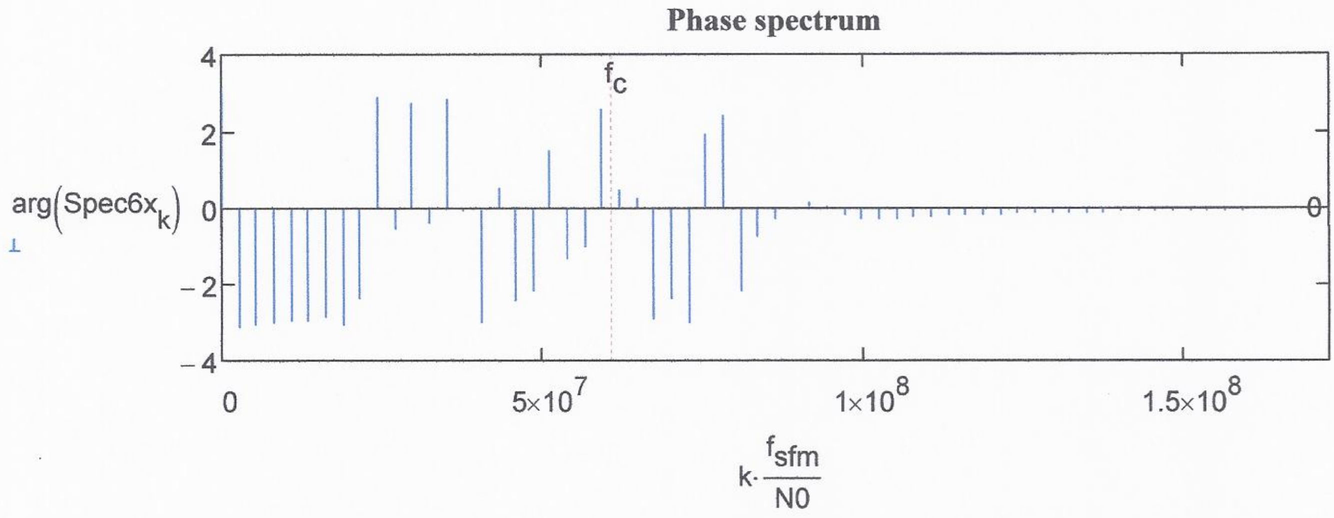


fig.:5.3.2.6.8

Frequency Responses for sampling period T_{sfm}

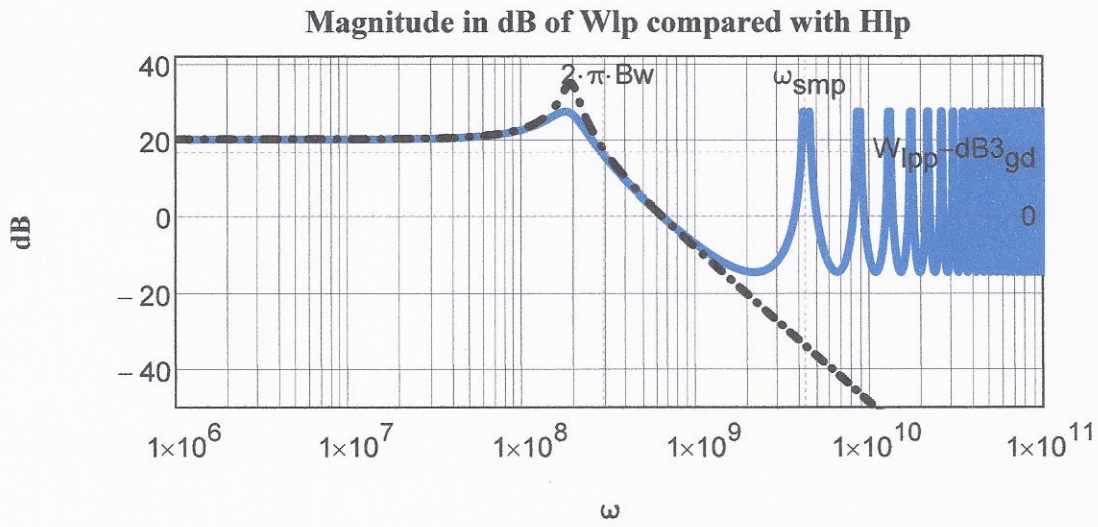


fig.:5.3.2.6.9

5.3 *Equivalent Digital Low Pass Filter (II°order)*

5.3.2.7 Sequence of the (single tone) Phase Modulated carrier response.

$\omega_{mpm} = 76.651 \cdot \frac{Mrads}{sec}$ $\quad A_{pm} = 0.02V$ $\quad 2\cdot\pi\cdot f_c = 0.383 \cdot \frac{Grads}{sec}$ $\quad \omega_c = 0.383 \cdot \frac{Grads}{sec}$

$m_p = 8$ $$u\_10_k := \frac{v_{pm}(npm_k, \omega_c, \omega_{mpm}, A_{pm}, m_p)}{volt} \qquad (5.3.2.7.1)$$

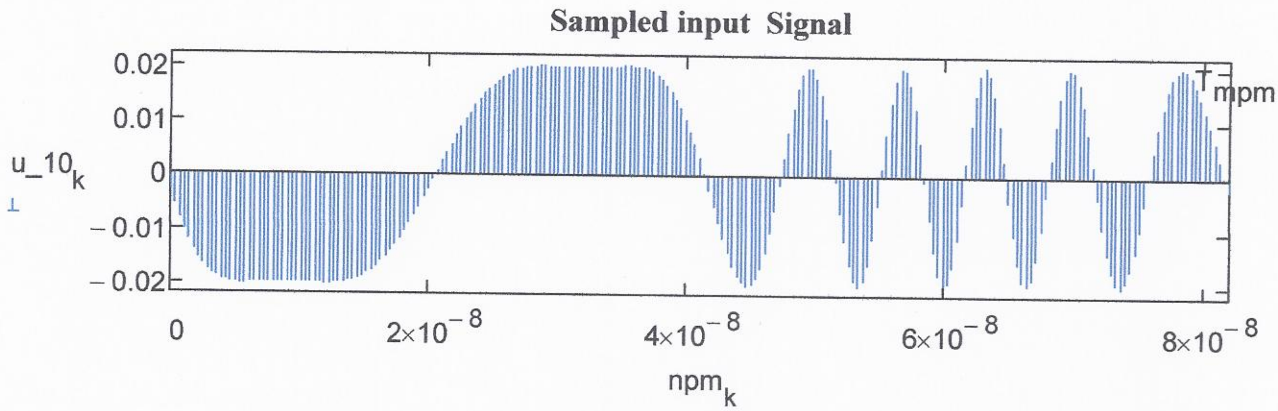


fig.:5.3.2.7.1

Place $T_s := T_{spm}$ $\quad T_s = 0.362 \cdot ns$ $\quad \omega_{smp} := \frac{2\cdot\pi}{T_s}$

$$v7_i(k) := u\_10_k \qquad (5.3.2.7.2)$$

$$svsr7 := CANONIC2LP(v7_i, A_5, \zeta, \omega_5, T_s, N0) \qquad (5.3.2.7.3)$$

$svsr7 = (-0.048 \;\; 2.013 \;\; 1.018 \;\; \{256,1\} \;\; \{256,1\} \;\; 1.069)$

$a7 := svsr7_{0,0}$ $\quad b7 := svsr7_{0,1}$ $\quad c7 := svsr7_{0,2}$ $\quad Glp7 := svsr7_{0,3}$ $\quad Y7 := svsr7_{0,4}$ $\quad d7 := svsr7_{0,5}$

$a7 = -0.04822531$ $\quad b7 = 2.0128600823$ $\quad c7 = 1.0176826132$ $\quad d7 = 1.069$

you get the following result for the t. f. as a function of z:

$T_s = 0.362 \cdot ns$
$$H_{lp}(z) := \begin{cases} \frac{a7}{z^{-2} - b7 \cdot z^{-1} + c7} & \text{if } \zeta \neq \omega_5 \\ \frac{a7}{(d7 - z^{-1})^2} & \text{otherwise} \end{cases} \qquad (5.3.2.7.4)$$

$$W1_{lpp} := 20 \cdot \log\left(\left|\frac{A_5 \cdot \omega_5}{2 \cdot \zeta}\right|\right)$$

BODE PLOTS (Low Pass (II° order)):

Frequency Responses for sampling period T_{spm}

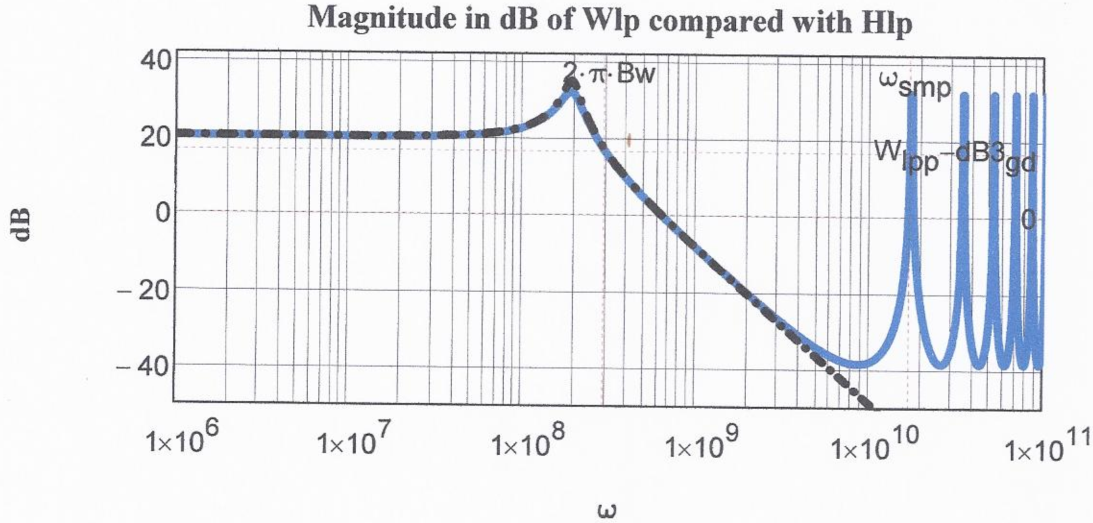


fig.:5.3.2.7.2

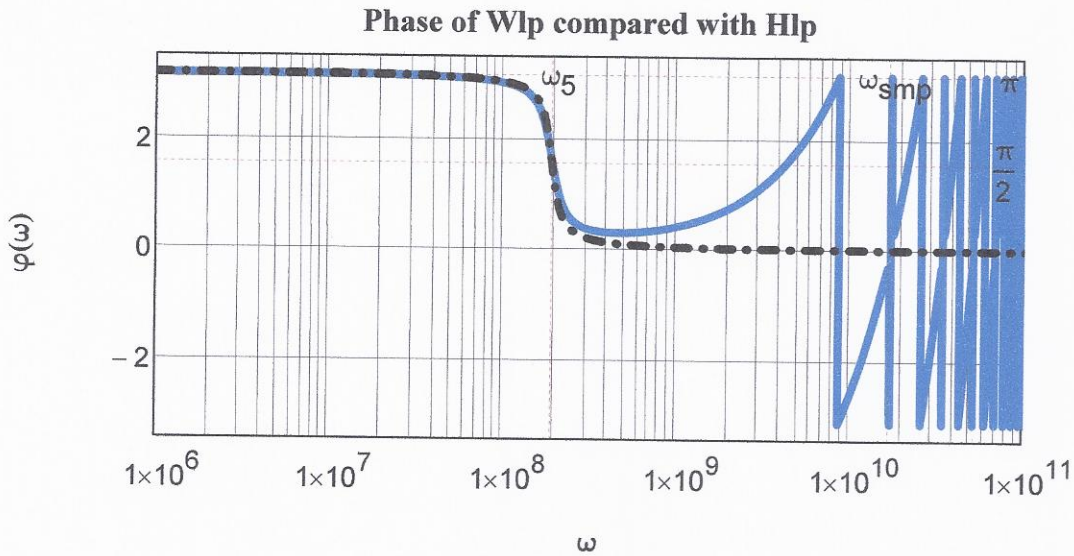


fig.:5.3.2.7.3

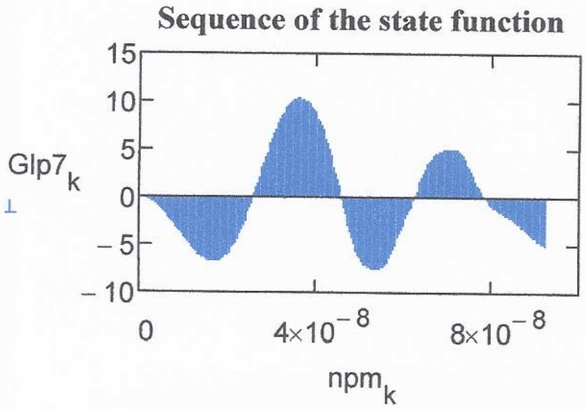


fig.:5.3.2.7.4

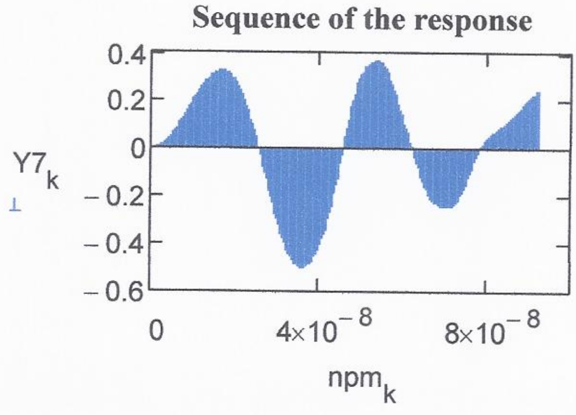


fig.:5.3.2.7.5

Block diagram of the difference equation algorithm for a second order system

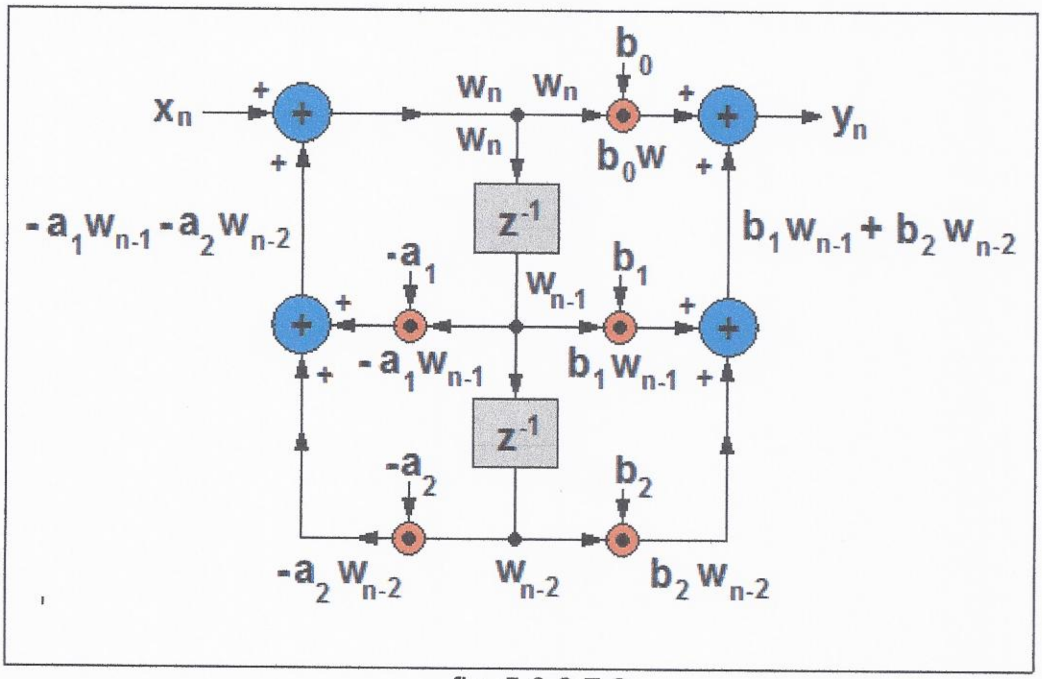


fig.:5.3.2.7.6

$$\text{Spec7x} := \text{FFT}(\text{Y7}) \qquad \max\left(\overrightarrow{|\text{Spec7x}|}\right) = 0.117 \tag{5.3.2.7.5}$$

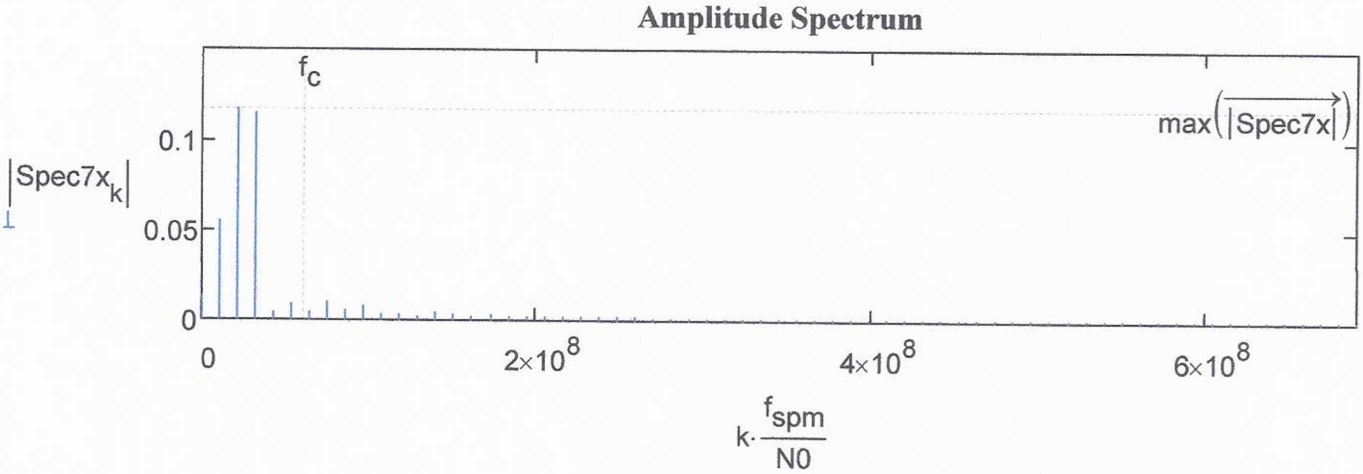


fig.:5.3.2.7.7

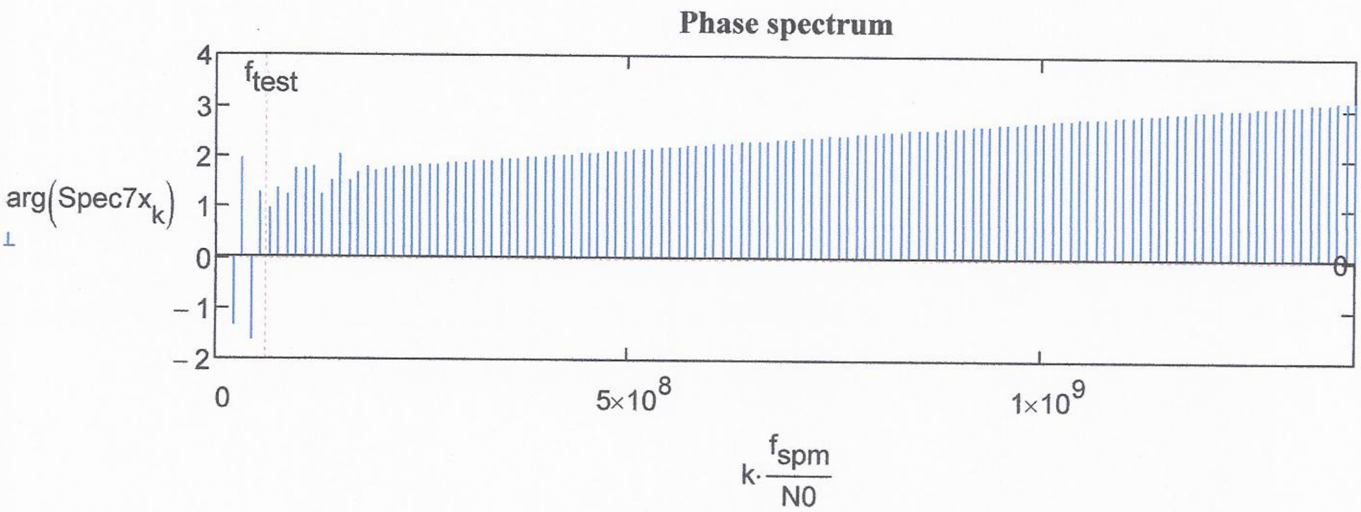


fig.:5.3.2.7.8

Frequency Responses for the sampling period T_{spm}

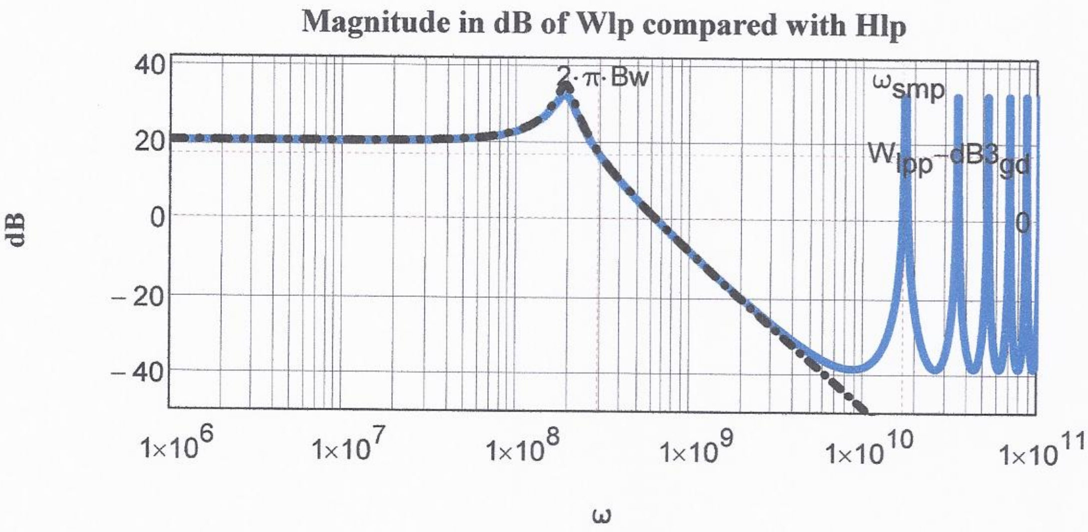


fig.:5.3.2.7.9

5.4
Synthetic Division Algorithm To Generate The Transfer Function Sequence. Output Produced By A Discrete Convolution .

Given the Z transform of the system's transfer function

$$H_{lp}(z) = \begin{cases} \dfrac{A_1}{z^{-2} - B_1 \cdot z^{-1} + C_1} & \text{if } \zeta \neq \omega_5 \\ \dfrac{A_1}{\left(D_1 - z^{-1}\right)^2} & \text{otherwise} \end{cases} \qquad (5.4.1)$$

it is possible to obtain the sequence of the pulse response by a synthetic division of the z transfer function. Once defined the orders of the Numerator $N_{num} := 0$ and of the Denominator of the t. f.: $M_{den} := 2$, then proceed with the calculation of the coefficients corresponding to the chosen sampling frequency:

$$A_1 := A_5 \cdot \omega_5^{\,2} \cdot T_s^{\,2} \qquad B_1 := 2 \cdot (1 + \zeta \cdot T_s) \qquad C_1 := T_s \cdot \left(\omega_5^{\,2} \cdot T_s + 2 \cdot \zeta\right) + 1 \qquad D_1 := T_s \cdot \omega_5 + 1$$

$$N1 := N_{num} + M_{den} \qquad \nu := 1 .. N0 - 1 \qquad (5.4.2)$$

Define two vectors with N0 rows, namely: "b" for the coefficients of the numerator and "a" for the coefficient the denominator:

Numerator Coeffs.	Denominator Co eff.		
$b_\nu := 0.0$	$a_\nu := 0.0$		
$b_0 := A_1$	$a_0 := \begin{cases} C_1 & \text{if } \zeta \neq \omega_5 \\ D_1^{\,2} & \text{otherwise} \end{cases}$	$a_0 = 1.018$	(5.4.3)
$b_1 := 0$	$a_1 := \begin{cases} -B_1 & \text{if } \zeta \neq \omega_5 \\ -2 \cdot D_1 & \text{otherwise} \end{cases}$	$a_1 = -2.013$	(5.4.4)
$b_2 := 0$	$a_2 := 1$	$a_2 = 1$	(5.4.5)

Then write the algorithm for the synthetic division:

$$N1 = 2 \qquad h_k := 0 \qquad h_0 := \frac{b_0}{a_0} \qquad h_\nu := \frac{1}{a_0} \cdot \left[b_\nu - \sum_{i=1}^{\nu} \left(h_{\nu-i} \cdot a_i\right) \right] \qquad (5.4.6)$$

The output as a response to an input signal can be obtained by a convolution integral as follows:

$$y30_\nu := \sum_{k=0}^{\nu} \left(\text{if}\left(\nu - k \geq 0, h_k \cdot u1_{\nu-k}, 0\right)\right) \qquad (5.4.7)$$

Now, to save space when applying the previous algorithm, it is convenient to call the following program ("SYNDIVC" acronym of: Synthetic Division and Convolution), it calculates the coefficient of the Z t. f. with correct sampling period linked to the signal bandwidth:

▶ SYNDIVC

5.4 Transfer Function Sequence Obtained by The Synthetic Division.

5.4.1 Sequence of the voltage Step response.

$u1_{10} = 5V$

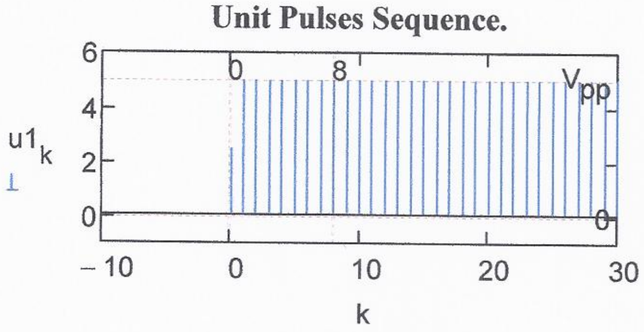


fig.:5.4.1.1

$T_{sstp} = 1.639 \cdot ns$ $\quad$ $conv1 := SYNDIVC\left(\frac{u1}{V}, A_5, \varsigma, \omega_5, T_{sstp}, N0\right)$ $\quad$ (5.4.1.1)

$conv1 = (-0.987 \quad 2.058 \quad 1.157 \quad \{256,1\} \quad \{256,1\} \quad \{256,1\} \quad \{256,1\} \quad 27.225 \quad 32.194)$

$a0 := conv1_{0,3}$ $\quad b0 := conv1_{0,4}$ $\quad h := conv1_{0,5}$ $\quad y10 := conv1_{0,6}$ $\quad S0 := conv1_{0,7}$ $\quad E0 := conv1_{0,8}$

T. F. Numerator coefficients:

$a0^T =$

	0	1	2	3	4	5	6	7
0	1.157	-2.058	1	0	0	0	0	...

T. F. Denominator coefficients:

$b0^T =$

	0	1	2	3	4	5	6	7
0	-0.987	0	0	0	0	0	0	...

Impulse response sequence

$h^T =$

	0	1	2	3	4
0	-0.853	-1.518	-1.963	-2.18	...

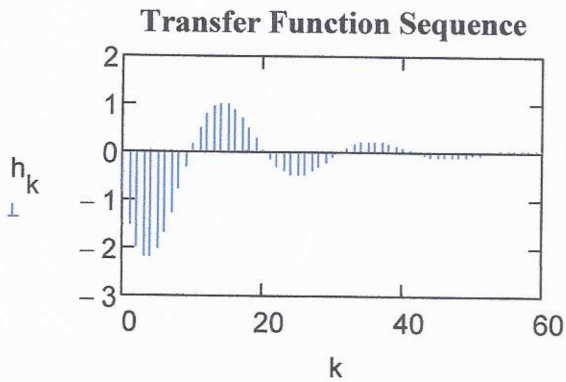


fig.:5.4.1.2

Stability (S3<∞): $\quad S0 = 27.225$

Energy of the sequence h3: $E0 = 32.194$

$$t := 0 \cdot T_{test}, 0 \cdot T_{test} + \frac{20 \cdot T_5 - 0 \cdot T_{test}}{1000} .. 20 \cdot T_5$$

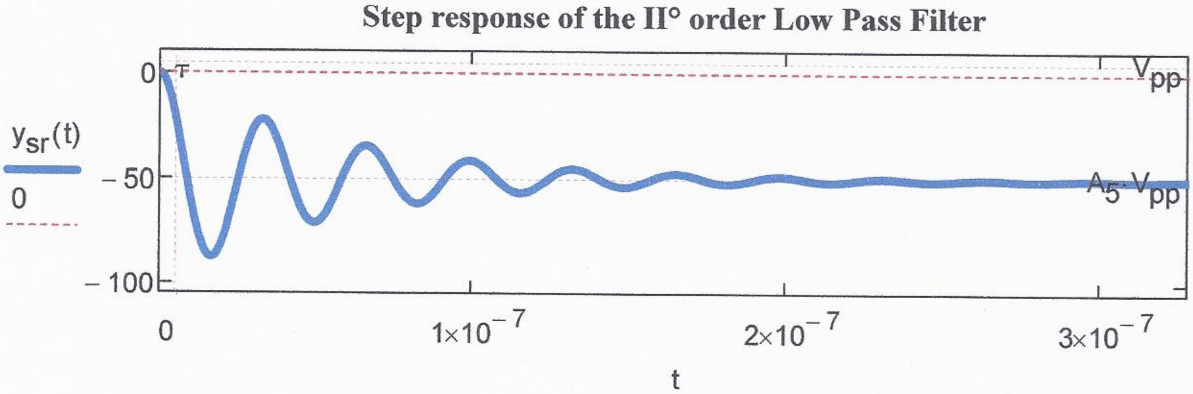


fig.:5.4.1.3

$V_{pp} = 5V$

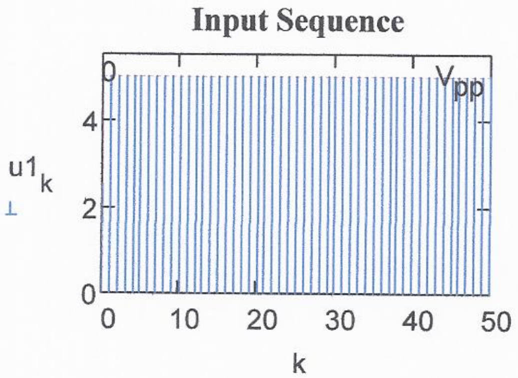


fig.:5.4.1.4

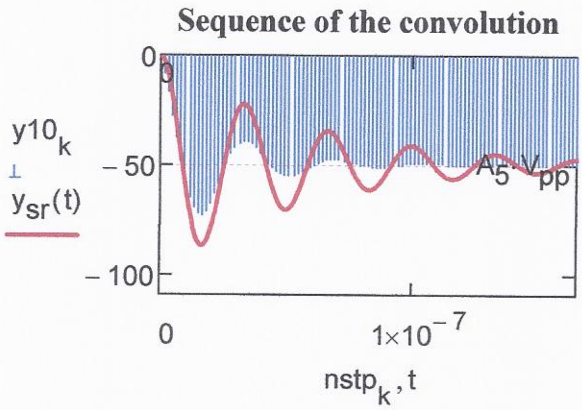


fig.:5.4.1.5

5.4 Transfer Function Sequence Obtained by The Synthetic Division.

5.4.2 Sequence of the Short Voltage Pulse response.

$$conv2 := SYNDIVC\left(u44, A_5, \zeta, \omega_5, T_{svp}, N0\right) \qquad (5.4.2.1)$$

$$conv2 = (-43.865 \quad 2.388 \quad 5.774 \quad \{256,1\} \quad \{256,1\} \quad \{256,1\} \quad \{256,1\} \quad 11.593 \quad 67.932)$$

$a2 := conv2_{0,3}$ $\quad b2 := conv2_{0,4}$ $\quad h2 := conv2_{0,5}$ $\quad y11 := conv2_{0,6}$

T. F. Numerator coefficients:

$a2^T =$

	0	1	2	3	4	5	6	7
0	5.774	-2.388	1	0	0	0	0	...

T. F. Denominator coefficients:

$b2^T =$

	0	1	2	3	4	5	6
0	-43.865	0	0	0	0	0	...

Impulse response sequence

$h2^T =$

	0	1	2	3	4
0	-7.597	-3.141	0.017	0.551	...

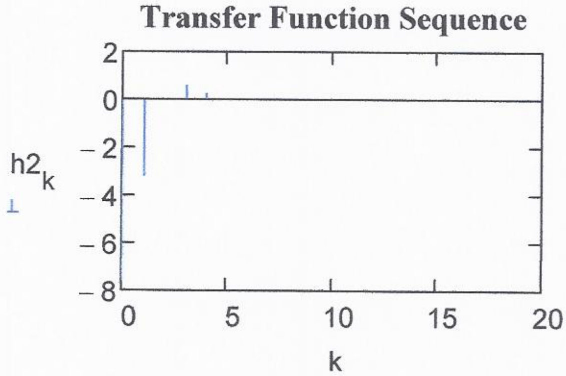


fig.:5.4.2.1

$$t := -2\cdot\tau_{pw}, -2\cdot\tau_{pw} + \frac{4\cdot\tau_{pw}}{5000} .. 2\cdot\tau_{pw}$$

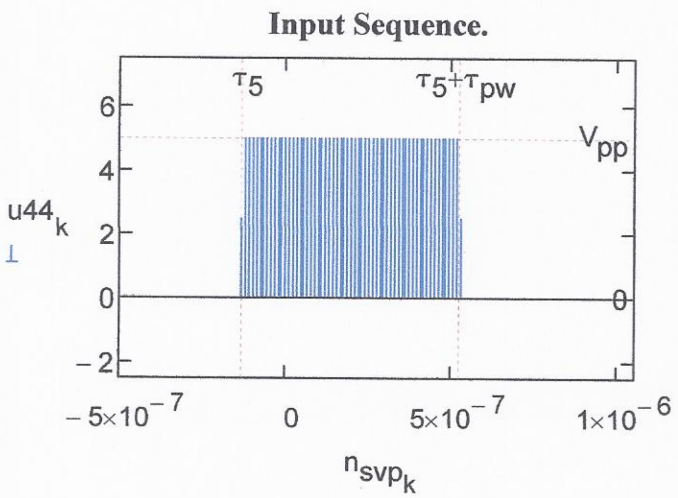


fig.:5.4.2.2

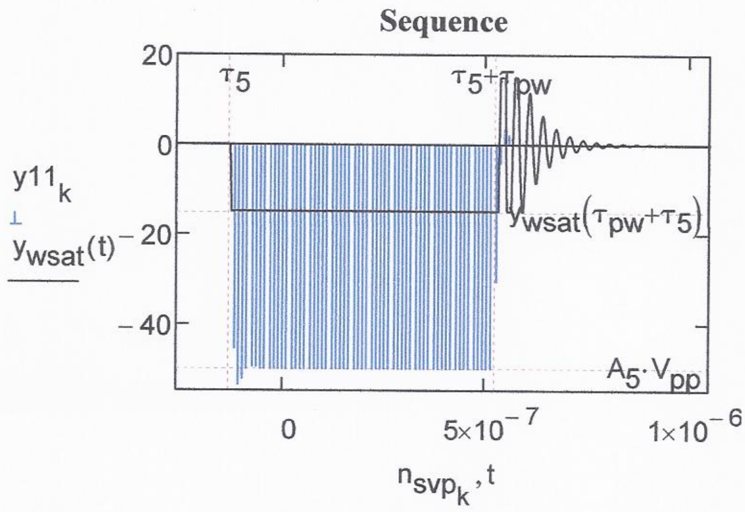


fig.:5.4.2.3

Spec11x := fft(y11)

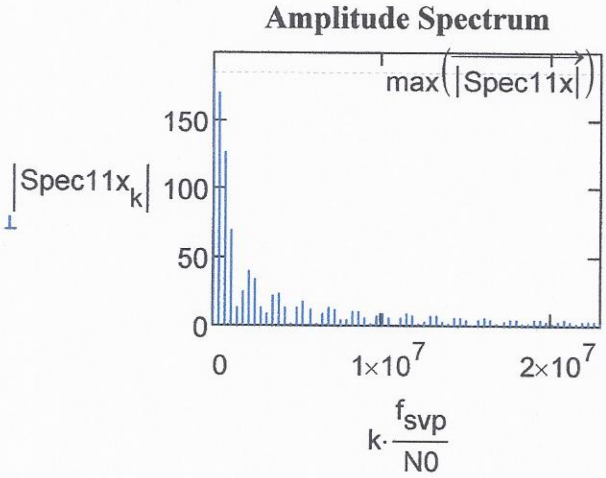


fig.:5.4.2.4

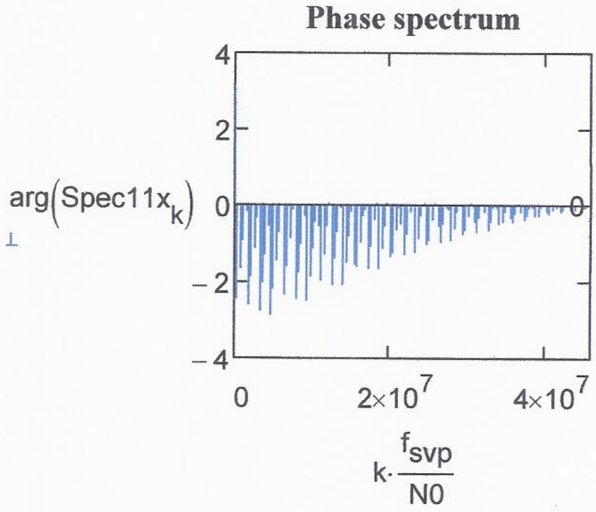


fig.:5.4.2.5

5.4 Transfer Function Sequence Obtained by The Synthetic Division.

5.4.3 Sequence of the Sawtooth response:

$$conv3 := SYNDIVC\left(\frac{u55}{V}, A_5, \zeta, \omega_5, T_{ssw}, N0\right) \qquad (5.4.3.1)$$

conv3 = (−0.011 2.006 1.007 {256,1} {256,1} {256,1} {256,1} 35.693 6.382)

$a03 := conv1_{0,3}$ $\quad b03 := conv1_{0,4}$ $\quad h3 := conv1_{0,5}$ $\quad y12 := conv3_{0,6}$

T. F. Numerator coefficients:

$a03^T =$

	0	1	2	3	4	5	6	7
0	1.157	-2.058	1	0	0	0	0	...

T. F. Denominator coefficients:

$b03^T =$

	0	1	2	3	4	5	6	7
0	-0.987	0	0	0	0	0	0	...

Impulse response sequence

$h3^T =$

	0	1	2	3	4
0	-0.853	-1.518	-1.963	-2.18	...

Transfer Function Sequence

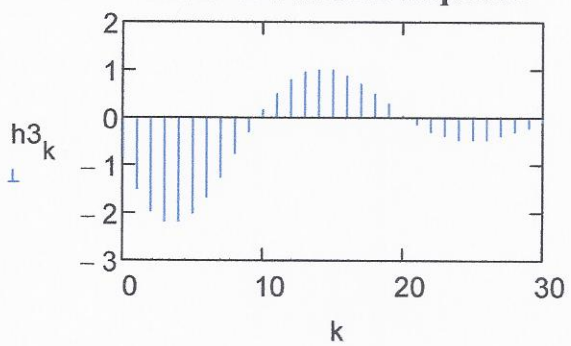


fig.:5.4.3.1

Input Sequence.

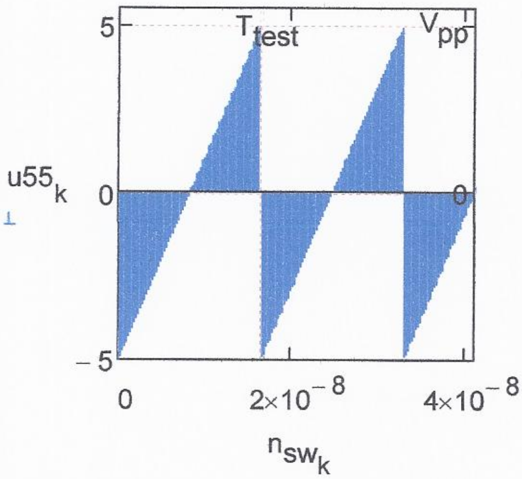


fig.:5.4.3.2

Output Sequence

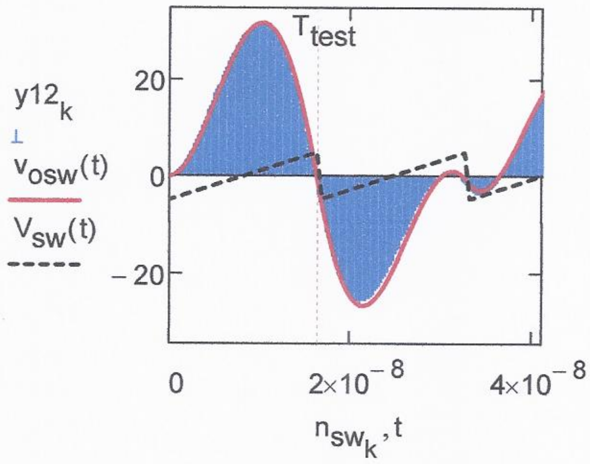


fig.:5.4.3.3

$Spec12x := fft(y12)$

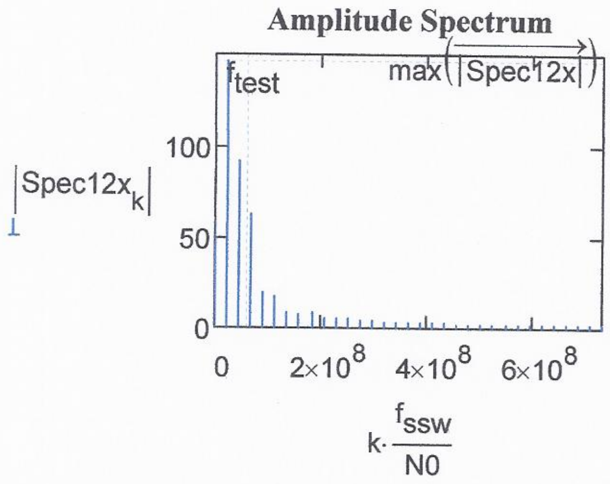


fig.:5.4.3.4

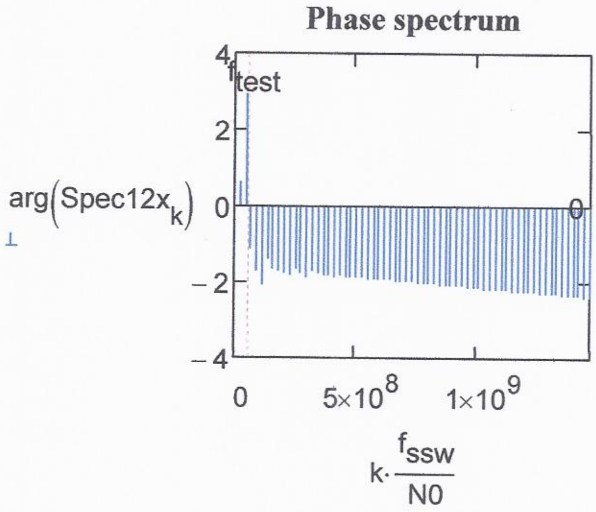


fig.:5.4.3.5

5.4 Transfer Function Sequence Obtained by The Synthetic Division.

5.4.4 Sequence of the Bipolar Square Wave response.

$$v11_i(k) := sqw_k \qquad (5.4.4.1)$$

$$conv4 := SYNDIVC\left(sqw, A_5, \zeta, \omega_5, T_{ssqw}, N0\right) \qquad (5.4.4.2)$$

$$conv4 = (-0.011 \quad 2.006 \quad 1.007 \quad \{256,1\} \quad \{256,1\} \quad \{256,1\} \quad \{256,1\} \quad 35.693 \quad 6.382)$$

$$a04 := conv4_{0,3} \qquad b04 := conv4_{0,4} \quad h4 := conv4_{0,5} \qquad y13 := conv4_{0,6}$$

T. F. Numerator coefficients:

$a04^T =$

	0	1	2	3	4	5	6	7
0	1.007	-2.006	1	0	0	0	0	...

T. F. Denominator coefficients:

$b04^T =$

	0	1	2	3	4	5	6	7
0	-0.011	0	0	0	0	0	0	...

Impulse response sequence

$h4^T =$

	0	1	2	3	4
0	-0.011	-0.021	-0.032	-0.042	...

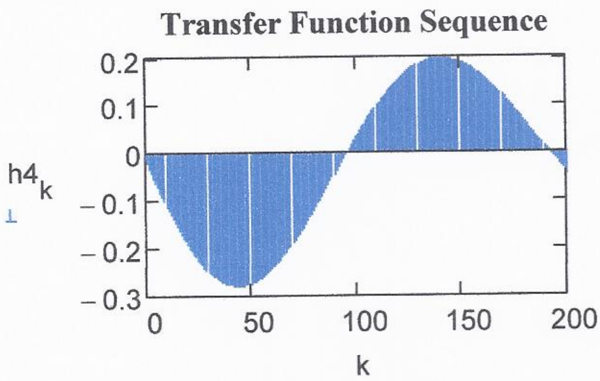


fig.:5.4.4.1

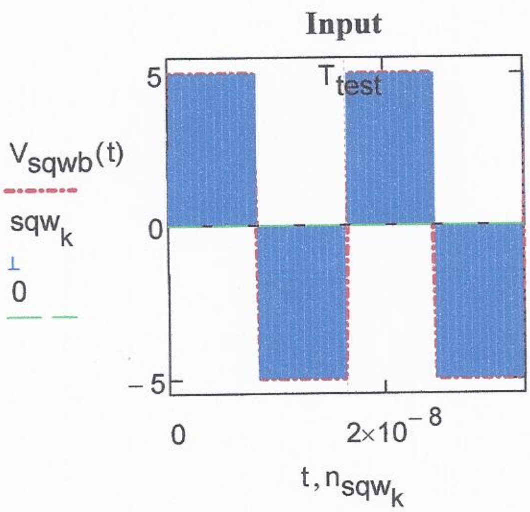


fig.:5.4.4.2

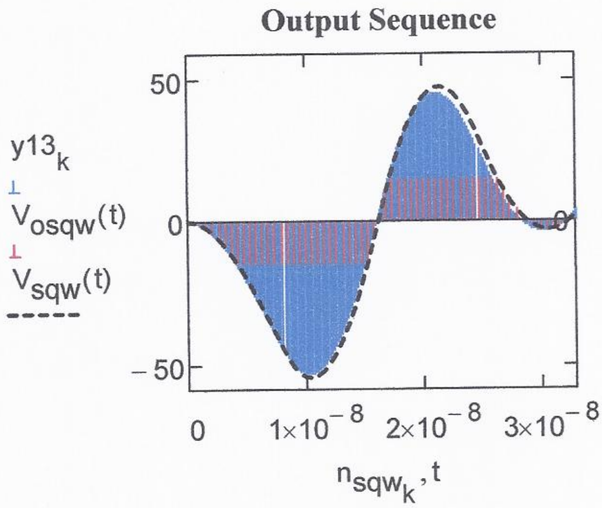


fig.:5.4.4.3

$Spec13x := fft(y13)$

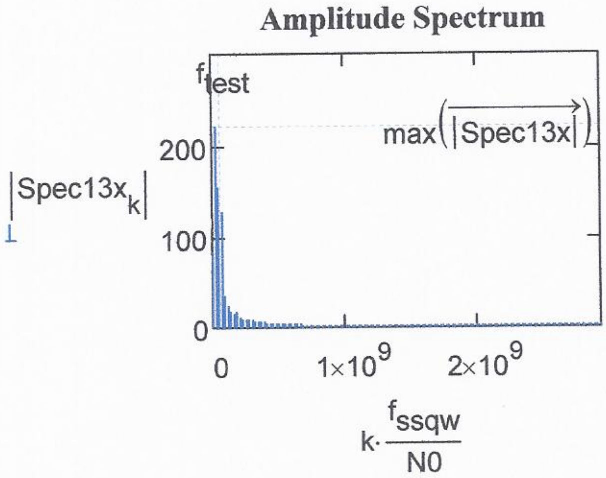


fig.:5.4.4.4

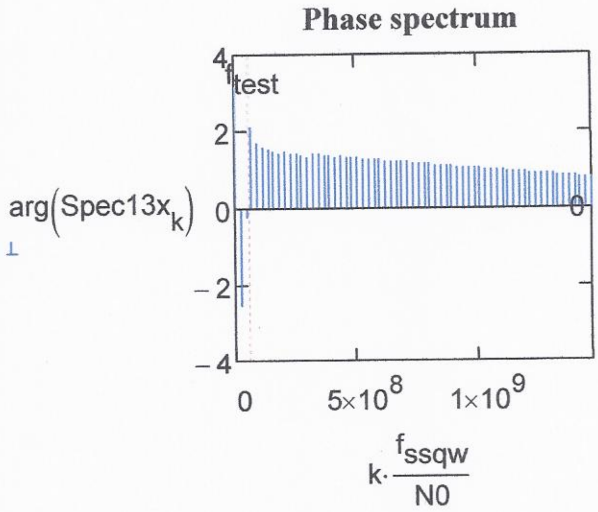


fig.:5.4.4.5

5.4 Transfer Function Sequence Obtained by The Synthetic Division.

5.4.5 Sequence of the AM (single tone) Signal response.

$$conv5 := SYNDIVC\left(u\_7, A_5, \zeta, \omega_5, T_{sam}, N0\right) \qquad (5.4.5.1)$$

$$conv5 = (-17.135 \quad 2.242 \quad 2.956 \quad \{256,1\} \quad \{256,1\} \quad \{256,1\} \quad \{256,1\} \quad 13.627 \quad 55.912)$$

$a05 := conv5_{0,3} \qquad b05 := conv5_{0,4} \quad h5 := conv5_{0,5} \qquad y14 := conv5_{0,6}$

T. F. Numerator coefficients:

$a05^T =$

	0	1	2	3	4	5	6	7
0	2.956	-2.242	1	0	0	0	0	...

T. F. Denominator coefficients:

$b05^T =$

	0	1	2	3	4	5	6
0	-17.135	0	0	0	0	0	...

Impulse response sequence

$h5^T =$

	0	1	2	3	4
0	-5.797	-4.398	-1.375	0.445	...

$m_{am} = 55 \cdot \%$

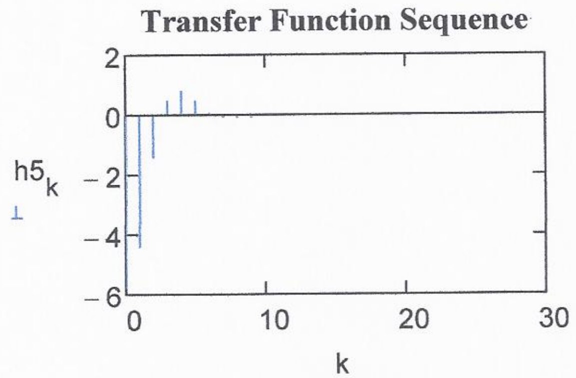


fig.:5.4.5.1

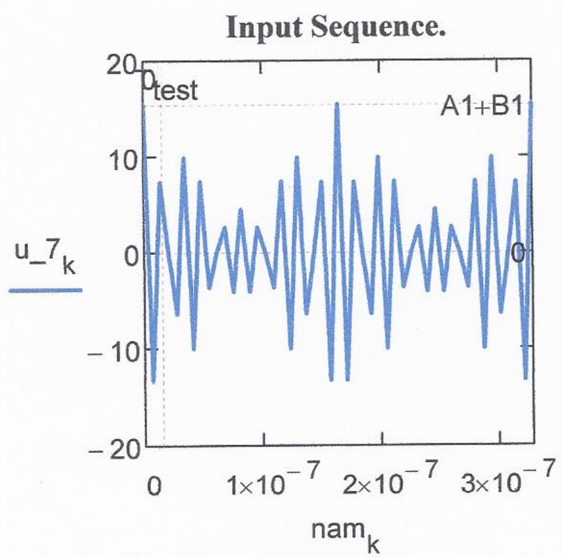


fig.:5.4.5.2

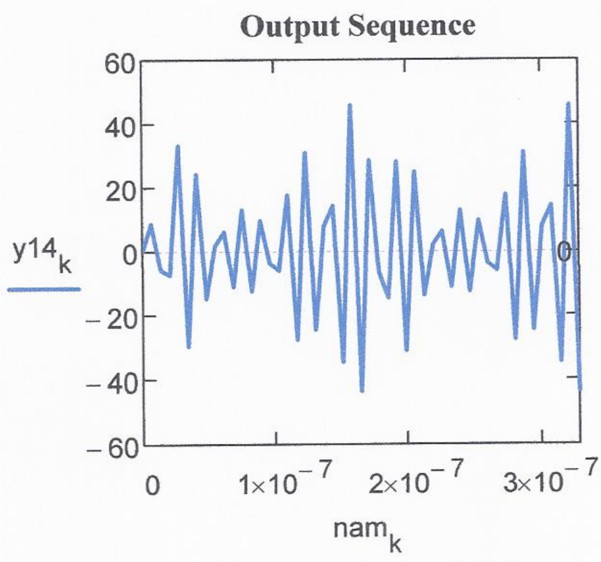


fig.:5.4.5.3

Sampled signal: $\quad Spec14x := fft(y14) \quad$ (5.4.5.2)

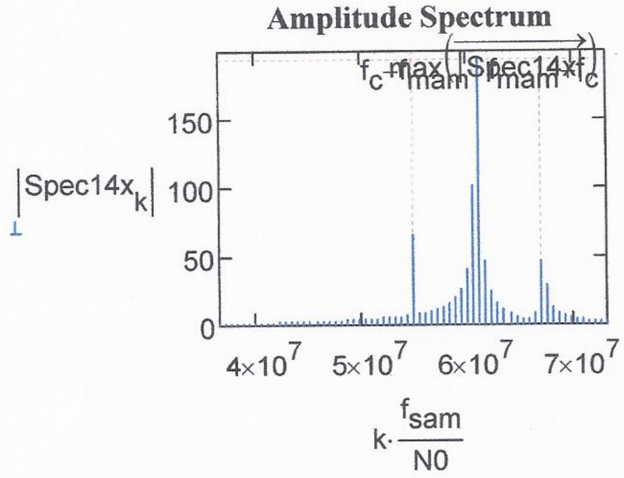


fig.:5.4.5.4

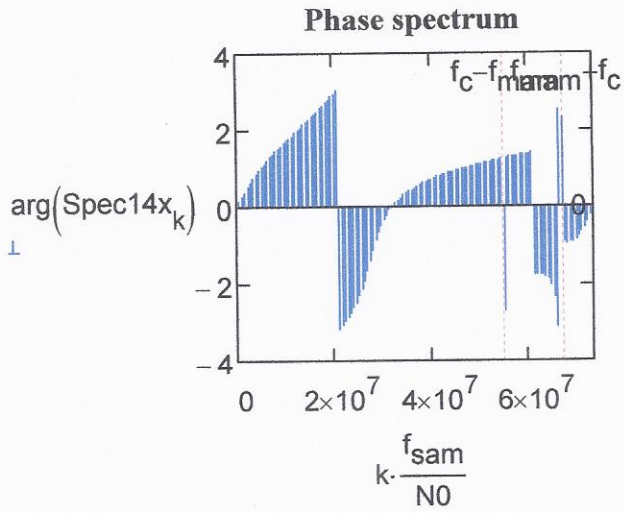


fig.:5.4.5.5

5.4.6 Sequence of the (single tone) Frequency Modulated carrier response.

$$v13_i(k) := u8_k \quad (5.4.6.1)$$

$$conv6 := SYNDIVC\left(u8, A_5, \zeta, \omega_5, T_{sfm}, N0\right) \quad (5.4.6.2)$$

$$conv6 = (-0.772 \quad 2.051 \quad 1.129 \quad \{256,1\} \quad \{256,1\} \quad \{256,1\} \quad \{256,1\} \quad 28.99 \quad 30.554)$$

$a06 := conv6_{0,3}$ $b06 := conv6_{0,4}$ $h6 := conv6_{0,5}$ $y15 := conv6_{0,6}$

T. F. Numerator coefficients:

$a06^T =$

	0	1	2	3	4	5	6	7
0	1.129	-2.051	1	0	0	0	0	...

T. F. Denominator coefficients:

$b06^T =$

	0	1	2	3	4	5	6	7
0	-0.772	0	0	0	0	0	0	...

Impulse response sequence

$h6^T =$

	0	1	2	3	4
0	-0.684	-1.243	-1.653	-1.904	...

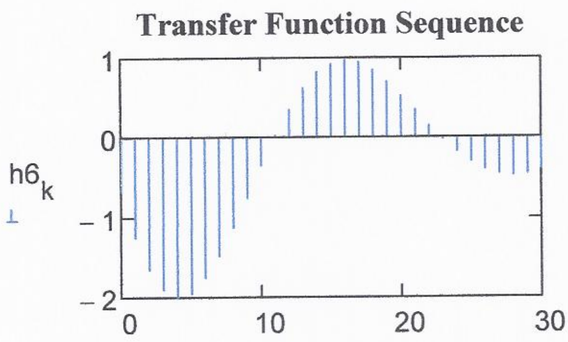


fig.:5.4.6.1

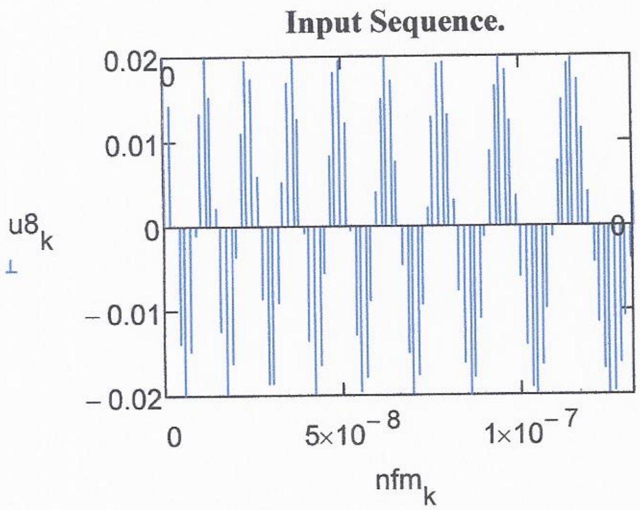


fig.:5.4.6.2

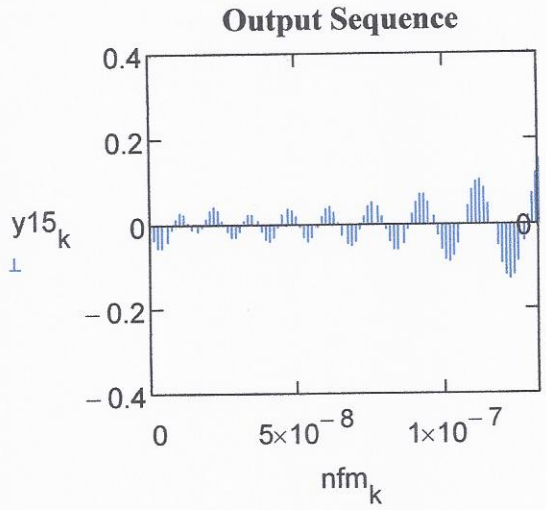


fig.:5.4.6.3

$m_f = 8$

$$Spec15x := fft(y15) \qquad (5.4.6.3)$$

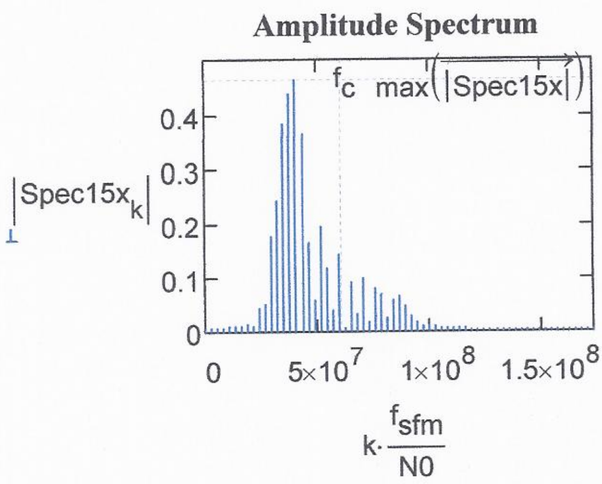


fig.:5.4.6.4

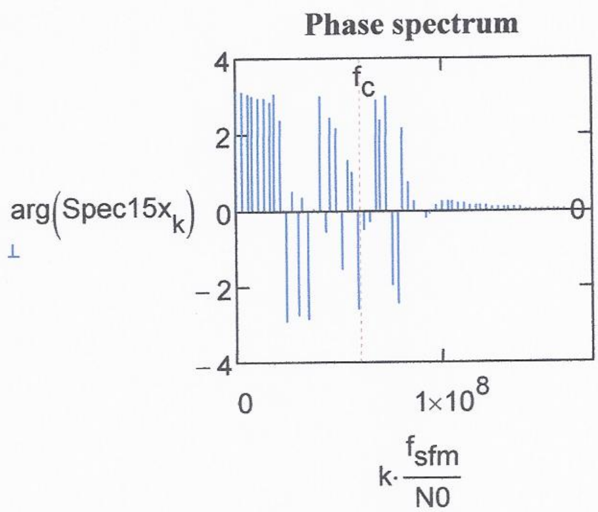


fig.:5.4.6.5

$A_{fm} \cdot A_5 = -0.2V \qquad \omega_{mfm} = 0.019 \cdot \frac{Grads}{sec} \qquad A_5 = -10 \qquad m_f = 8$

5.4.7 Sequence of the (single tone) Phase Modulated carrier response.

$$v14_i(k) := u9_k \qquad (5.4.7.1)$$

$$conv7 := SYNDIVC(u9, A_5, \zeta, \omega_5, T_{spm}, N0) \qquad (5.4.7.2)$$

conv7 = (−0.048 2.013 1.018 {256,1} {256,1} {256,1} {256,1} 45.268 13.507)

$a07 := conv7_{0,3}$ $b07 := conv7_{0,4}$ $h7 := conv7_{0,5}$ $y16 := conv7_{0,6}$

T. F. Numerator coefficients:

$a07^T =$

	0	1	2	3	4	5	6	7
0	1.018	-2.013	1	0	0	0	0	...

T. F. Denominator coefficients:

$b07^T =$

	0	1	2	3	4	5	6	7
0	-0.048	0	0	0	0	0	0	...

Impulse response sequence

$h7^T =$

	0	1	2	3	4
0	-0.047	-0.094	-0.139	-0.182	...

$m_p = 8$

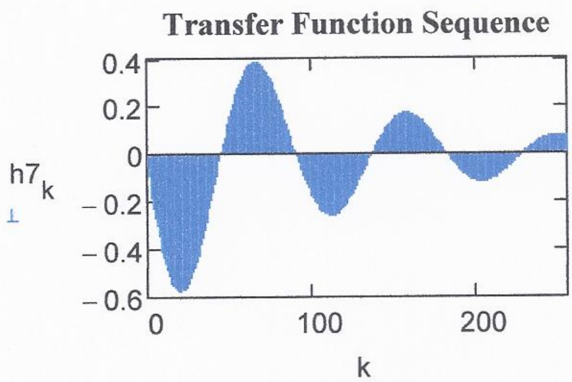
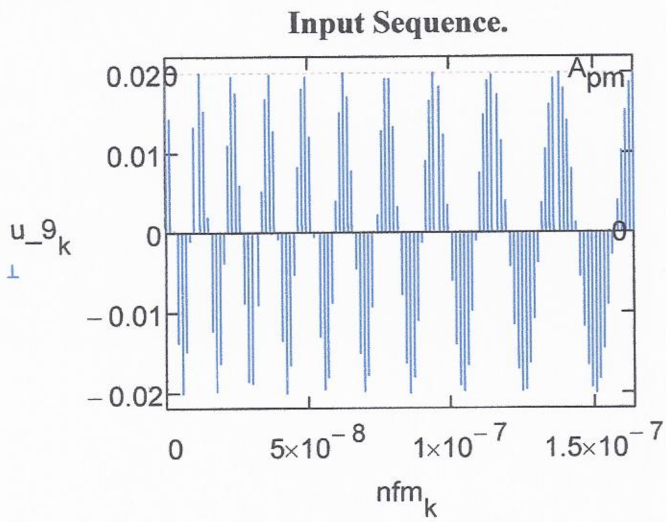


fig.:5.4.7.1

Sampled signal: Spec16x := fft(y16)



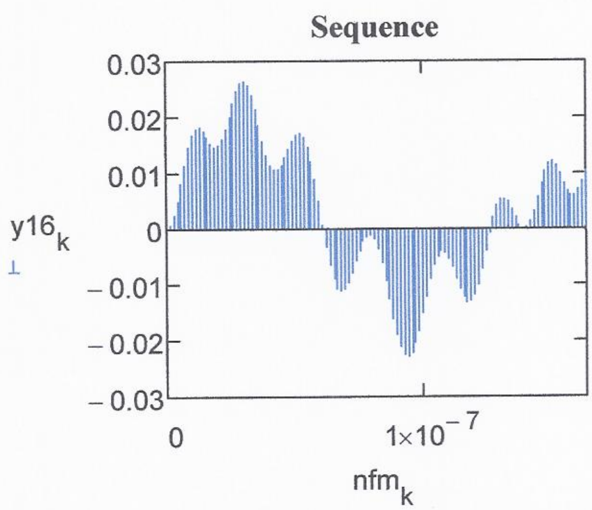


fig.:5.4.7.2

fig.:5.4.7.3

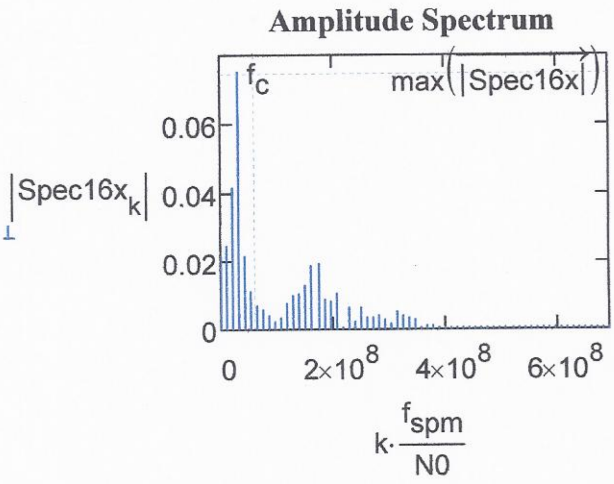


fig.:5.4.7.4

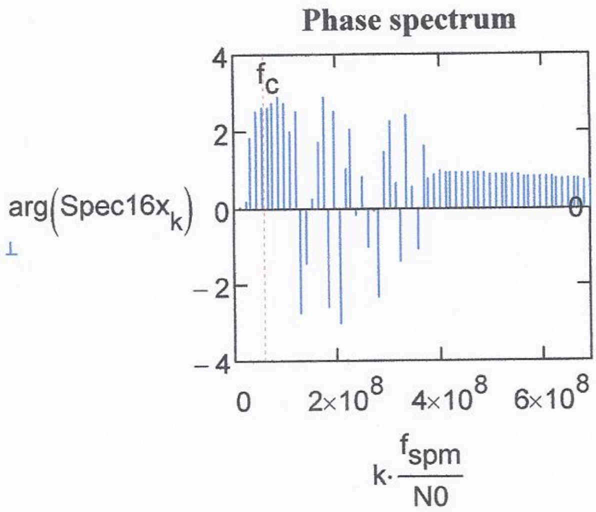


fig.:5.4.7.5

$\omega_c = 0.383 \cdot \frac{\text{Grads}}{\text{sec}}$ $\omega_{mpm} = 0.077 \cdot \frac{\text{Grads}}{\text{sec}}$ $A_5 = -10$ $m_p = 8$

5.5

Search of the discrete time sequence of the output by a discrete convolution

$$T_s := T_{ssw}$$

$A_0 := A_5 \cdot \omega_5^2 \cdot T_s^2$ $\quad B_0 := 2 \cdot (1 + \zeta \cdot T_s)$ $\quad C_0 := T_s \cdot (\omega_5^2 \cdot T_s + 2 \cdot \zeta) + 1$ $\quad D_0 := T_s \cdot \omega_5 + 1$

The sequence corresponding to the transfer function

$$H_{lp}(z) := \begin{cases} \dfrac{A_0}{z^{-2} - B_0 \cdot z^{-1} + C_0} & \text{if } \zeta \neq \omega_5 \\ \dfrac{A_0}{(D_0 - z^{-1})^2} & \text{otherwise} \end{cases} \qquad (5.5.1)$$

can be found using the "**invztrans**" MATHCAD's operator as follows:

$A_0 := A_0 \quad B_0 := B_0 \quad k := k \quad C_0 := C_0$

First case: $\zeta \neq \omega_5$

$$h1_k := \frac{A_0}{z^{-2} - B_0 \cdot z^{-1} + C_0} \text{ invztrans}, z, k \rightarrow \qquad (5.5.2)$$

$$h1_k := \left[\left(\frac{B_0 - \sqrt{B_0^2 - 4 \cdot C_0}}{C_0}\right)^{k-1} \cdot \left(2 \cdot C_0 - B_0^2 + B_0 \cdot \sqrt{B_0^2 - 4 \cdot C_0}\right) \ldots \atop + \left(\frac{B_0 + \sqrt{B_0^2 - 4 \cdot C_0}}{C_0}\right)^{k-1} \cdot \left(B_0^2 - 2 \cdot C_0 + B_0 \cdot \sqrt{B_0^2 - 4 \cdot C_0}\right)\right] \cdot \left[\left(\frac{1}{2}\right)^k \cdot \frac{A_0}{\left(C_0^2 \cdot \sqrt{B_0^2 - 4 \cdot C_0}\right)}\right]$$

Second case: $\zeta = \omega_5$

$A_0 := A_0 \quad D_0 := D_0 \quad k := k$

$$h1_k := \frac{A_0}{(D_0 - z^{-1})^2} \text{ invztrans}, z, k \rightarrow$$

$$h1_k := \frac{A_0 \cdot \left(\frac{1}{D_0}\right)^k \cdot (k+1)}{D_0^2}$$

$$K1 := \frac{B_0 - \sqrt{B_0^2 - 4 \cdot C_0}}{C_0} \qquad K2 := \left(2 \cdot C_0 - B_0^2 + B_0 \cdot \sqrt{B_0^2 - 4 \cdot C_0}\right)$$

$$K3 := \left(\frac{B_0 + \sqrt{B_0^2 - 4 \cdot C_0}}{C_0}\right) \qquad K4 := \left(B_0^2 - 2 \cdot C_0 + B_0 \cdot \sqrt{B_0^2 - 4 \cdot C_0}\right)$$

$$K5 := \text{if}\left[\zeta \neq \omega_5, \frac{A_0}{\left(C_0^2 \cdot \sqrt{B_0^2 - 4 \cdot C_0}\right)}, 0\right]$$

$$h1_k := \begin{cases} \left(K1^{k-1} \cdot K2 + K3^{k-1} \cdot K4\right) \cdot \left[\left(\frac{1}{2}\right)^k \cdot K5\right] & \text{if } \zeta \neq \omega_5 \\ A_0 \cdot D_0^{-(k+2)} \cdot (k+1) & \text{otherwise} \end{cases} \tag{5.5.3}$$

The result is the sequence of the impulse response, here depicted:

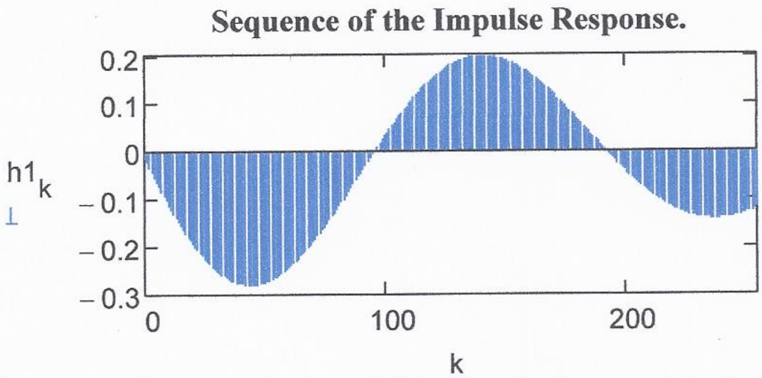


fig.:5.5.1

The Output of the Digital System is given by the **discrete convolution** between the discrete time input signal (the discrete time sequence of the sawtooth wave for this example) and the discrete impulse response of the System:

$$\nu := 1 .. N0 - 1 \qquad y15_\nu := \sum_{k=0}^{\nu} \left(\text{if}\left(\nu - k \geq 0, h1_k \cdot \frac{u55_{\nu-k}}{V}, 0\right)\right) \tag{5.5.4}$$

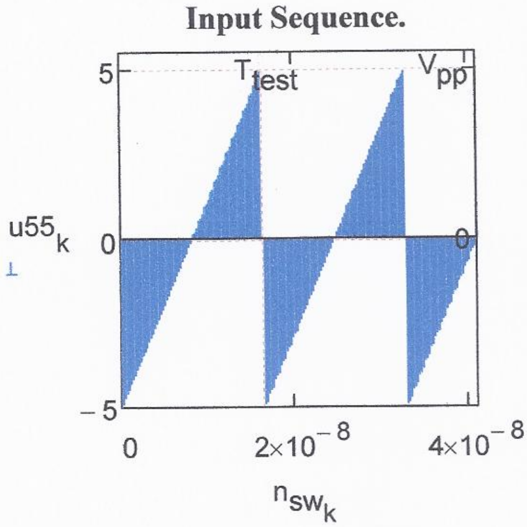


fig.:5.5.2

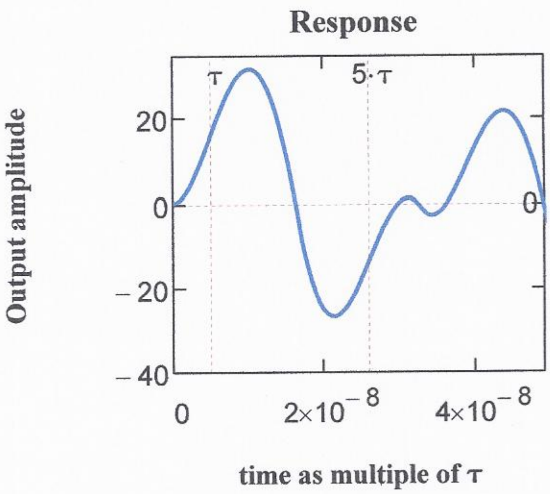


fig.:5.5.3

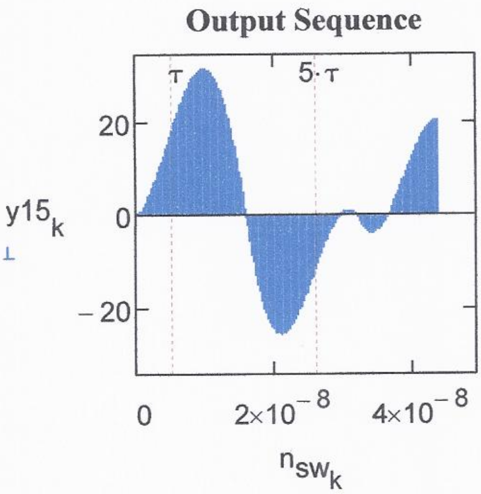


fig.:5.5.4

Knowing the sequences of any input $u1_\nu$ and that of the impulse response $h1_\nu$, one can obtain the relative Z transforms namely : $X_{lp}(z) := \sum_{\nu=0}^{N1-1} \left(u1_\nu \cdot z^{-\nu}\right)$ and $H_{lps}(z) := \sum_{\nu=0}^{N0-1} \left(h1_\nu \cdot z^{-\nu}\right)$. The **z-inverse transform of the product** $Y_{lp}(z) := H_{lps}(z) \cdot X_{lp}(z)$, which corresponds to the convolution of the two given sequences $u1_\nu$, $h1_\nu$, provides the sought output.

Example 1: voltage ramp as input::
Since the t. f. is already known, you just have to calculate the input's Z transform as follows:

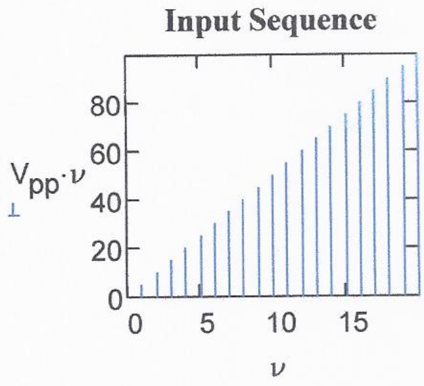


fig.:5.5.5

$$V_{pp} := V_{pp},\ \nu := \nu, \qquad V_{pp}\cdot\nu \text{ ztrans}, \nu \;\rightarrow\; \frac{V_{pp}\cdot z}{(z-1)^2}$$

$$\text{results:} \quad \mathcal{Z}\left(V_{pp}\cdot\nu\right) = \frac{V_{pp}\cdot z}{(z-1)^2} \qquad (5.5.5)$$

Output sequence corresponding to the z-inverse transform of the product H(z)Vi(z):

$$V_{pp} := V_{pp} \quad A_0 := A_0 \quad B_0 := B_0 \quad C_0 := C_0 \quad z := z$$

First case: $\zeta \neq \omega_5$

$$y16_k := \frac{A_0}{z^{-2} - B_0\cdot z^{-1} + C_0}\cdot\frac{V_{pp}\cdot z}{(z-1)^2} \;\left|\begin{array}{l}\text{invztrans}, z, k \\ \text{simplify}, \max \rightarrow \\ \text{combine}\end{array}\right.$$

To simplify place:

$$K6 := \frac{B_0 - 2}{\left(C_0 - B_0 + 1\right)^2} \qquad K7 := \frac{B_0 + \sqrt{B_0^{\,2} - 4\cdot C_0}}{C_0} \qquad K88 := B_0^{\,3} - 2\cdot B_0^{\,2} + \left(1 - 3\cdot C_0\right)\cdot B_0 + 4\cdot C_0$$

$$K8 := \left[\left(B_0^{\,2} - 2\cdot B_0 - C_0 + 1\right)\cdot\sqrt{B_0^{\,2} - 4\cdot C_0} + K88\right] \qquad K9 := \frac{B_0 - \sqrt{B_0^{\,2} - 4\cdot C_0}}{C_0}$$

$$K10 := \left[\left(2\cdot B_0 - B_0^{\,2} + C_0 - 1\right)\cdot\sqrt{B_0^{\,2} - 4\cdot C_0} + K88\right] \qquad K11 := C_0\cdot\sqrt{B_0^{\,2} - 4\cdot C_0}\cdot\left(C_0 - B_0 + 1\right)^2$$

Resulting sequence: $$y16_k := (A_0 \cdot V_{pp}) \cdot \left[\frac{k}{C_0 - B_0 + 1} - K6 \; ... \; + \frac{\left(\frac{1}{2}\right)^k \cdot K7^{k-1} \cdot K8}{K11} \; ... \; + (-1) \cdot \frac{\left(\frac{1}{2}\right)^k \cdot K9^{k-1} \cdot K10}{K11} \right] \quad (5.5.6)$$

Second case: $\zeta = \omega_5$

$V_{pp} := V_{pp} \quad A_0 := A_0 \quad D_0 := D_0 \quad C_0 := C_0 \quad z := z$

$$y16_k := \frac{A_0}{\left(D_0 - z^{-1}\right)^2} \cdot \frac{V_{pp} \cdot z}{(z-1)^2} \;\Bigg|\; \begin{array}{l} \text{invztrans}, z, k \\ \text{simplify}, \max \\ \text{collect}, \left(\frac{1}{D_0}\right)^k \\ \text{collect}, A_0 \cdot V_{pp} \end{array} \rightarrow$$

$$y16_k := \left[\frac{\left(\frac{1}{D_0}\right)^k \cdot \left(D_0 - 1\right) \cdot k + 2 \cdot D_0}{D_0} - \left[-k \cdot \left(D_0 - 1\right) + 2\right] \right] \cdot \frac{\left(A_0 \cdot V_{pp}\right)}{\left(D_0 - 1\right)^3 \cdot V} \quad (5.5.7)$$

General result:

$$y_{o_k} := \begin{cases} (A_0 \cdot V_{pp}) \cdot \frac{k}{C_0 - B_0 + 1} + \frac{\left(\frac{1}{2}\right)^k \cdot \left(K7^{k-1} \cdot K8 - K9^{k-1} \cdot K10\right)}{K11} - K6 & \text{if } \zeta \neq \omega_5 \\ \left[\frac{\left(\frac{1}{D_0}\right)^k \cdot \left(D_0 - 1\right) \cdot k + 2 \cdot D_0}{D_0} - \left[-k \cdot \left(D_0 - 1\right) + 2\right] \right] \cdot \frac{A_0 \cdot V_{pp}}{\left(D_0 - 1\right)^3} & \text{otherwise} \end{cases} \quad (5.5.8)$$

$\zeta = 0.018 \cdot \frac{\text{Grads}}{\text{sec}} \qquad \omega_5 = 0.192 \cdot \frac{\text{Grads}}{\text{sec}} \qquad Q_5 = 5.4$

$$Y_{o_k} := \text{if}\left(-V_{sat} \le y_{o_k} \le V_{sat}, y_{o_k}, \text{if}\left(y_{o_k} \le 0.0 \cdot V, -V_{sat}, V_{sat}\right)\right) \quad (5.5.9)$$

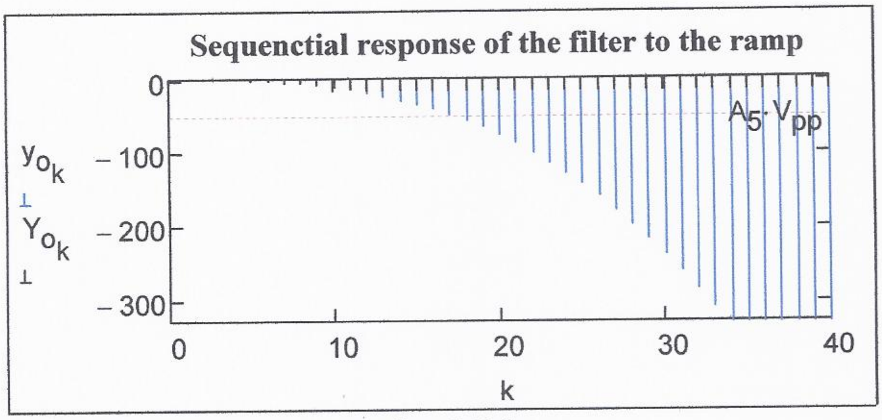


fig.:5.5.6

Example 2: Sinusoidal input:

$$\Delta T := \frac{2 \cdot T_{test}}{N0 - 1} \qquad \Delta T = 0.129 \cdot ns$$

Z transform of the input signal: $\omega := \omega_{test}$ $\qquad \omega = 0.383 \cdot \frac{Grads}{sec}$

$\Delta T := \Delta T \qquad \omega := \omega \qquad V_{pp} = 5V$

$$\nu := \nu \quad V_{pp} := V_{pp} \quad V_{pp} \cdot \sin(\omega \cdot \nu \cdot \Delta T) \text{ ztrans}, \nu \rightarrow \frac{V_{pp} \cdot z \cdot \sin(\omega \cdot \Delta T)}{z^2 - 2 \cdot \cos(\omega \cdot \Delta T) \cdot z + 1}$$

Place: $K2 := \sin(\Delta T \cdot \omega) \qquad \cos(\Delta T \cdot \omega) = \sqrt{1 - K2^2} \qquad \sqrt{1 - K2^2} = 0.999$

$$K2 := K2 \qquad poles1 := 1 - 2 \cdot \sqrt{1 - K2^2} \cdot z^{-1} + z^{-2} \text{ solve}, z \rightarrow \begin{pmatrix} \sqrt{1 - K2^2} + K2 \cdot j \\ \sqrt{1 - K2^2} - K2 \cdot j \end{pmatrix}$$

$$p1 := poles1_0 \qquad p1 = 0.999 + 0.049j$$

$$p2 := poles1_1 \qquad p2 = 0.999 - 0.049j$$

$$\frac{V_{pp} \cdot z \cdot K2}{z^2 - 2 \cdot \sqrt{1 - K2^2} \cdot z + 1} = \frac{K2 \cdot V_{pp} \cdot z}{(p1 - z) \cdot (p1 - \overline{p1})} - \frac{K2 \cdot V_{pp} \cdot z}{(p1 - \overline{p1}) \cdot (z - \overline{p1})}$$

$$\frac{K2 \cdot V_{pp} \cdot z}{(p1 - z) \cdot (p1 - \overline{p1})} = \left[1 + \frac{p1}{(z - p1)}\right] \cdot \frac{(K2 \cdot V_{pp})}{(\overline{p1} - p1)}$$

$$\frac{K2 \cdot V_{pp} \cdot z}{(p1 - \overline{p1}) \cdot (z - \overline{p1})} = -\left[1 + \frac{\overline{p1}}{(z - \overline{p1})}\right] \cdot \frac{K2 \cdot V_{pp}}{\overline{p1} - p1}$$

$$\frac{A_0}{z^{-2} - B_0 \cdot z^{-1} + C_0} = \frac{A_0}{\left(z - \frac{B_0 + \sqrt{B_0^2 - 4 \cdot C_0}}{2 \cdot C_0}\right) \cdot \left(z - \frac{B_0 - \sqrt{B_0^2 - 4 \cdot C_0}}{2 \cdot C_0}\right)}$$

$$q1 := \frac{B_0 + \sqrt{B_0^{\ 2} - 4 \cdot C_0}}{2 \cdot C_0} \qquad q2 := \frac{B_0 - \sqrt{B_0^{\ 2} - 4 \cdot C_0}}{2 \cdot C_0}$$

$$q1 = 0.996 + 0.032j \qquad q2 = 0.996 - 0.032j$$

$$\frac{A_0}{(z-q1)\cdot(z-q2)} = \left[\frac{1}{(z-q1)} - \frac{1}{(z-q2)}\right]\cdot\frac{A_0}{(q1-q2)}$$

First case: $\zeta \neq \omega_5$

$$y17_k := \frac{A_0 \cdot K2 \cdot V_{pp}}{(q1-q2)\cdot\left(p1-\overline{p1}\right)} \cdot \left[\frac{1}{(z-q1)} - \frac{1}{(z-q2)}\right] \cdot \left[\frac{z}{(p1-z)} \cdots + \frac{-z}{\left(z0-\overline{p1}\right)}\right] \left| \begin{array}{l} \text{invztrans}, z, \text{using}, \nu = k \\ \text{simplify} \end{array} \right. \rightarrow$$

Computing the corresponding sequence the result returned for the symbolic operation is too large to be displayed.

$q1 := q1 \quad p1 := p1 \qquad z := z$

$$\left[\frac{1}{(z-q1)} \cdot \frac{z}{(p1-z)}\right] \text{invztrans} \rightarrow -\frac{p1^n - q1^n}{p1 - q1}$$

$q1 := q1 \quad p1 := p1 \qquad z := z$

$$\left[\frac{1}{(z-q1)} \cdot \frac{-z}{\left(z-\overline{p1}\right)}\right] \text{invztrans} \rightarrow \frac{\left(\overline{p1}\right)^n - q1^n}{q1 - \overline{p1}}$$

$q2 := q2 \quad p1 := p1 \qquad z := z \qquad n := n \qquad k := k$

$$\left[\frac{1}{(z-q2)} \cdot \frac{z}{(p1-z)}\right] \text{invztrans} \rightarrow -\frac{p1^n - q2^n}{p1 - q2}$$

$q2 := q2 \quad p1 := p1 \qquad z := z$

$$\left[\frac{1}{(z-q2)} \cdot \frac{z}{\left(z-\overline{p1}\right)}\right] \text{invztrans} \rightarrow -\frac{\left(\overline{p1}\right)^n - q2^n}{q2 - \overline{p1}}$$

$$y17_k := \frac{A_0 \cdot K2 \cdot V_{pp}}{(q1-q2)\cdot\left(p1-\overline{p1}\right)} \cdot \left[\frac{p1^k - q1^k}{q1 - p1} + \frac{\left(\overline{p1}\right)^k - q1^k}{q1 - \overline{p1}} - \frac{p1^k - q2^k}{q2 - p1} - \frac{\left(\overline{p1}\right)^k - q2^k}{q2 - \overline{p1}}\right]$$

Proof

Proof: **First case:** $\zeta \neq \omega_5$

$q1 := q1 \quad q2 := q2 \qquad p1 := p1 \qquad \overline{p1} := \overline{p1}$

$$\frac{p1^k - q1^k}{q1 - p1} + \frac{(\overline{p1})^k - q1^k}{q1 - \overline{p1}} - \frac{p1^k - q2^k}{q2 - p1} - \frac{(\overline{p1})^k - q2^k}{q2 - \overline{p1}} \text{ ztrans}, k \rightarrow -\frac{z\cdot(q1-q2)\cdot(p1 - 2\cdot z + \overline{p1})}{(p1-z)\cdot(q1-z)\cdot(q2-z)\cdot(z-\overline{p1})}$$

$$\mathcal{Z}\left[\frac{p1^k - q1^k}{q1 - p1} + \frac{(\overline{p1})^k - q1^k}{q1 - \overline{p1}} - \frac{p1^k - q2^k}{q2 - p1} - \frac{(\overline{p1})^k - q2^k}{q2 - \overline{p1}}\right] = -\frac{z\cdot(q1-q2)\cdot(p1 - 2\cdot z + \overline{p1})}{(p1-z)\cdot(q1-z)\cdot(q2-z)\cdot(z-\overline{p1})}$$

$$\left[-\frac{z\cdot(q1-q2)\cdot(p1 - 2\cdot z + \overline{p1})}{(p1-z)\cdot(q1-z)\cdot(q2-z)\cdot(z-\overline{p1})}\right] \Bigg| \begin{matrix}\text{parfrac}, z \\ \text{simplify}\end{matrix} \rightarrow -\frac{z\cdot(q1-q2)\cdot(p1 - 2\cdot z + \overline{p1})}{(p1-z)\cdot(q1-z)\cdot(q2-z)\cdot(z-\overline{p1})}$$

$$\mathcal{Z}\left[\frac{p1^k - q1^k}{q1 - p1} + \frac{(\overline{p1})^k - q1^k}{q1 - \overline{p1}} - \frac{p1^k - q2^k}{q2 - p1} - \frac{(\overline{p1})^k - q2^k}{q2 - \overline{p1}}\right] = \frac{\overline{p1}\cdot(q1-q2)}{-(q1-\overline{p1})\cdot(q2-\overline{p1})\cdot(z-\overline{p1})} \cdots$$
$$+ \frac{p1\cdot(q1-q2)\cdot(p1-\overline{p1})}{(p1-q1)\cdot(p1-q2)\cdot(p1-z)\cdot(p1-\overline{p1})} \cdots$$
$$+ (-1)\cdot\frac{q1\cdot(p1 - 2\cdot q1 + \overline{p1})}{-(p1-q1)\cdot(q1-z)\cdot(q1-\overline{p1})} \cdots$$
$$+ (-1)\cdot\frac{q2\cdot(q1-q2)\cdot(p1 - 2\cdot q2 + \overline{p1})}{(p1-q2)\cdot(q1-q2)\cdot(q2-z)\cdot(q2-\overline{p1})}$$

Proof

Second case case: $\zeta = \omega_5$

$$y17_k := \frac{A_0\cdot K2\cdot V_{pp}}{(p1_0 - \overline{p1_0})}\cdot\frac{1}{(D_0 - z^{-1})^2}\cdot\left[\frac{z}{(p1_0 - z)} \cdots + \frac{-z}{(z - \overline{p1_0})}\right] \Bigg| \begin{matrix}\text{invztrans}, z, \text{using}, n = k \\ \text{simplify}\end{matrix} \rightarrow$$

Computing the corresponding sequence the result returned for the symbolic operation is too large to be displayed.

$$\frac{1}{(D_0 - z^{-1})^2}\cdot\frac{z}{(p1_0 - z)} \text{ invztrans}, \text{using}, n = k \rightarrow$$

$$\frac{1}{(D_0 - z^{-1})^2}\cdot\frac{-z}{(z - \overline{p1_0})} \text{ invztrans}, \text{using}, n = k \rightarrow$$

$$y17_k = \frac{A_0 \cdot K2 \cdot V_{pp}}{\left(p1 - \overline{p1}\right)} \cdot \left[\frac{D_0 \cdot (k+2) \cdot p1 - k - 1}{D_0^{\,2} \cdot \left(D_0 \cdot p1 - 1\right)^2} \cdot \left(\frac{1}{D_0}\right)^k - \frac{(p1)^{k+2}}{\left(D_0 \cdot p1 - 1\right)^2} \cdots + \frac{\overline{p1} \cdot (k+2) \cdot D_0 - (k+1)}{D_0^{\,2} \cdot \left(D_0 \cdot \overline{p1} - 1\right)^2} \cdot \left(\frac{1}{D_0}\right)^k - \frac{\left(\overline{p1}\right)^{k+2}}{\left(D_0 \cdot \overline{p1} - 1\right)^2} \right]$$

Proof

<u>Proof:</u> Second case: $\zeta = \omega_5$

$$\frac{A_0 \cdot K2 \cdot V_{pp}}{\left(p1 - \overline{p1}\right)} \cdot \left[\frac{D_0 \cdot (k+2) \cdot p1 - k - 1}{D_0^{\,2} \cdot \left(D_0 \cdot p1 - 1\right)^2} \cdot \left(\frac{1}{D_0}\right)^k - \frac{(p1)^{k+2}}{\left(D_0 \cdot p1 - 1\right)^2} \cdots + \frac{\overline{p1} \cdot (k+2) \cdot D_0 - (k+1)}{D_0^{\,2} \cdot \left(D_0 \cdot \overline{p1} - 1\right)^2} \cdot \left(\frac{1}{D_0}\right)^k - \frac{\left(\overline{p1}\right)^{k+2}}{\left(D_0 \cdot \overline{p1} - 1\right)^2} \right] \text{ztrans}, k \rightarrow -\frac{A_0 \cdot K2 \cdot V_{pp} \cdot z^3 \cdot (p1 -}{\left(D_0 \cdot z - 1\right)^2 \cdot (p1 - z) \cdot (p1}$$

$$\mathcal{Z}\left[\frac{A_0 \cdot K2 \cdot V_{pp}}{\left(p1_0 - \overline{p1_0}\right)} \cdot \left[\frac{D_0 \cdot (k+2) \cdot p1_0 - k - 1}{D_0^{\,2} \cdot \left(D_0 \cdot p1_0 - 1\right)^2} \cdot \left(\frac{1}{D_0}\right)^k - \frac{(p1_0)^{k+2}}{\left(D_0 \cdot p1_0 - 1\right)^2} \cdots + \frac{\overline{p1_0} \cdot (k+2) \cdot D_0 - (k+1)}{D_0^{\,2} \cdot \left(D_0 \cdot \overline{p1_0} - 1\right)^2} \cdot \left(\frac{1}{D_0}\right)^k - \frac{\left(\overline{p1_0}\right)^{k+2}}{\left(D_0 \cdot \overline{p1_0} - 1\right)^2} \right]\right] = -\frac{A_0 \cdot K2 \cdot V_{pp} \cdot z^3 \cdot (p1 - 2 \cdot z}{\left(D_0 \cdot z - 1\right)^2 \cdot (p1 - z) \cdot (p1 - \overline{p}}$$

Proof

General result:

$$y17_k := \frac{A_0 \cdot K2 \cdot V_{pp}}{\left(p1 - \overline{p1}\right)} \cdot \begin{cases} \dfrac{1}{(q1 - q2)} \cdot \left[\dfrac{(p1)^k - q1^k}{q1 - p1} \cdots + \dfrac{\left(\overline{p1}\right)^k - q1^k}{q1 - \overline{p1}} - \dfrac{(p1)^k - q2^k}{q2 - p1} - \dfrac{\left(\overline{p1}\right)^k - q2^k}{q2 - \overline{p1}} \right] & \text{if } \zeta \neq \omega_5 \\ \left[\dfrac{D_0 \cdot (k+2) \cdot p1 - k - 1}{D_0^{\,2} \cdot \left(D_0 \cdot p1 - 1\right)^2} \cdot \left(\dfrac{1}{D_0}\right)^k - \dfrac{(p1)^{k+2}}{\left(D_0 \cdot p1 - 1\right)^2} \cdots + \dfrac{\overline{p1} \cdot (k+2) \cdot D_0 - (k+1)}{D_0^{\,2} \cdot \left(D_0 \cdot \overline{p1} - 1\right)^2} \cdot \left(\dfrac{1}{D_0}\right)^k - \dfrac{\left(\overline{p1}\right)^{k+2}}{\left(D_0 \cdot \overline{p1} - 1\right)^2} \right] & \text{otherwise} \end{cases}$$

(5.5.10)

$y17^T =$

	0	1	2	3	4	5	6	7	8
0	0	0	-0.054j	-0.16j	-0.319j	-0.53j	-0.79j	-1.101j	...

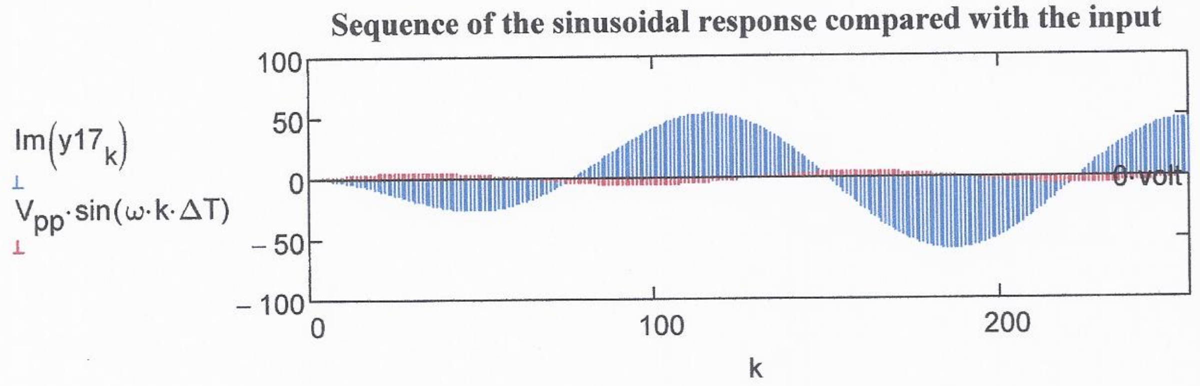


fig.:5.5.7

Bode plots:

$$H_{lp}(z) := \begin{cases} \dfrac{A_0}{z^{-2} - B_0 \cdot z^{-1} + C_0} & \text{if } \zeta \neq \omega_5 \\ \dfrac{A_0}{\left(D_0 - z^{-1}\right)^2} & \text{otherwise} \end{cases} \qquad (5.5.11)$$

$$1.1 \cdot \frac{A_0 \cdot K2 \cdot V_{pp}}{\left(p1 - \overline{p1}\right)} = 0.029j\,V$$

$A_5 = -10$ $\qquad 20 \cdot \log\left(\left|W_{lp}\left(j \cdot \sqrt{\omega_5 \cdot \omega_{smp}}\right)\right|\right) = -19.036$

$$H_{lpdB}(\omega) := 20 \cdot \log\left(\left|H_{lp}\left(e^{j \cdot \omega \cdot T_s}\right)\right|\right)$$

$$\omega := \frac{\omega_5}{U \cdot 10^{10}}, \frac{\omega_5}{U \cdot 10^{10}} + \frac{\omega_5 \cdot 10 \cdot U - \dfrac{\omega_5}{U \cdot 10^{10}}}{4 \cdot U^2} \,..\, 10 \cdot U \cdot \omega_5$$

BODE Plots of H(z) compared with that of W(jω)

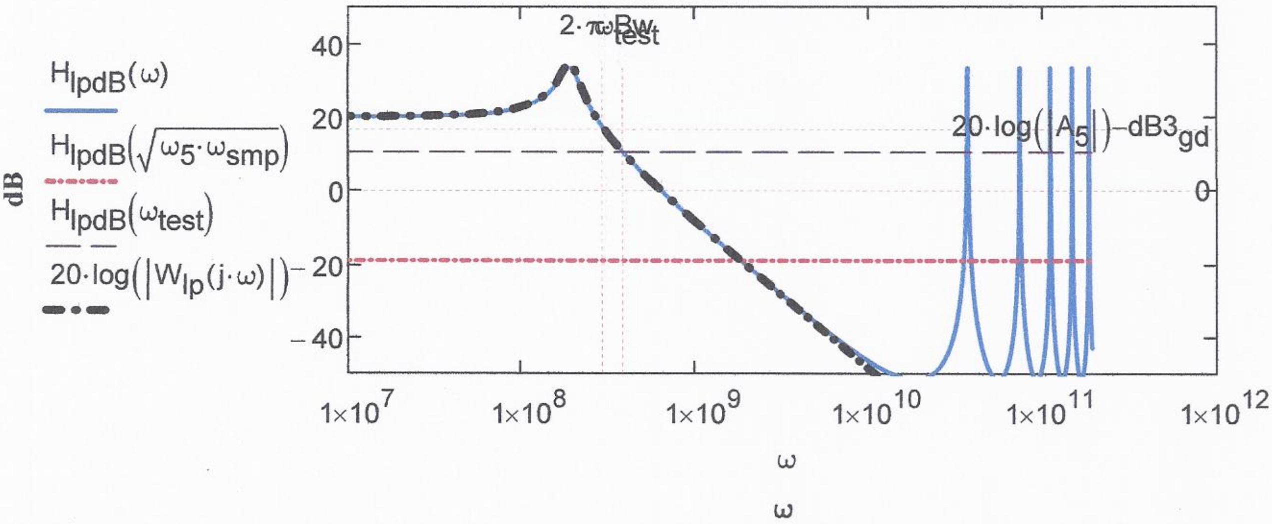


fig.:5.5.8

5.6
The bilinear transformation

5.6.1 Z-transfer function of the II° Order Low Pass Digital Filter.

On the other hand, using the bilinear transformation: $s = \frac{2}{T_s}\cdot\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$, (5.6.1)

we get a new t. f. for the given system, in the z domain.

$A_5 := A_5 \qquad \omega_5 := \omega_5 \qquad \zeta := \zeta \quad T_s := T_s \qquad \omega_5 = 0.192\cdot\frac{\text{Grads}}{\text{sec}}$

$$H4(z) := \frac{A_5\cdot\omega_5^2}{s^2 + 2\cdot\zeta\cdot s + \omega_5^2}\left|\begin{array}{l}\text{substitute}, s = \frac{2}{T_s}\cdot\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\\ \text{collect}, z\\ \text{collect}, A_5\cdot(T_s\cdot\omega_5)^2\end{array}\right. \rightarrow \tag{5.6.2}$$

$$H4(z) = \frac{(z^{-1}+1)^2}{z^{-2} + z^{-1}\cdot\frac{2\cdot T_s^2\cdot\omega_5^2 - 8}{T_s^2\cdot\omega_5^2 - 4\cdot\zeta\cdot T_s + 4} + \frac{4\cdot T_s\cdot\zeta + T_s^2\cdot\omega_5^2 + 4}{T_s^2\cdot\omega_5^2 - 4\cdot\zeta\cdot T_s + 4}}\cdot\frac{A_5\cdot T_s^2\cdot\omega_5^2}{T_s^2\cdot\omega_5^2 - 4\cdot\zeta\cdot T_s + 4}$$

$$H4(z) := \frac{(z^{-1}+1)^2}{z^{-2} + z^{-1}\cdot\frac{2\cdot T_s^2\cdot\omega_5^2 - 8}{T_s^2\cdot\omega_5^2 - 4\cdot\zeta\cdot T_s + 4} \dots + \frac{4\cdot T_s\cdot\zeta + T_s^2\cdot\omega_5^2 + 4}{T_s^2\cdot\omega_5^2 - 4\cdot\zeta\cdot T_s + 4}}\cdot\frac{A_5\cdot T_s^2\cdot\omega_5^2}{T_s^2\cdot\omega_5^2 - 4\cdot\zeta\cdot T_s + 4} \tag{5.6.3}$$

Parameters necessary for the design of the digital filter:

Consider the sampling time $T_s = 1.708\times 10^{-4}\cdot\mu s$

$$\alpha_1 := \frac{A_5\cdot\omega_5^2\cdot T_s^2}{T_s^2\cdot\omega_5^2 - 4\cdot\zeta\cdot T_s + 4} \qquad \beta_1 := \frac{2\cdot T_s^2\cdot\omega_5^2 - 8}{T_s^2\cdot\omega_5^2 - 4\cdot\zeta\cdot T_s + 4} \qquad \gamma_1 := \frac{T_s^2\cdot\omega_5^2 + 4\cdot\zeta\cdot T_s + 4}{T_s^2\cdot\omega_5^2 - 4\cdot\zeta\cdot T_s + 4}$$

$\alpha_1 = -0.0026847177 \qquad \beta_1 = -2.005003071 \qquad \gamma_1 = 1.0060769577$

result for the t. f. as a function of z^{-1}: $\omega_{smp} = 17.338\cdot\frac{\text{Grads}}{\text{sec}}$

$$\omega_5 = 0.192 \cdot \frac{Grads}{sec} \qquad \boxed{H4(z) := \alpha_1 \cdot \frac{\left(z^{-1}+1\right)^2}{z^{-2}+\beta_1 \cdot z^{-1}+\gamma_1}} \qquad (5.6.4)$$

If $\zeta = \omega_5$

$$x = z^{-1} \qquad H_{lp}(x) := A_5 \cdot \frac{\omega_5^{\,2}}{\left(s+\omega_5\right)^2} \left| \begin{array}{l} \text{substitute}, s = \dfrac{2}{T_s} \cdot \left(\dfrac{1-x}{1+x}\right) \\ \text{collect}, x \\ \text{factor} \end{array} \right. \rightarrow$$

$$\text{z transfer function:} \quad H_{lp}(z) = \frac{\left(A_5 \cdot T_s^{\,2} \cdot \omega_5^{\,2}\right)}{\left(T_s \cdot \omega_5 - 2\right)^2} \cdot \frac{\left(z^{-1}+1\right)^2}{\left[z^{-1} + \dfrac{T_s \cdot \omega_5 + 2}{\left(T_s \cdot \omega_5 - 2\right)}\right]^2} \qquad (5.6.5)$$

$$\alpha_{11} := \frac{\left(A_5 \cdot T_s^{\,2} \cdot \omega_5^{\,2}\right)}{\left(T_s \cdot \omega_5 - 2\right)^2} \qquad \beta_{22} := \frac{T_s \cdot \omega_5 + 2}{\left(T_s \cdot \omega_5 - 2\right)}$$

$$H4(z) := \left[\left| \begin{array}{ll} \alpha_1 \cdot \dfrac{\left(z^{-1}+1\right)^2}{z^{-2}+\beta_1 \cdot z^{-1}+\gamma_1} & \text{if } \zeta \neq \omega_5 \\ \alpha_{11} \cdot \dfrac{\left(z^{-1}+1\right)^2}{\left(z^{-1}+\beta_{22}\right)^2} & \text{otherwise} \end{array} \right. \right] \qquad (5.6.6)$$

Now compare the Bode diagrams of H4(z) with those of $H_{lp}(z)$. To do that, the coefficients of $H_{lp}(z)$ must redefined with the given sampling time T_s. Therefore they are rewritten here below:

$$A0 := A_5 \cdot \omega_5^{\,2} \cdot T_s^{\,2} \qquad B0 := 2 \cdot \left(1 + \zeta \cdot T_s\right) \qquad C0 := T_s \cdot \left(\omega_5^{\,2} \cdot T_s + 2 \cdot \zeta\right) + 1 \qquad D0 := T_s \cdot \omega_5 + 1$$

$$A0 = -0.01070921 \qquad B0 = 2.006060171 \qquad C0 = 1.0071310916 \qquad D0 = 1.033$$

$$H_{lp}(z) := \left| \begin{array}{ll} \dfrac{A0}{z^{-2} - B0 \cdot z^{-1} + C0} & \text{if } \zeta \neq \omega_5 \\ \dfrac{A0}{\left(D0 - z^{-1}\right)^2} & \text{otherwise} \end{array} \right. \qquad (5.6.7)$$

BODE PLOTS (Low Pass (II° order)):

$$\omega := \frac{\omega_5}{20\cdot U}, \frac{\omega_5}{20\cdot U} + \frac{\omega_5\cdot U - \frac{\omega_5}{20\cdot U}}{U^2} .. 10\cdot U\cdot \omega_5$$

Frequency response of the II° order Low Pass Filter

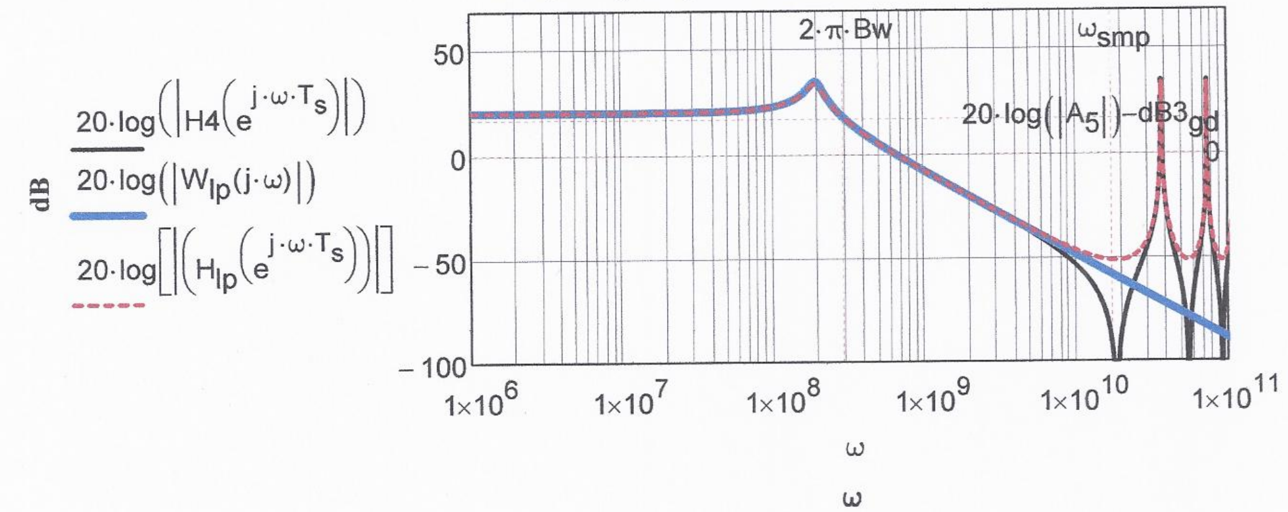


fig.:5.6.1

Phase Response of the II° order Low Pass Filter

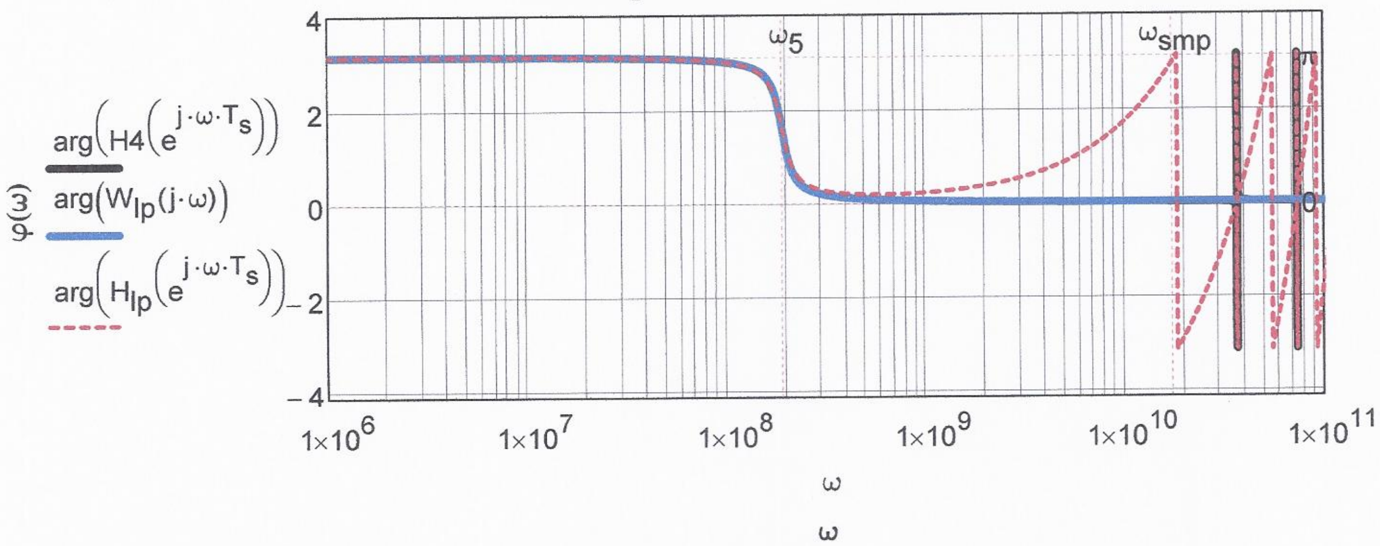


fig.:5.6.2

5.6.2 Difference equations (II°order Low Pass filter). Canonical form.

$$H4(z) = \alpha_1 \cdot \frac{\left(z^{-1}+1\right)^2}{z^{-2}+\beta_1 \cdot z^{-1}+\gamma_1} \qquad (5.6.2.1)$$

$$H4(z) = \frac{Y2(z)}{X(z)} = \frac{Y2(z)}{W1(z)} \cdot \frac{W1(z)}{X(z)} \qquad (5.6.2.2)$$

$$\frac{Y2(z)}{W1(z)} = \alpha_1 \cdot \left(1+2 \cdot z^{-1}+z^{-2}\right) \qquad (5.6.2.3)$$

$$Y2(z) = \alpha_1 \cdot \left(1+2 \cdot z^{-1}+z^{-2}\right) \cdot W1(z) \qquad (5.6.2.4)$$

$$\boxed{y2(\nu) = \alpha_1 \cdot ((w1(\nu)+2 \cdot w1(\nu-1)+w1(\nu-2)))} \qquad (5.6.2.5)$$

$$\frac{W1(z)}{X(z)} = \frac{1}{z^{-2}+\beta_1 \cdot z^{-1}+\gamma_1} \qquad (5.6.2.6)$$

$$X(z) = \left(z^{-2}+\beta_1 \cdot z^{-1}+\gamma_1\right) \cdot W1(z) \qquad (5.6.2.7)$$

$$X(z) = \gamma_1 \cdot W1(z)+\beta_1 \cdot z^{-1} \cdot W1(z)+z^{-2} \cdot W1(z) \qquad (5.6.2.8)$$

$$\boxed{x(\nu) = \gamma_1 \cdot w1(\nu)+\beta_1 \cdot w1(\nu-1)+w1(\nu-2)} \qquad (5.6.2.9)$$

The corresponding set of difference equations is:

$$w1(\nu) = \frac{x(\nu)}{\gamma_1} - \frac{\beta_1}{\gamma_1} \cdot w1(\nu-1) - \frac{1}{\gamma_1} \cdot w1(\nu-2) \qquad (5.6.2.10)$$

$$y2(\nu) = \alpha_1 \cdot (w1(\nu)+2 \cdot w1(\nu-1)+w1(\nu-2)) \qquad (5.6.2.11)$$

$$\alpha_1 := \alpha_1 \qquad \beta_1 := \beta_1 \qquad \gamma_1 := \gamma_1$$

Z T. Initial value theorem:

$$\lim_{z \to \infty} \left(\alpha_1 \cdot \frac{1+2 \cdot z^{-1}+z^{-2}}{z^{-2}+\beta_1 \cdot z^{-1}+\gamma_1}\right) \to \frac{\alpha_1}{\gamma_1}$$

Z T. Final value theorem:

$$\lim_{z \to 0} \left(\alpha_1 \cdot \frac{1+2 \cdot z^{-1}+z^{-2}}{z^{-2}+\beta_1 \cdot z^{-1}+\gamma_1}\right) \to \alpha_1$$

5.6.2.1 Sequence of the voltage Step response.

Recurrence relations:

$$v_i(\nu) := u1_\nu \qquad (5.6.2.1.1)$$

1) $$w1(\nu) := \begin{cases} \dfrac{v_i(\nu) - \beta_1 \cdot w1(\nu-1) - w1(\nu-2)}{\gamma_1} & \text{if } \nu > 1 \\ \dfrac{\alpha_1}{\gamma_1} \cdot v_i(\nu) & \text{if } \nu = 0 \\ \left(\dfrac{v_i(1) - \beta_1 \cdot w1(0)}{\gamma_1}\right) & \text{if } \nu = 1 \end{cases} \qquad (5.6.2.1.2)$$

2) $$y2(\nu) := \begin{cases} \alpha_1 \cdot ((w1(\nu) + 2 \cdot w1(\nu-1) + w1(\nu-2))) & \text{if } \nu > 1 \\ \alpha_1 \cdot ((w1(1) + 2 \cdot w1(0))) & \text{if } \nu = 1 \\ \alpha_1 \cdot w1(0) & \text{otherwise} \end{cases} \qquad (5.6.2.1.3)$$

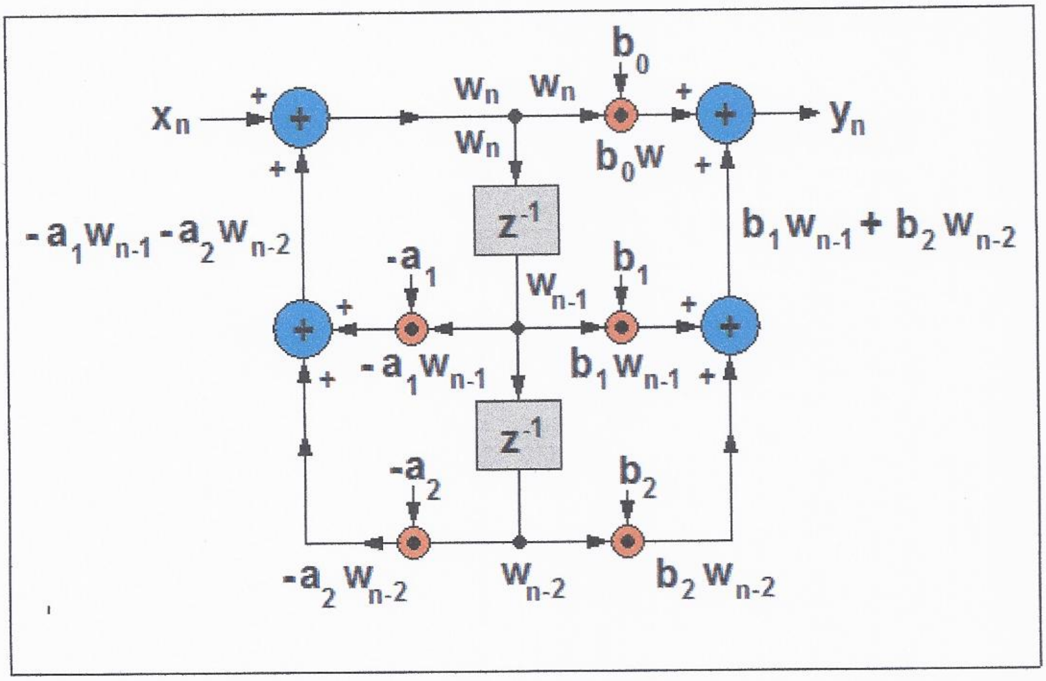
Same relations but vectorized: $W1_\nu := 0.0$

1) $$W1_\nu := \begin{cases} \dfrac{\dfrac{u1_\nu}{volt} - \beta_1 \cdot W1_{\nu-1} - W1_{\nu-2}}{\gamma_1} & \text{if } \nu > 1 \\ \dfrac{1}{\gamma_1} \cdot \dfrac{u1_\nu}{volt} & \text{if } \nu = 0 \\ \dfrac{\dfrac{u1_1}{volt} - \beta_1 \cdot W1_0}{\gamma_1} & \text{otherwise} \end{cases} \qquad (5.6.2.1.4)$$

2) $$Y2_\nu := \begin{cases} \alpha_1 \cdot \left(W1_\nu + 2 \cdot W1_{\nu-1} + W1_{\nu-2}\right) & \text{if } \nu > 1 \\ \alpha_1 \cdot W1_0 & \text{if } \nu = 0 \\ \alpha_1 \cdot \left(W1_1 + 2 \cdot W1_0\right) & \text{otherwise} \end{cases} \qquad (5.6.2.1.5)$$

▶ BILINEAR

Block diagram of the difference equation algorithm for a second order system



Input signal sequence

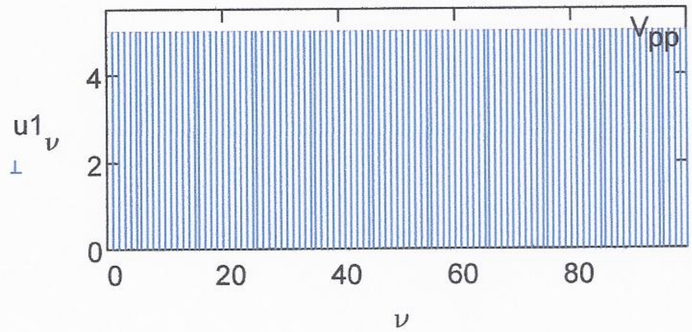


fig.:5.6.2.1.1

$A_5 = -10 \qquad bl1 := \text{BILINEAR}\left(\frac{u1}{V}, A_5, \zeta, \omega_5, T_{sstp}, N0\right)$ (5.6.2.1.6)

$bl1 = (-0.248 \quad -1.959 \quad 1.058 \quad \{256,1\} \quad \{256,1\})$

$a00 := bl1_{0,0} \qquad b00 := bl1_{0,1} \qquad c00 := bl1_{0,2} \qquad W00 := bl1_{0,3} \qquad Y11 := bl1_{0,4}$

$a5 = -17.13472986 \qquad b5 = 2.2424068406 \qquad c5 = 2.9558798269$

Sequence of the state function

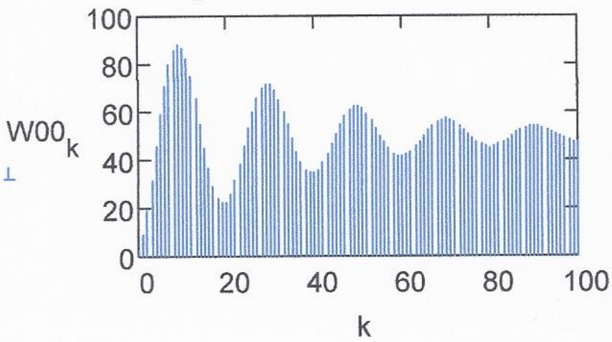


fig.:5.6.2.1.2

Output Sequence

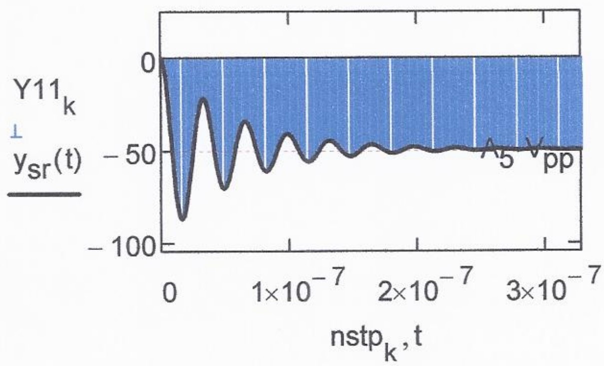


fig.:5.6.2.1.3

5.6.2.2 Sequence of the Short Voltage Pulse response.

$T_{test} = 16.394 \cdot ns$ $\quad T_{svp} = 10.93 \cdot ns \quad \tau = 5.218 \times 10^{-3} \cdot \mu s$

Chosen Test signal period, $T_{test} = 16.394 \cdot ns$ $\quad \frac{1}{T_{test}} = 0.061 \cdot GHz$

Short pulse sequence of amplitude V_i:

Pulse Sequence.

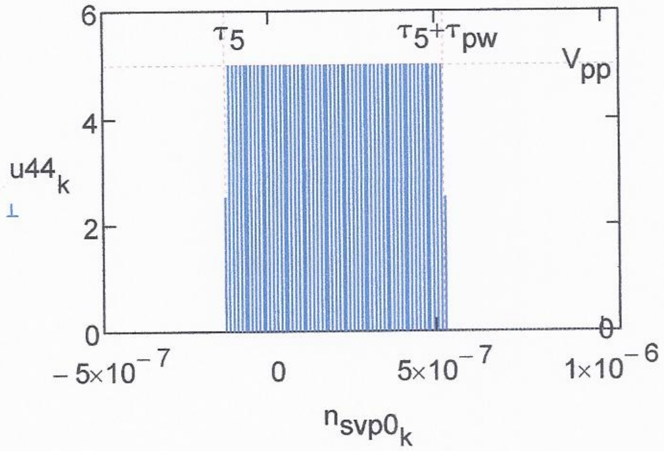


fig.:5.6.2.2.1

$$bl2 := BILINEAR(u44, A_5, \zeta, \omega_5, T_{svp}, N0) \qquad (5.6.2.2.1)$$

$$bl2 = (-5.764 \quad 0.102 \quad 1.204 \quad \{256,1\} \quad \{256,1\})$$

$a11 := bl2_{0,0}$ $\quad b11 := bl2_{0,1}$ $\quad c11 := bl2_{0,2}$ $\quad W11 := bl2_{0,3}$ $\quad Y22 := bl2_{0,4}$

$a11 = -5.7635166$ $\quad b11 = 0.101563937$ $\quad c11 = 1.2038427013$

Block diagram of the difference equation algorithm for a second order system

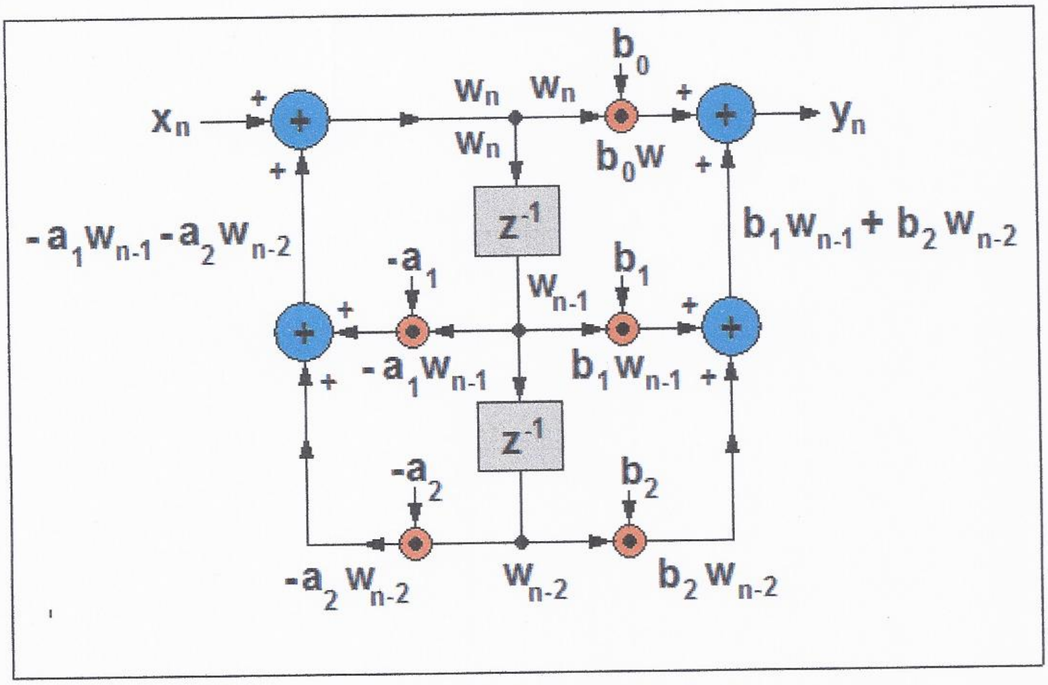


fig.:5.6.2.2.2

Sequence of the voltage Step response.

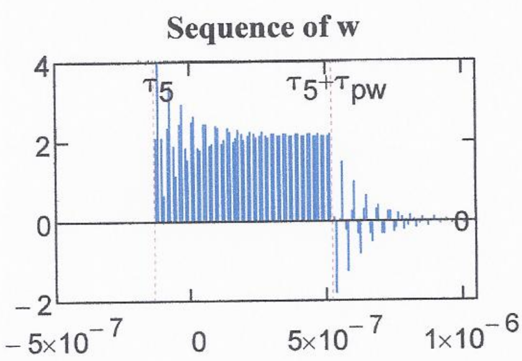


fig.:5.6.2.2.3

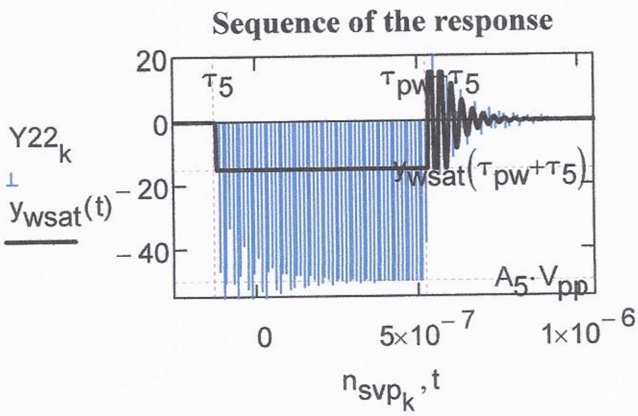


fig.:5.6.2.2.4

$Spec18x := fft(Y22)$

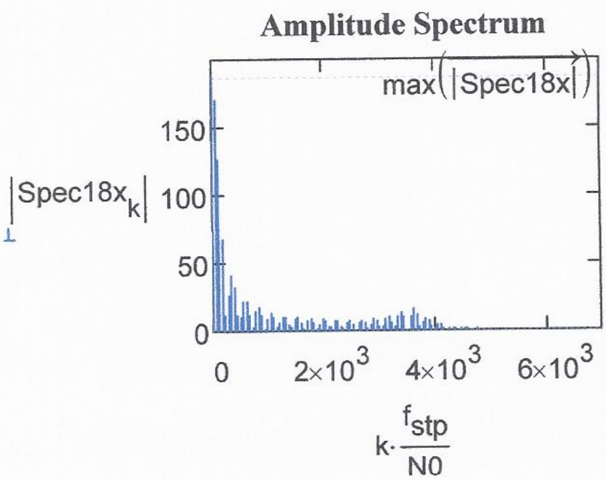


fig.:5.6.2.2.5

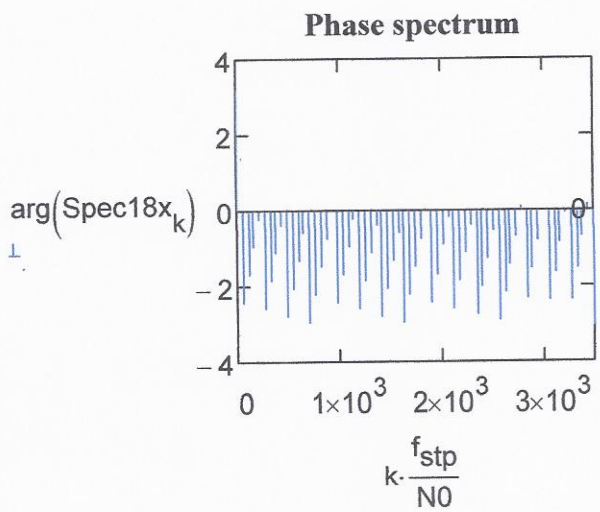
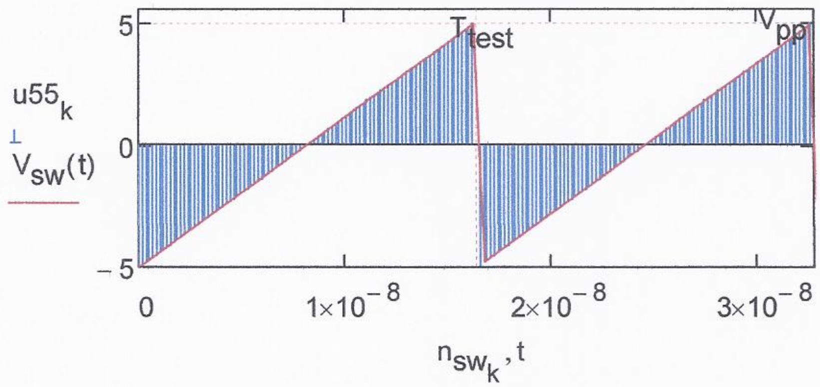


fig.:5.6.2.2.6

5.6.2.3 Sequence of the Sawtooth Response



$$\frac{T_5}{T_{test}} = 2$$

fig.:5.6.2.3.1

$\omega_5 = 191.627 \cdot \frac{Mrads}{sec}$ Signal bandwidth: $\boxed{B_{sw} = 2.928 \times 10^3 \cdot MHz}$ $f_{ssaw} := 2 \cdot B_{sw}$

$f_{ssaw} = 5.856 \times 10^3 \cdot MHz$ $\omega_{ssaw} := 2 \cdot \pi \cdot f_{ssaw}$

$\boxed{Parseval_{sw} = 16.466}$ $\boxed{Average1 \cdot volt = 0 \cdot V}$ $\boxed{RMS1 \cdot volt = 2.887 \times 10^3 \cdot mV}$

$T_{test} = 1.639 \times 10^{-8} s$ $\zeta = 0.018 \cdot \frac{Grads}{sec}$ $T_{ssaw} := \frac{1}{f_{ssaw}}$ $\omega_{ssaw} = 36.792 \cdot \frac{Grads}{sec}$

$\zeta \cdot T_s = 3.03 \times 10^{-3}$ $T_{ssaw} = 1.708 \times 10^{-10} s$ $A_5 = -10$ $Q_5 = 5.4$

$T_5 = 0.033 \cdot \mu s$ $\frac{f_{ssaw}}{f_{test}} = 96$ $T_{test} = 0.016 \cdot \mu s$

$A_5 = -10$ $bl3 := BILINEAR\left(\frac{u10}{V}, A_5, \zeta, \omega_5, T_{ssaw}, N0\right)$ (5.6.2.3.1)

$bl3 = \left(-2.685 \times 10^{-3} \quad -2.005 \quad 1.006 \quad \{256,1\} \quad \{256,1\}\right)$

$a22 := bl3_{0,0}$ $b22 := bl3_{0,1}$ $c22 := bl3_{0,2}$ $W22 := bl3_{0,3}$ $Y33 := bl3_{0,4}$

$a22 = -0.00268472$ $b22 = -2.0050030707$ $c22 = 1.0060769577$

Block diagram of the difference equation algorithm for a second order system

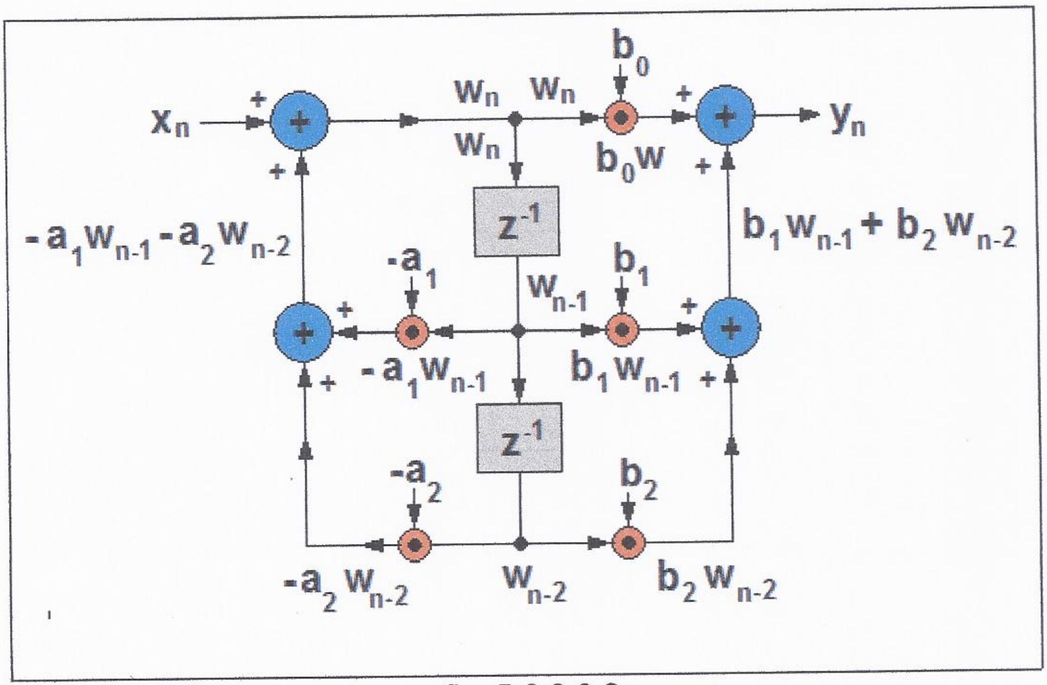


fig.:5.6.2.3.2

$A_5 = -10$ ***Sequence of the Sawtooth response.*** $Q_5 = 5.4$

$T_{test} = 0.016 \cdot \mu s$

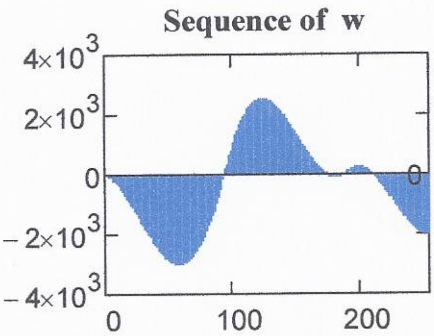


fig.:5.6.2.3.3

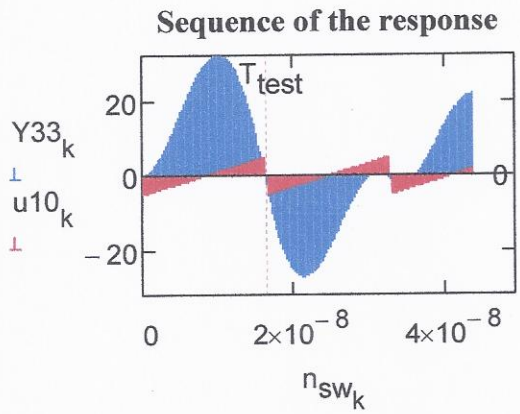


fig.:5.6.2.3.4

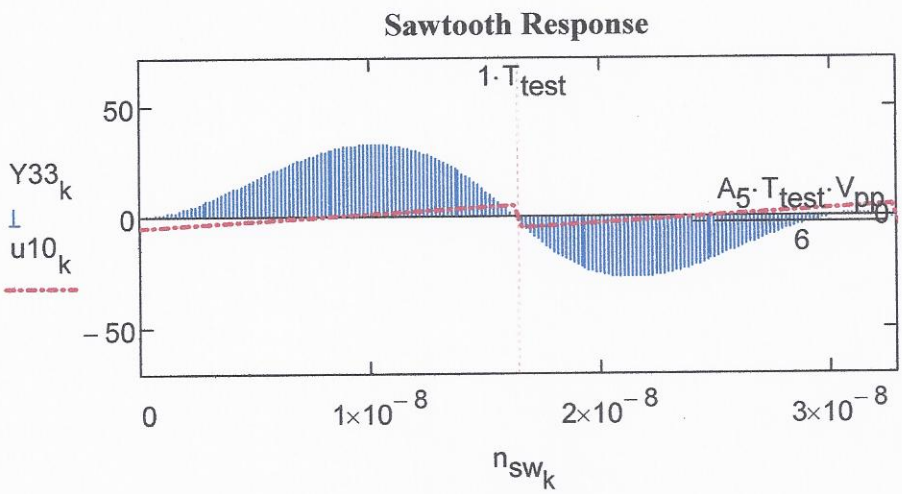


fig.:5.6.2.3.5

$$SpeC_{00x} := fft(Y33)$$

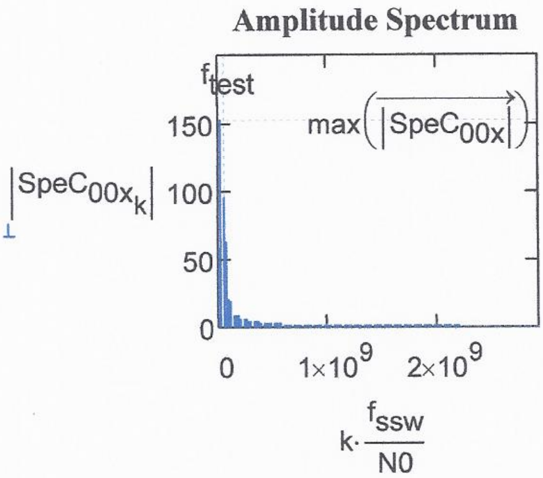


fig.:5.6.2.3.6

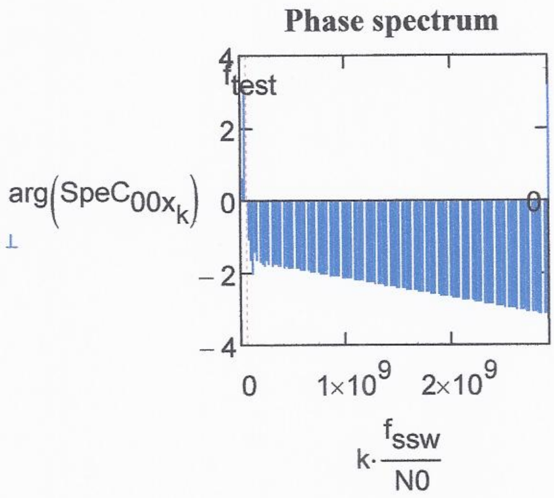


fig.:5.6.2.3.7

$$max\left(\overrightarrow{|SpeC_{00x}|}\right) = 153.4\frac{1}{V}\cdot V$$

5.6 *The bilinear transformation*

5.6.2.4 Sequence of the Bipolar Square Wave response.

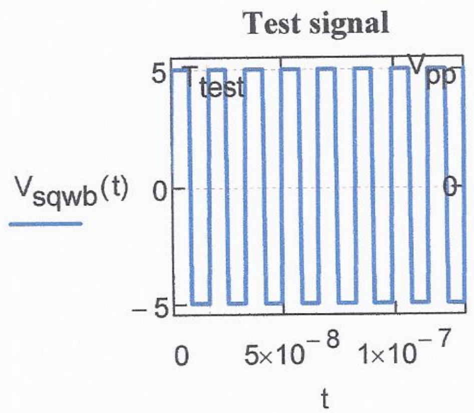


fig.:5.6.2.4.1

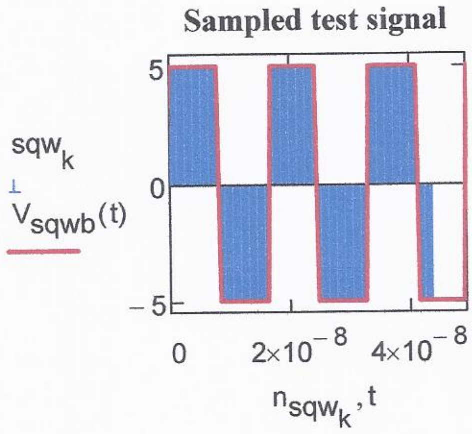


fig.:5.6.2.4.2

$$bl4 := BILINEAR(sqw, A_5, \zeta, \omega_5, T_{ssqw}, N0) \qquad (5.6.2.4.1)$$

$$bl4 = \left(-2.685 \times 10^{-3} \quad -2.005 \quad 1.006 \quad \{256,1\} \quad \{256,1\}\right)$$

$a33 := bl4_{0,0}$ $\quad b33 := bl4_{0,1}$ $\quad c33 := bl4_{0,2}$ $\quad W33 := bl4_{0,3}$ $\quad Y44 := bl4_{0,4}$

$a33 = -0.00268472$ $\quad b33 = -2.0050030707$ $\quad c33 = 1.0060769577$

Block diagram of the difference equation algorithm for a second order system

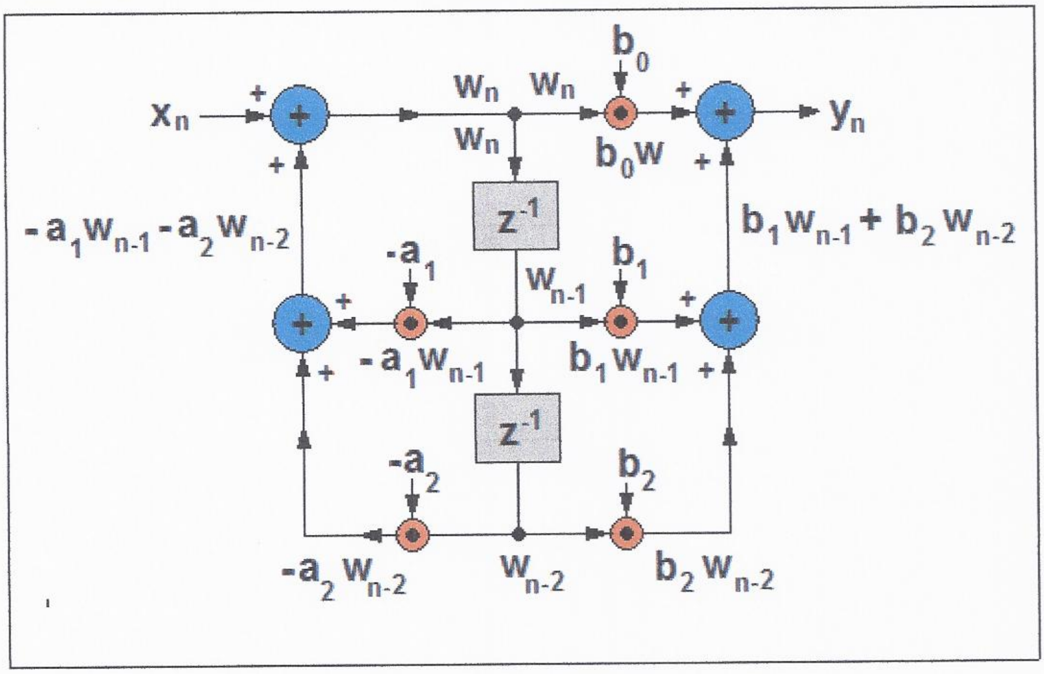


fig.:5.6.2.4.3

Sequence of the Bipolar Square Wave response.

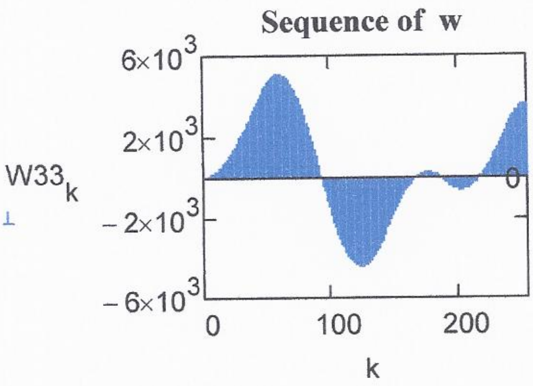


fig.:5.6.2.4.4

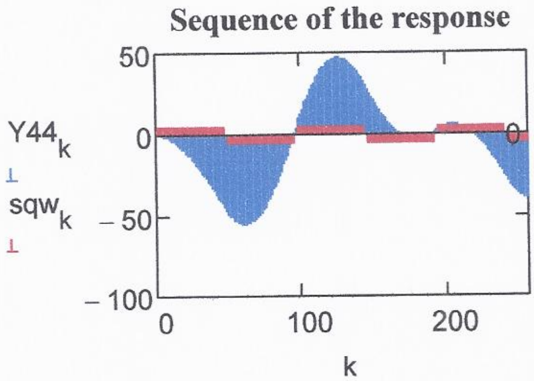


fig.:5.6.2.4.5

$$SpeC_{01x} := \mathrm{fft}(Y44) \qquad (5.6.2.4.2)$$

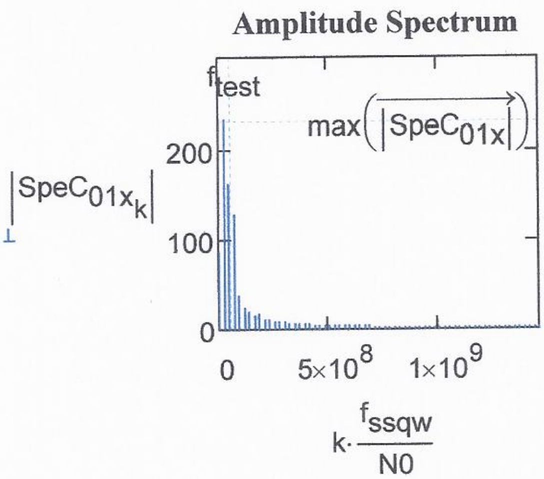


fig.:5.6.2.4.6

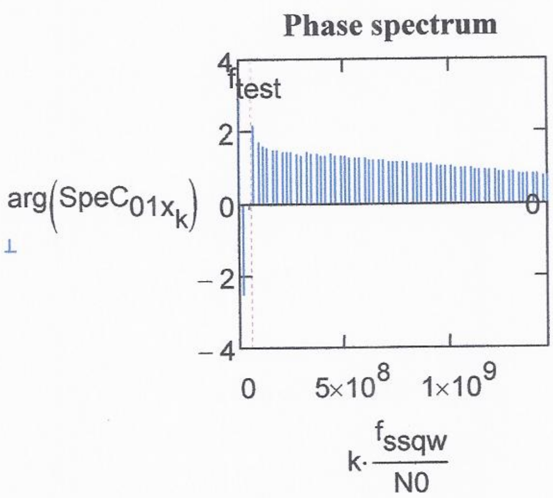


fig.:5.6.2.4.7

$$\max\left(\overrightarrow{|SpeC_{01x}|}\right) = 230.965$$

5.6.2.5 Sequence of the (single tone) AM Signal response.

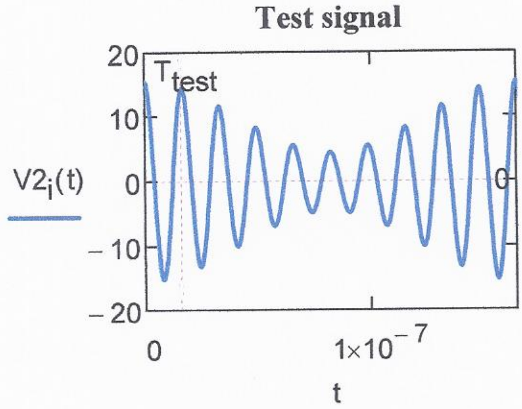


fig.:5.6.2.5.1

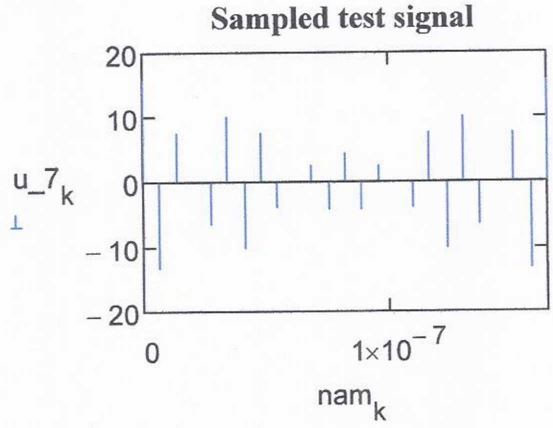


fig.:5.6.2.5.2

$$bl5 := BILINEAR(u\_7, A_5, \zeta, \omega_5, T_{sam}, N0) \qquad (5.6.2.5.1)$$

$$bl5 = (-3.277 \quad -0.875 \quad 1.185 \quad \{256,1\} \quad \{256,1\})$$

$a44 := bl5_{0,0}$ $\quad b44 := bl5_{0,1}$ $\quad c44 := bl5_{0,2}$ $\quad W44 := bl5_{0,3}$ $\quad Y55 := bl5_{0,4}$

$a44 = -3.27707905$ $\quad b44 = -0.8746131198$ $\quad c44 = 1.1854447394$

Block diagram of the difference equation algorithm for a second order system

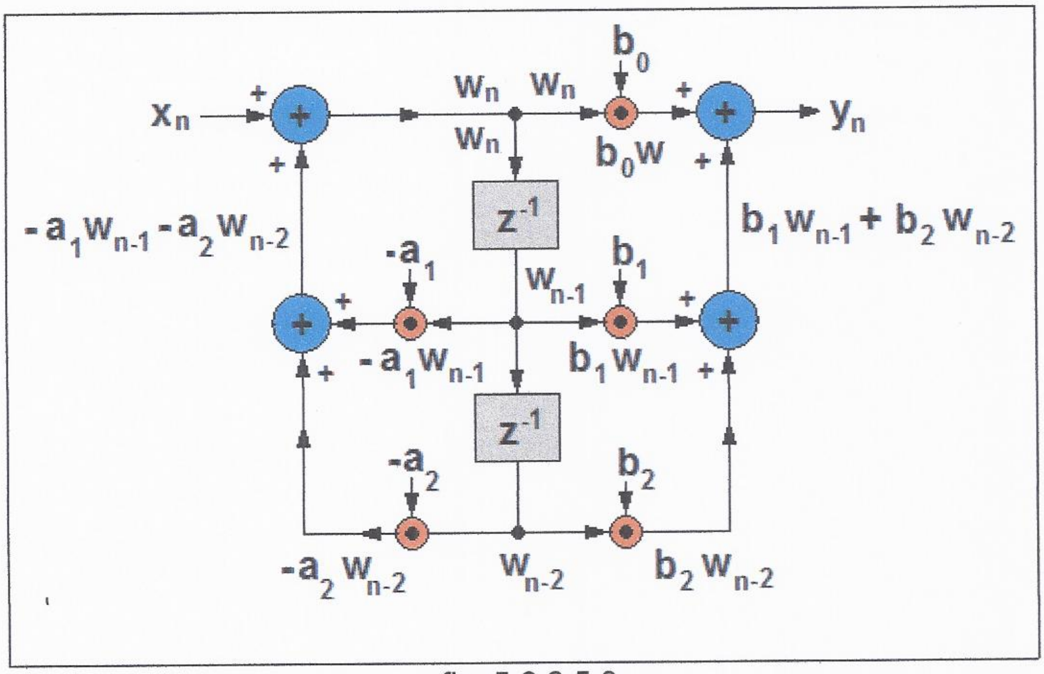


fig.:5.6.2.5.3

Sequence of the AM (single tone) Signal response.

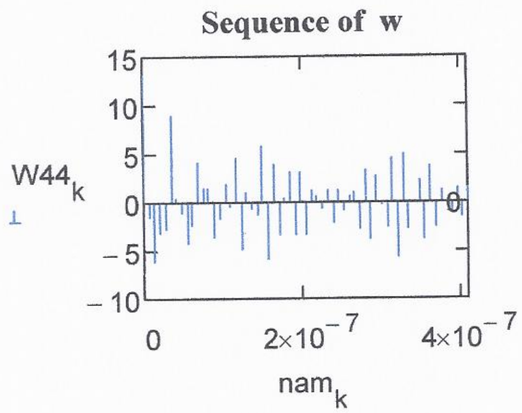


fig.:5.6.2.5.4

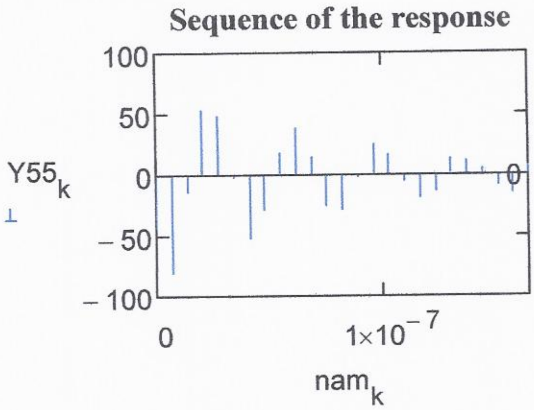


fig.:5.6.2.5.5

$SpeC_{01x} := \mathrm{fft}(Y55)$

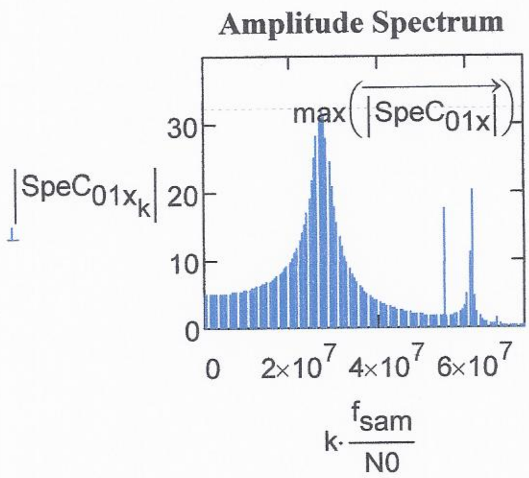


fig.:5.6.2.5.6

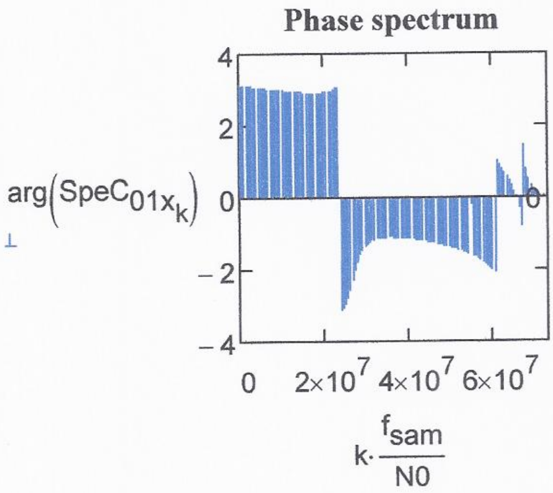


fig.:5.6.2.5.7

$$\max\left(\overrightarrow{\left|SpeC_{01x}\right|}\right) = 32.497$$

5.6.2.6 Sequence of the (single tone) Frequency Modulated carrier response .

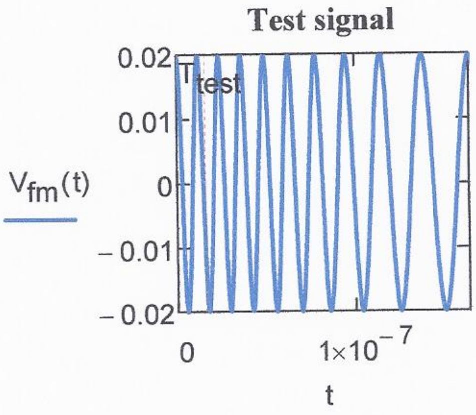


fig.:5.6.2.6.1

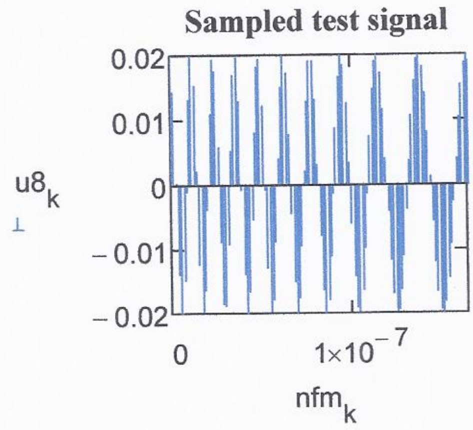


fig.:5.6.2.6.2

$$bl6 := \text{BILINEAR}\left(u8, A_5, \zeta, \omega_5, T_{sfm}, N0\right) \qquad (5.6.2.6.1)$$

$bl6 = (-0.194 \quad -1.974 \quad 1.052 \quad \{256,1\} \quad \{256,1\})$

$a55 := bl6_{0,0}$ $\quad b55 := bl6_{0,1}$ $\quad c55 := bl6_{0,2}$ $\quad W5 := bl6_{0,3}$ $\quad Y66 := bl6_{0,4}$

$a55 = -0.19414962$ $\quad b55 = -1.9741133834$ $\quad c55 = 1.0517732332$

Block diagram of the difference equation algorithm for a second order system

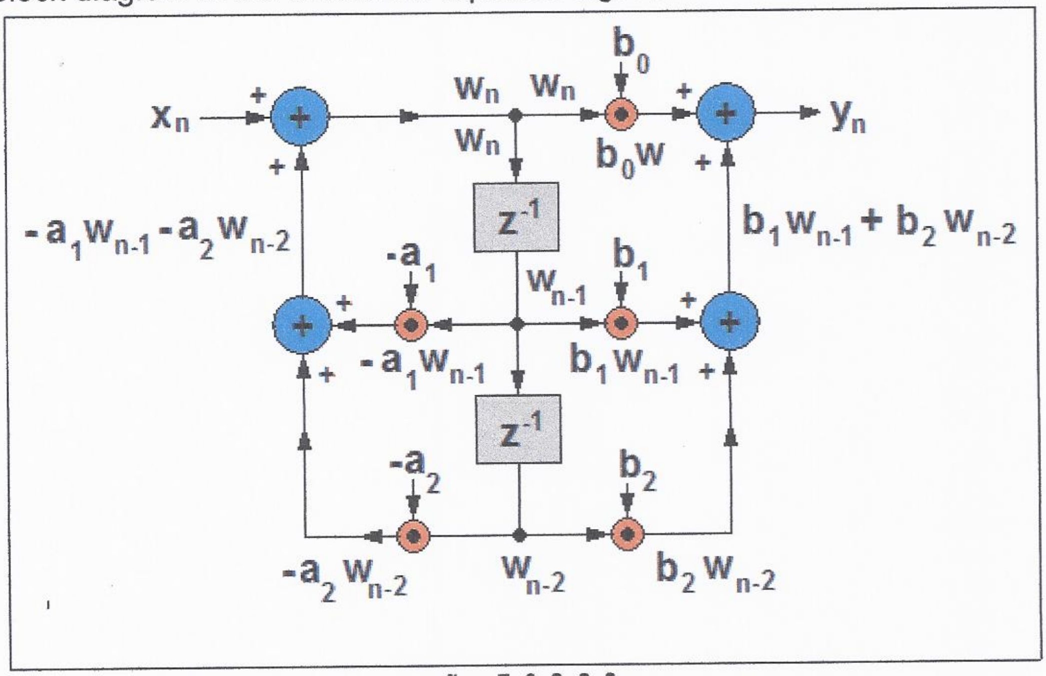


fig.:5.6.2.6.3

Sequence of the Frequency (single tone) Modulated carrier response.

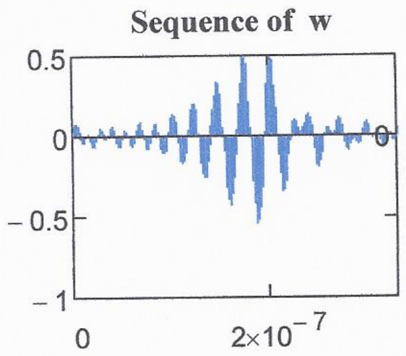


fig.:5.6.2.6.4

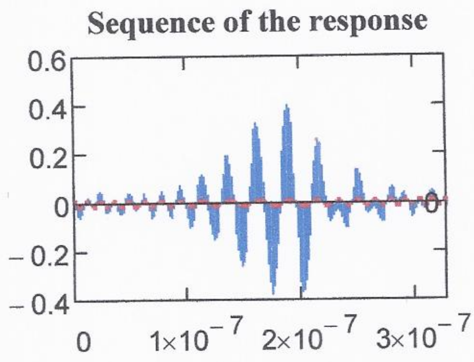


fig.:5.6.2.6.5

$$SpeC_{02x} := \mathrm{fft}(Y66)$$

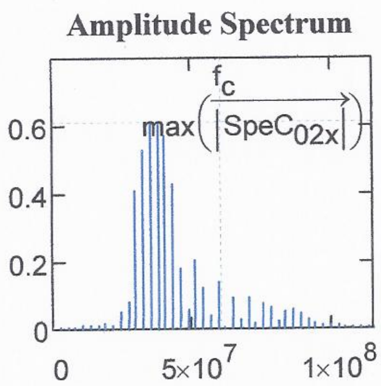


fig.:5.6.2.6.6

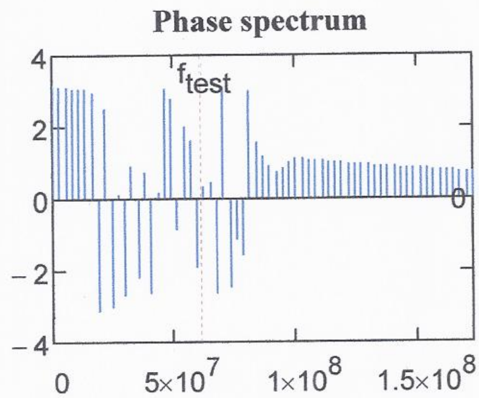


fig.:5.6.2.6.7

$$\max\left(\overrightarrow{\left|SpeC_{02x}\right|}\right) = 0.614$$

5.6.2.7 Sequence of the (single tone) Phase Modulated carrier response.

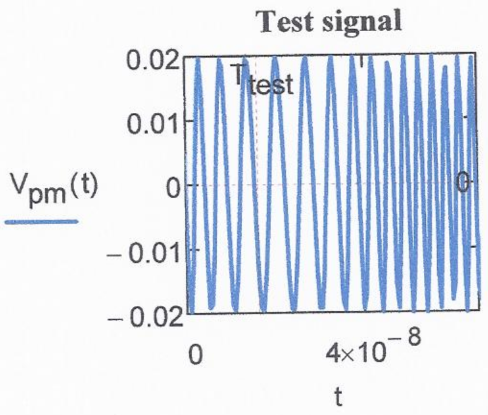


fig.:5.6.2.7.1

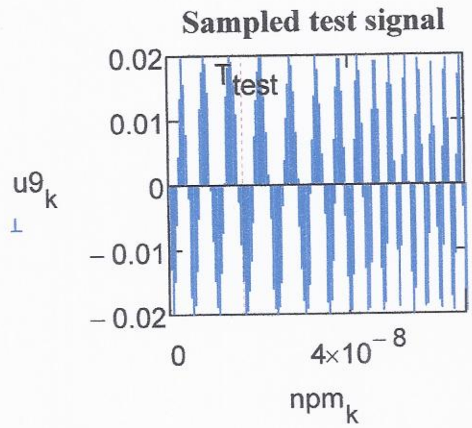


fig.:5.6.2.7.2

$$bl7 := \text{BILINEAR}\left(u9, A_5, \zeta, \omega_5, T_{spm}, N0\right) \tag{5.6.2.7.1}$$

$$bl7 = (-0.012 \quad -2.008 \quad 1.013 \quad \{256,1\} \quad \{256,1\})$$

$a66 := bl7_{0,0}$ $\quad b66 := bl7_{0,1}$ $\quad c66 := bl7_{0,2}$ $\quad W66 := bl7_{0,3}$ $\quad Y77 := bl7_{0,4}$

$a66 = -0.01211965$ $\quad b66 = -2.0080797634$ $\quad c66 = 1.0129276215$

Block diagram of the difference equation algorithm for a second order system

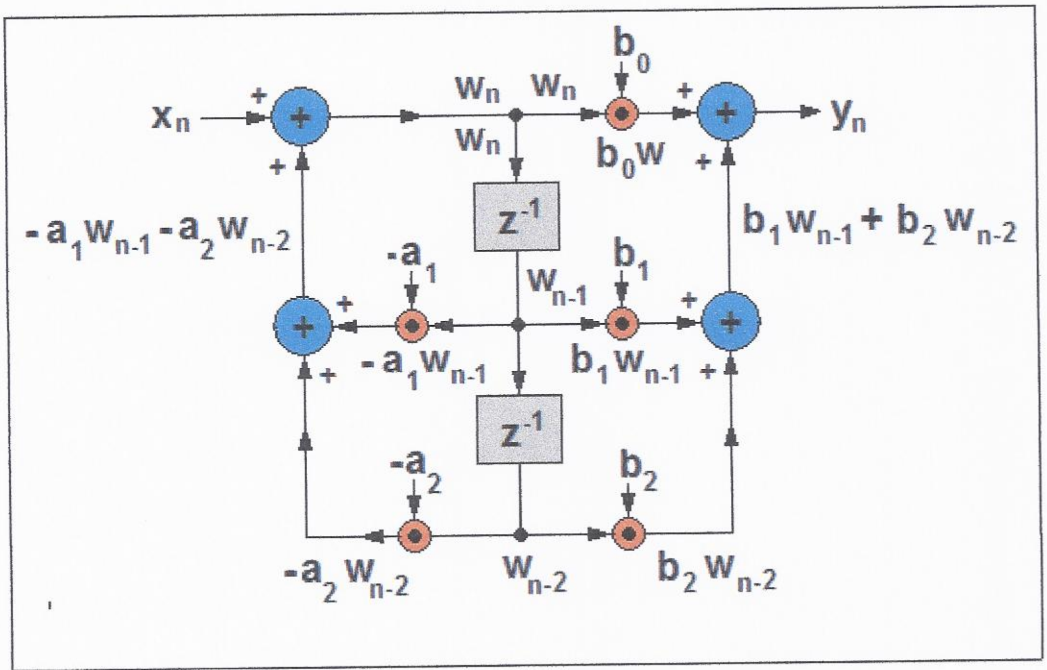


fig.:5.6.2.7.3

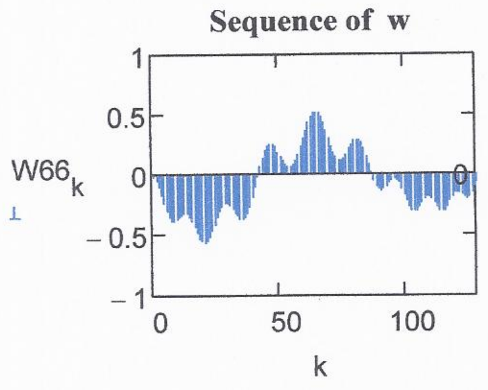


fig.:5.6.2.7.4

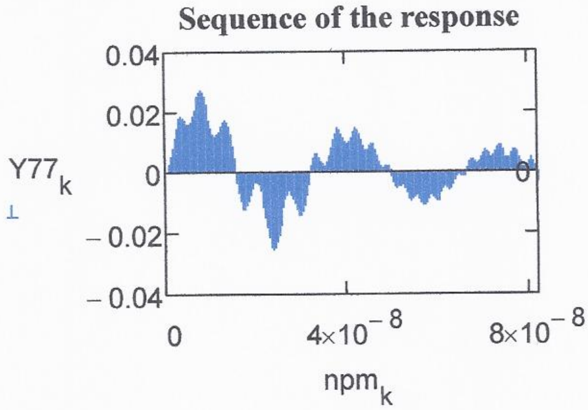


fig.:5.6.2.7.5

$$SpeC_{03x} := \mathrm{fft}(Y77)$$

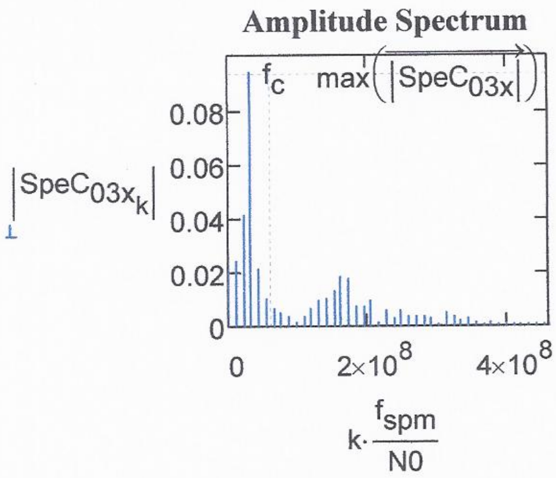


fig.:5.6.2.7.6

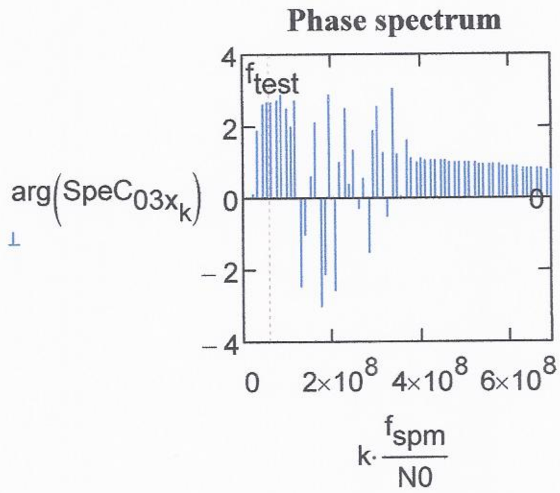


fig.:5.6.2.7.7

$$\max\left(\overrightarrow{\left|SpeC_{00x}\right|}\right) = 153.4$$

5.7
Synthetic Division algorithm (considering the bilinear transformation).

Search of the sequence corresponding to the z t. f. knowing its numerator and denominator coefficients:

$$\alpha_2 := \begin{vmatrix} \dfrac{A_5 \cdot \omega_5{}^2 \cdot T_s{}^2}{T_s{}^2 \cdot \omega_5{}^2 - 4 \cdot \zeta \cdot T_s + 4} & \text{if } \zeta \neq \omega_5 \\ \dfrac{A_5 \cdot T_s{}^2 \cdot \omega_5{}^2}{(T_s \cdot \omega_5 - 2)^2} & \text{otherwise} \end{vmatrix} \quad , \quad \beta_2 := \begin{vmatrix} \dfrac{2 \cdot T_s{}^2 \cdot \omega_5{}^2 - 8}{T_s{}^2 \cdot \omega_5{}^2 - 4 \cdot \zeta \cdot T_s + 4} & \text{if } \zeta \neq \omega_5 \\ 2 \cdot \dfrac{T_s{}^2 \cdot \omega_5{}^2 - 4}{(T_s \cdot \omega_5 - 2)^2} & \text{otherwise} \end{vmatrix} \quad ,$$

$$\gamma_2 := \begin{vmatrix} \dfrac{T_s{}^2 \cdot \omega_5{}^2 + 4 \cdot \zeta \cdot T_s + 4}{T_s{}^2 \cdot \omega_5{}^2 - 4 \cdot \zeta \cdot T_s + 4} & \text{if } \zeta \neq \omega_5 \\ \dfrac{(T_s \cdot \omega_5 + 2)^2}{(T_s \cdot \omega_5 - 2)^2} & \text{otherwise} \end{vmatrix}$$

$\zeta - \omega_5 = -173.884 \cdot \dfrac{\text{Mrads}}{\text{sec}}$ $\qquad \alpha_2 = -2.685 \times 10^{-3}$ $\qquad \beta_2 = -2.005$

$\gamma_2 = 1.006$

The z t. f. is:

$$H_{lp}(z) := \alpha_2 \cdot \frac{1 + 2 \cdot z^{-1} + z^{-2}}{z^{-2} + \beta_2 \cdot z^{-1} + \gamma_2}, \qquad (5.7.1)$$

Numerator $N := 2$ $\qquad$ Denominator $M := 2$

$N1 := N + M$

Numerator Coeffs.	Denominator Coeffs.	
$b_\nu := 0.0$	$a_\nu := 0.0$	$h3_k := 0$
$b_0 := \alpha_2$	$a_0 := \gamma_2$	$a_0 = ▮$
$b_1 := 2 \cdot \alpha_2$	$a_1 := \beta_2$	$a_1 = ▮$
$b_2 := \alpha_2$	$a_2 := 1$	$a_2 = ▮$

$N1 = 4$ $\qquad h3_0 := \dfrac{b_0}{a_0}$ $\qquad$ $$h3_\nu := \frac{1}{a_0} \cdot \left[b_\nu - \sum_{i=1}^{\nu} \left(h3_{\nu-i} \cdot a_i \right) \right] \qquad (5.7.2)$$

In this worksheet will be used the following program:

SYNDIVBL

5.7 Synthetic Division algorithm (considering the bilinear transformation).

5.7.1 Sequence of the voltage Step response.

$T_s := T_{sstp}$ $\qquad$ $syndbl := SYNDIVBL\left(\frac{u1}{V}, A_5, \zeta, \omega_5, T_{sstp}, N0\right)$ $\qquad$ (5.7.1.1)

$syndbl = (-0.248 \quad -1.959 \quad 1.058 \quad \{256,1\} \quad \{256,1\})$

$a5 := syndbl_{0,0}$ $\quad$ $b5 := syndbl_{0,1}$ $\quad$ $c5 := syndbl_{0,2}$ $\quad$ $h3 := syndbl_{0,3}$ $\quad$ $Y24 := syndbl_{0,4}$

$a5 = -0.24783425$ $\quad$ $b5 = -1.9593019235$ $\quad$ $c5 = 1.0584356239$

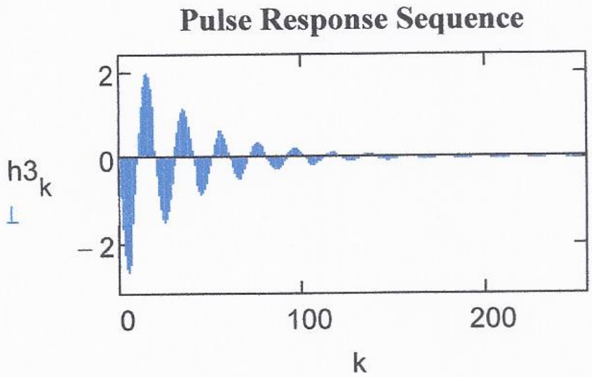


fig.:5.7.1.1

$t := 0\cdot\tau, \frac{40\cdot\tau}{1000} .. 40\cdot\tau$ $\qquad$ $y24_\nu := \sum_{k=0}^{\nu} \left(\text{if}\left(\nu - k \geq 0, h3_k \cdot u1_{\nu-k}, 0\right)\right)$ $\qquad$ (5.7.1.2)

$V_{pp} = 5V$

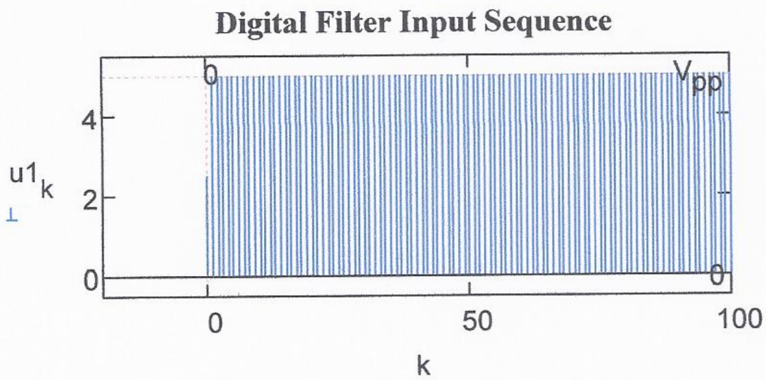


fig.:5.7.1.2

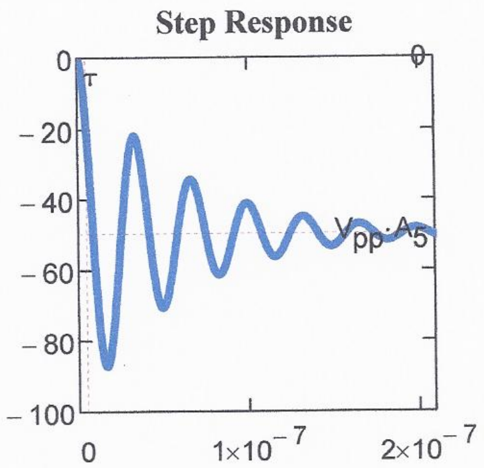


fig.:5.7.1.3

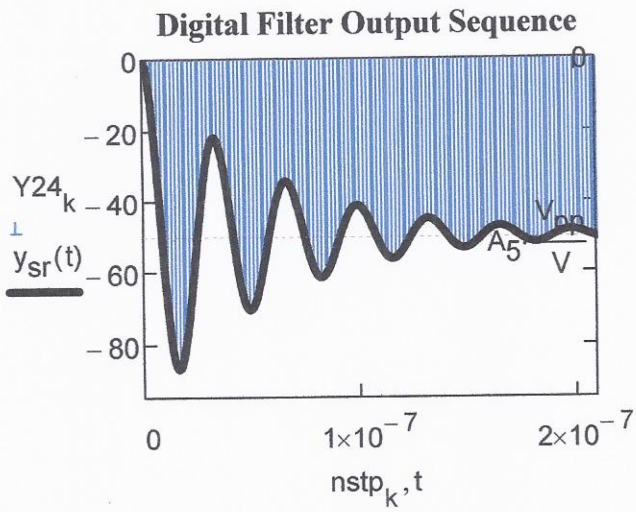


fig.:5.7.1.4

5.7 Synthetic Division algorithm (considering the bilinear transformation).

5.7.2 Sequence of the Short Voltage Pulse response.

$$syndbl1 := SYNDIVBL\left(u44, A_5, \zeta, \omega_5, T_{svp}, N0\right) \qquad (5.7.2.1)$$

$$syndbl1 = (-5.764 \quad 0.102 \quad 1.204 \quad \{256,1\} \quad \{256,1\})$$

$$h4 := syndbl1_{0,3} \qquad Y25 := syndbl1_{0,4} \qquad (5.7.2.2)$$

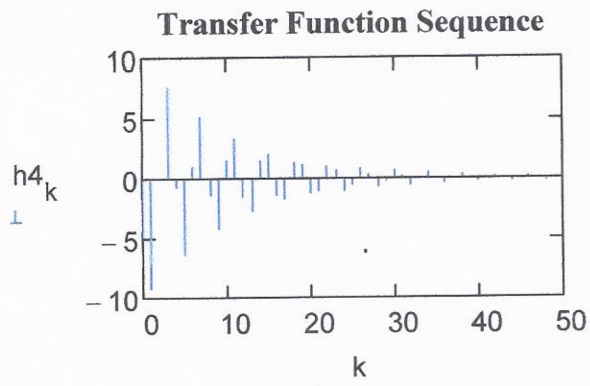


fig.:5.7.2.1

$$\tau_5 = -131.154 \cdot ns \qquad t := 1\cdot\tau_5, \frac{\tau_5 + 8\cdot\left(\tau_{pw} + \tau_5\right) - 1\cdot\tau_5}{10^{15}} .. \tau_5 + 8\cdot\left(\tau_{pw} + \tau_5\right)$$

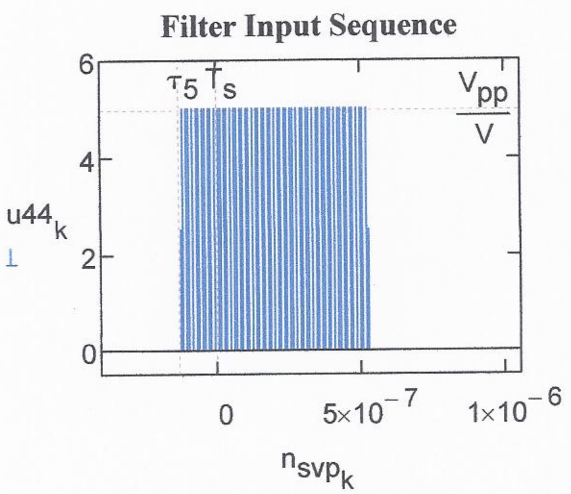


fig.:5.7.2.2

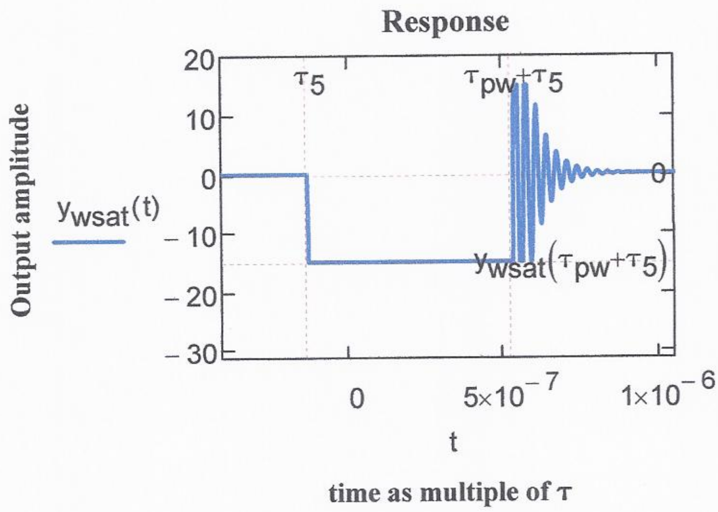


fig.:5.7.2.3

$Q_5 = 5.4$

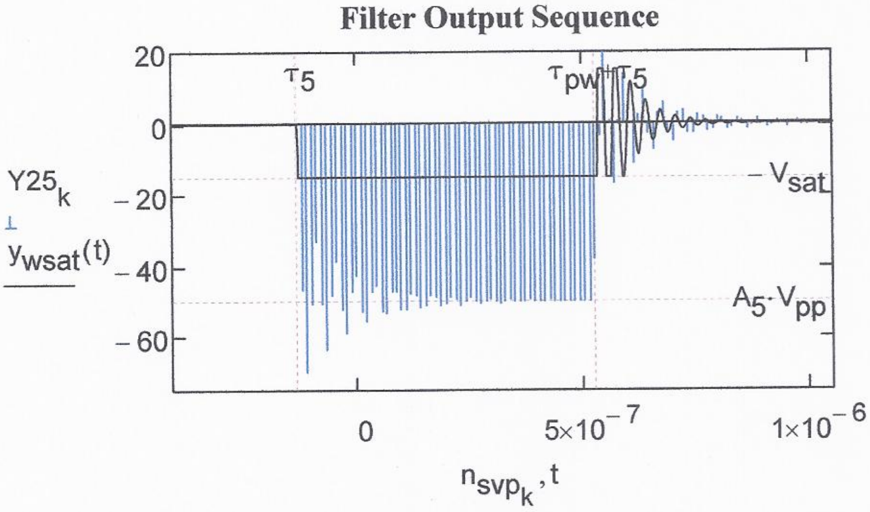


fig.:5.7.2.4

5.7 Synthetic Division algorithm (considering the bilinear transformation).

5.7.3 Sequence of the Sawtooth response:

$$\mathrm{syndbl2} := \mathrm{SYNDIVBL}\left(\frac{u55}{V}, A_5, \zeta, \omega_5, T_{ssw}, N0\right) \quad (5.7.3.1)$$

$$\mathrm{syndbl2} = \left(-2.685\times10^{-3} \quad -2.005 \quad 1.006 \quad \{256,1\} \quad \{256,1\}\right)$$

$$h5 := \mathrm{syndbl2}_{0,3} \qquad Y27 := \mathrm{syndbl2}_{0,4} \quad (5.7.3.2)$$

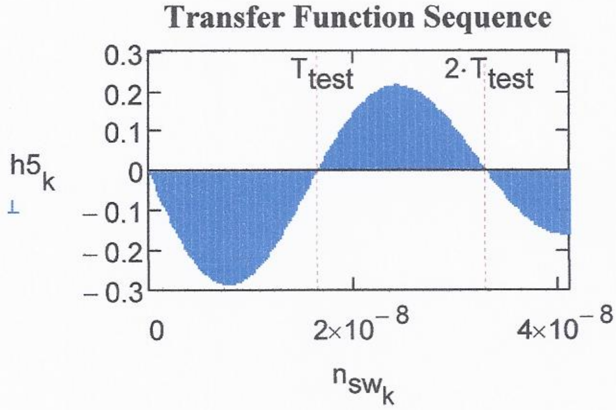


fig.:5.7.3.1

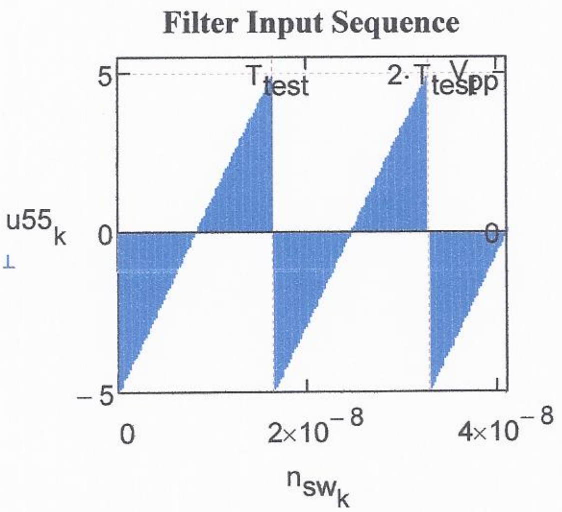


fig.:5.7.3.2

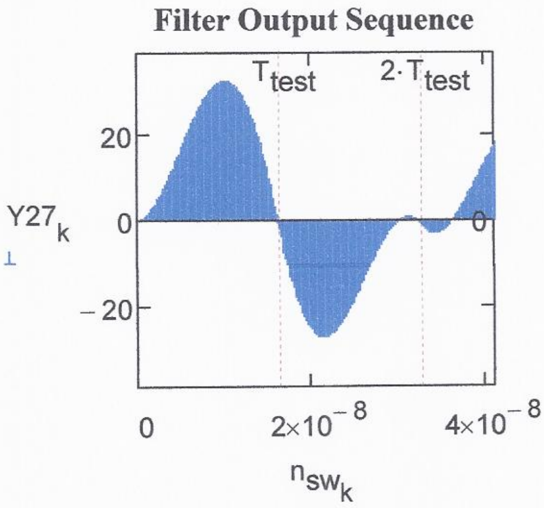


fig.:5.7.3.3

$$A_5 \cdot V_{pp} = -50V$$

5.7 Synthetic Division algorithm (considering the bilinear transformation).

5.7.4 Sequence of the Bipolar Square Wave response.

$$\text{syndbl3} := \text{SYNDIVBL}\left(\text{sqw}, A_5, \zeta, \omega_5, T_{ssqw}, \text{N0}\right) \qquad (5.7.4.1)$$

$$\text{syndbl3} = \left(-2.685\times 10^{-3} \quad -2.005 \quad 1.006 \quad \{256,1\} \quad \{256,1\}\right)$$

$$h6 := \text{syndbl3}_{0,3} \qquad Y26 := \text{syndbl3}_{0,4} \qquad (5.7.4.2)$$

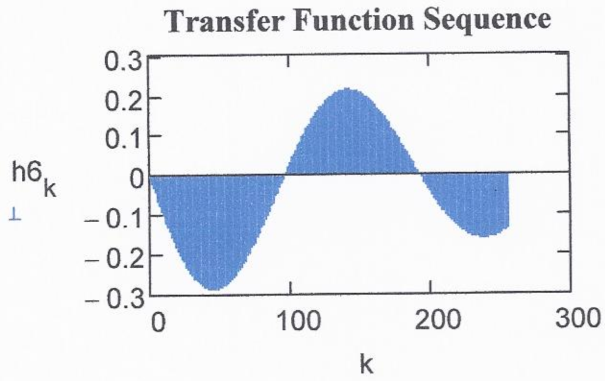


fig.:5.7.4.1

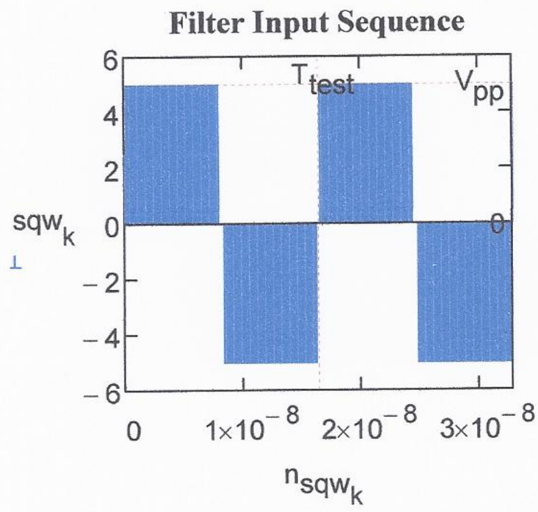


fig.:5.7.4.2

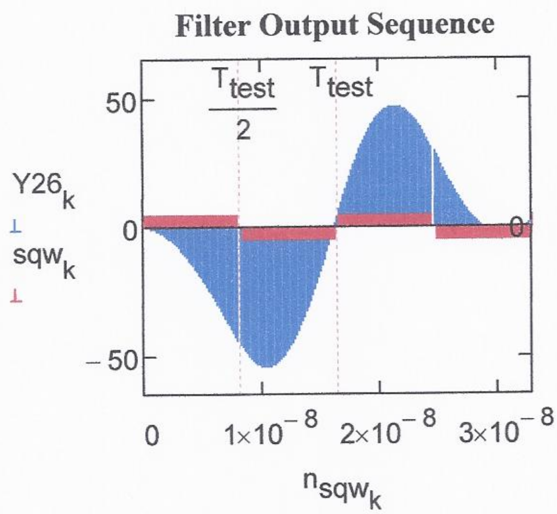


fig.:5.7.4.3

5.7 Synthetic Division algorithm (considering the bilinear transformation).

5.7.5 Sequence of the (single tone) AM Signal response.

$$\text{syndbl4} := \text{SYNDIVBL}\left(u\_7, A_5, \zeta, \omega_5, T_{sam}, N0\right) \quad (5.7.5.1)$$

$$\text{syndbl4} = (-3.277 \quad -0.875 \quad 1.185 \quad \{256,1\} \quad \{256,1\})$$

$$h7 := \text{syndbl4}_{0,3} \qquad Y28 := \text{syndbl4}_{0,4} \quad (5.7.5.2)$$

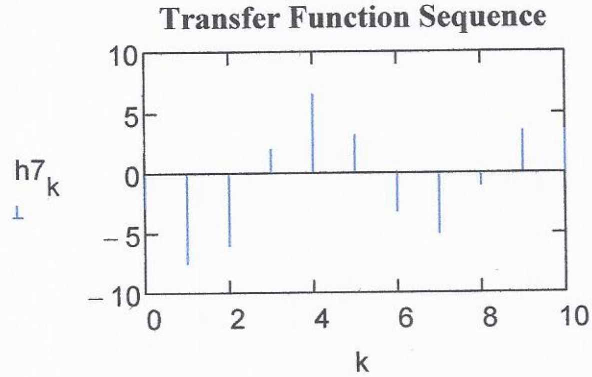


fig.:5.7.5.1

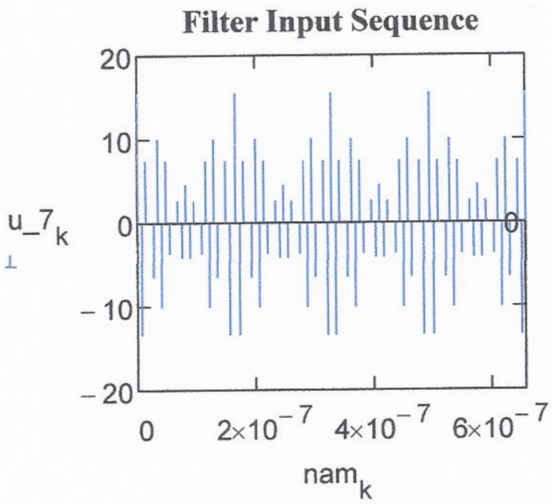


fig.:5.7.5.2

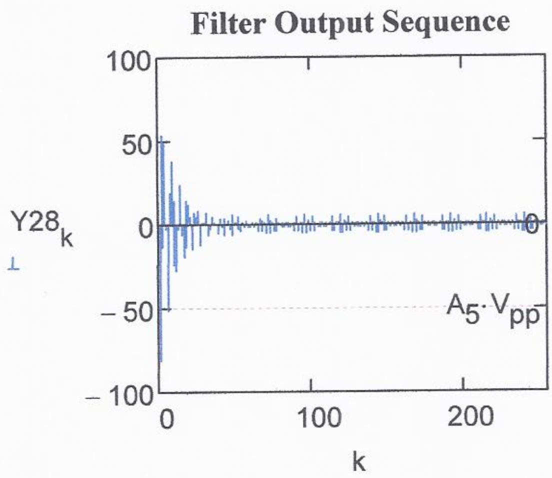


fig.:5.7.5.3

5.7 Synthetic Division algorithm (considering the bilinear transformation).

5.7.6 Sequence of the (single tone) Frequency Modulated carrier response.

$$syndbl5 := SYNDIVBL(u8, A_5, \zeta, \omega_5, T_{sfm}, N0) \qquad (5.7.6.1)$$

$$syndbl5 = (-0.194 \quad -1.974 \quad 1.052 \quad \{256,1\} \quad \{256,1\})$$

$$h8 := syndbl5_{0,3} \qquad Y29 := syndbl5_{0,4} \qquad (5.7.6.2)$$

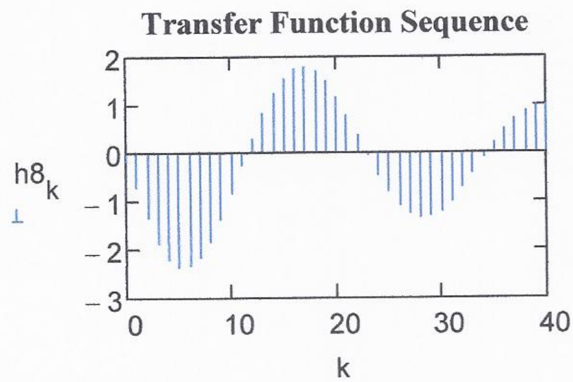


fig.:5.7.6.1

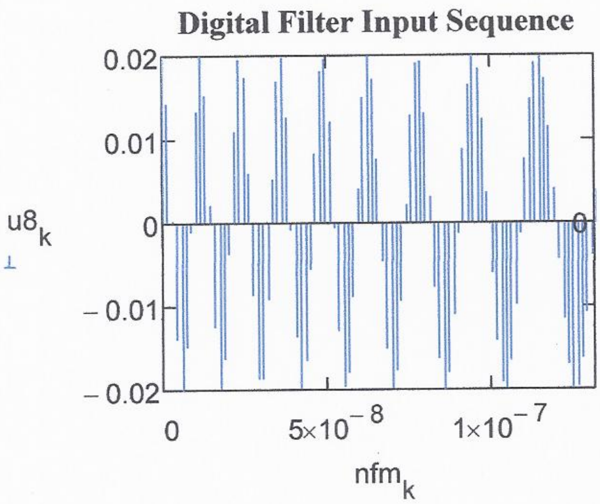


fig.:5.7.6.2

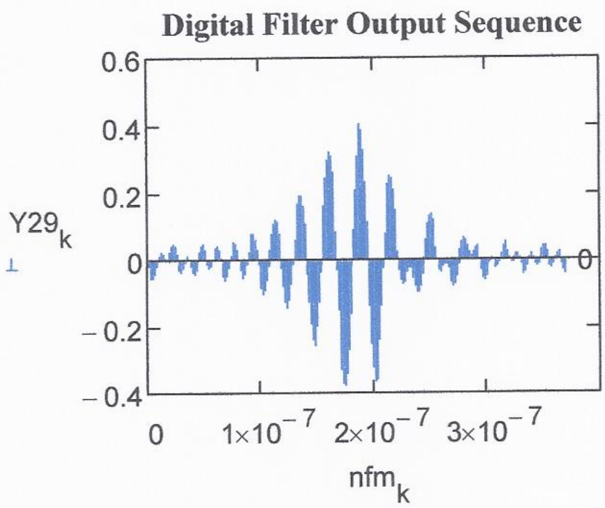


fig.:5.7.6.3

5.7 Synthetic Division algorithm (considering the bilinear transformation).

5.7.7 Sequence of the (single tone) Phase Modulated carrier response.

$$syndbl6 := SYNDIVBL\left(u\_9, A_5, \zeta, \omega_5, T_{spm}, N0\right) \qquad (5.7.7.1)$$

$m_p = 8$

$$syndbl6 = (-0.012 \quad -2.008 \quad 1.013 \quad \{256,1\} \quad \{256,1\})$$

$A_{fm} = 0.02V$

$$h9 := syndbl6_{0,3} \qquad Y30 := syndbl6_{0,4} \qquad (5.7.7.2)$$

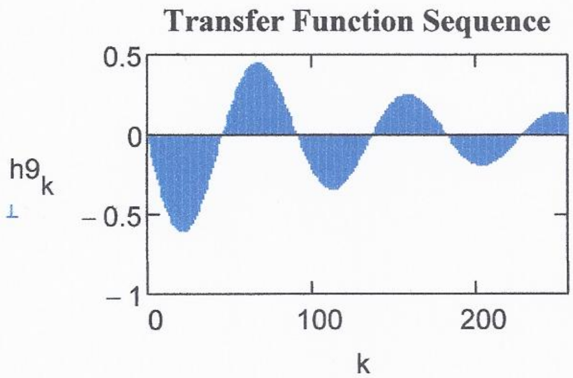


fig.:5.7.7.1

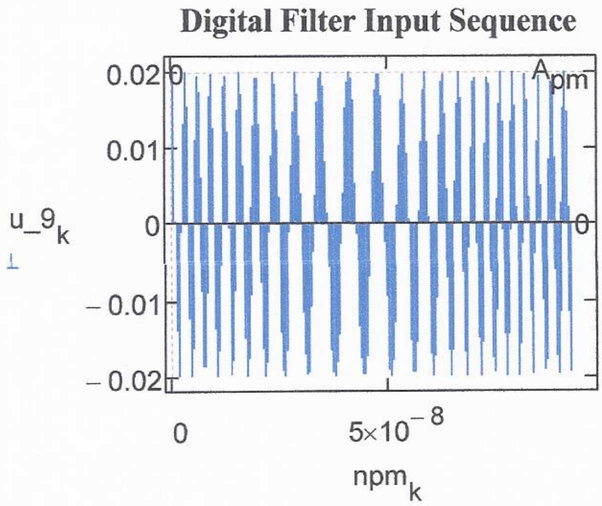


fig.:5.7.7.2

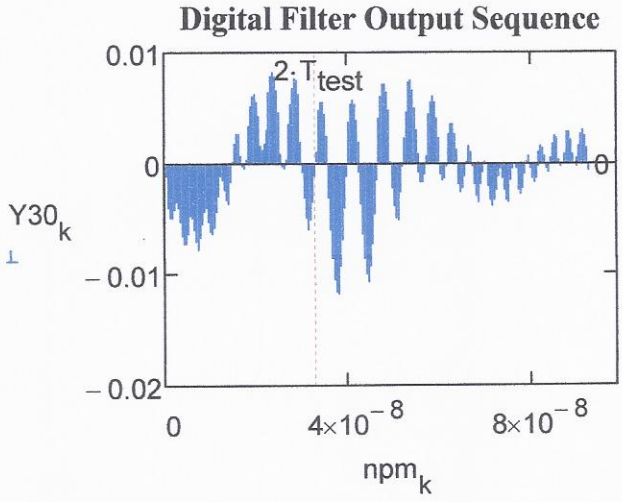


fig.:5.7.7.3

$$A_5 \cdot A_{pm} = -0.2V$$

5.8

Analytical search of the output sequence by means of the residues method (considering the bilinear transformation).

Search of the sequence corresponding to the filter's t. f. using the residues method.
Numerator and denominator's coefficients of the z t. f.:

$$T_s := T_{sstp}$$

$$\alpha_2 := \begin{cases} \dfrac{A_5 \cdot \omega_5^{\,2} \cdot T_s^{\,2}}{T_s^{\,2} \cdot \omega_5^{\,2} - 4 \cdot \zeta \cdot T_s + 4} & \text{if } \zeta \neq \omega_5 \\ \dfrac{A_5 \cdot T_s^{\,2} \cdot \omega_5^{\,2}}{\left(T_s \cdot \omega_5 - 2\right)^2} & \text{otherwise} \end{cases} \quad , \quad \beta_2 := \begin{cases} \dfrac{2 \cdot T_s^{\,2} \cdot \omega_5^{\,2} - 8}{T_s^{\,2} \cdot \omega_5^{\,2} - 4 \cdot \zeta \cdot T_s + 4} & \text{if } \zeta \neq \omega_5 \\ 2 \cdot \dfrac{T_s^{\,2} \cdot \omega_5^{\,2} - 4}{\left(T_s \cdot \omega_5 - 2\right)^2} & \text{otherwise} \end{cases} ,$$

$$\gamma_2 := \begin{cases} \dfrac{T_s^{\,2} \cdot \omega_5^{\,2} + 4 \cdot \zeta \cdot T_s + 4}{T_s^{\,2} \cdot \omega_5^{\,2} - 4 \cdot \zeta \cdot T_s + 4} & \text{if } \zeta \neq \omega_5 \\ \dfrac{\left(T_s \cdot \omega_5 + 2\right)^2}{\left(T_s \cdot \omega_5 - 2\right)^2} & \text{otherwise} \end{cases}$$

$\zeta - \omega_5 = -173.884 \cdot \dfrac{\text{Mrads}}{\text{sec}}$ $\qquad \alpha_2 = -0.248$ $\qquad \beta_2 = -1.959$ $\qquad \gamma_2 = 1.058$

The z t. f. is:

$$H_{lp}(z) := \alpha_2 \cdot \frac{1 + 2 \cdot z^{-1} + z^{-2}}{z^{-2} + \beta_2 \cdot z^{-1} + \gamma_2}, \tag{5.8.1}$$

First case $\zeta \neq \omega_5$ Define the function:

$$F10(z, n) = z^{n-1} \cdot H_{lp}(z) \tag{5.8.2}$$

namely:

$$F10(z, n) := \alpha_2 \cdot \frac{z^n \cdot (z+1)^2}{\gamma_2 \cdot z^3 + \beta_2 \cdot z^2 + z} \tag{5.8.3}$$

$\zeta = 0.018 \cdot \dfrac{\text{Grads}}{\text{sec}}$ $\qquad \omega_5 = 0.192 \cdot \dfrac{\text{Grads}}{\text{sec}}$

$z := z \qquad \beta_2 := \beta_2 \qquad \gamma_2 := \gamma_2$

Poles calculation: $\text{poles1} := \left(\gamma_2 \cdot z^3 + \beta_2 \cdot z^2 + z\right) \text{ solve}, z \rightarrow \begin{pmatrix} 0 \\ -\dfrac{\dfrac{\beta_2}{2} + \dfrac{\sqrt{\beta_2^{\,2} - 4\cdot\gamma_2}}{2}}{\zeta := \zeta?} \\ -\dfrac{\dfrac{\beta_2}{2} - \dfrac{\sqrt{\beta_2^{\,2} - 4\cdot\gamma_2}}{2}}{\gamma_2} \end{pmatrix}$ (5.8.4)

$z := z \qquad p0 := p0 \qquad p1 := p1 \qquad p2 := p2$

$\zeta \neq \omega_5 \qquad p0 := \text{poles1}_0 \qquad p1 := \text{poles1}_1 \qquad p2 := \text{poles1}_2$

$$p1 := -\left(\frac{\beta_2 + \sqrt{\beta_2^{\,2} - 4\cdot\gamma_2}}{2\cdot\gamma_2}\right) \qquad p2 := -\left(\frac{\beta_2 - \sqrt{\beta_2^{\,2} - 4\cdot\gamma_2}}{2\cdot\gamma_2}\right)$$

$p0 = 0 \qquad p1 = \blacksquare \qquad p2 = 0.926 + 0.297j$

$t := t \qquad r := \text{ceil}(\max(|\text{poles1}|))\cdot 1.0 \qquad r = 2$

$\xi(t) := r\cdot\cos(t)$

$\psi(t) := r\cdot\sin(t)$

$\phi(t) := \xi(t) + j\cdot\psi(t)$

$t_0 := 0 \quad t_{fin} := 2\cdot\pi \qquad t := t_0, t_0 + \dfrac{t_{fin} - t_0}{1000} .. t_{fin} \qquad n = 2$

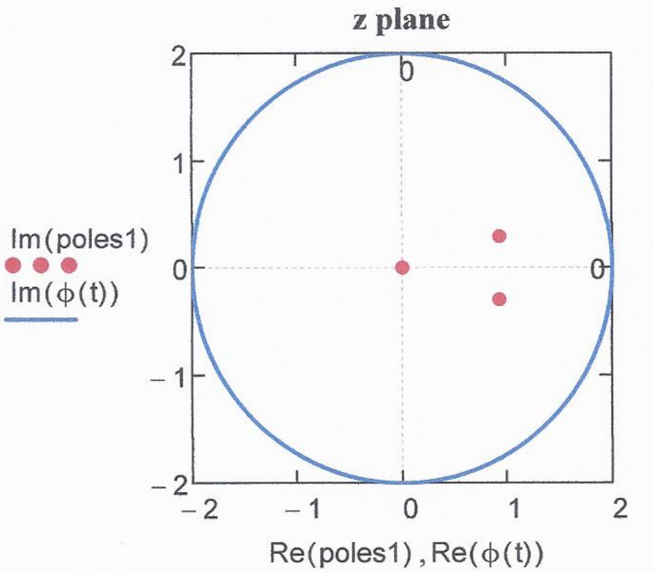


fig.:5.8.1

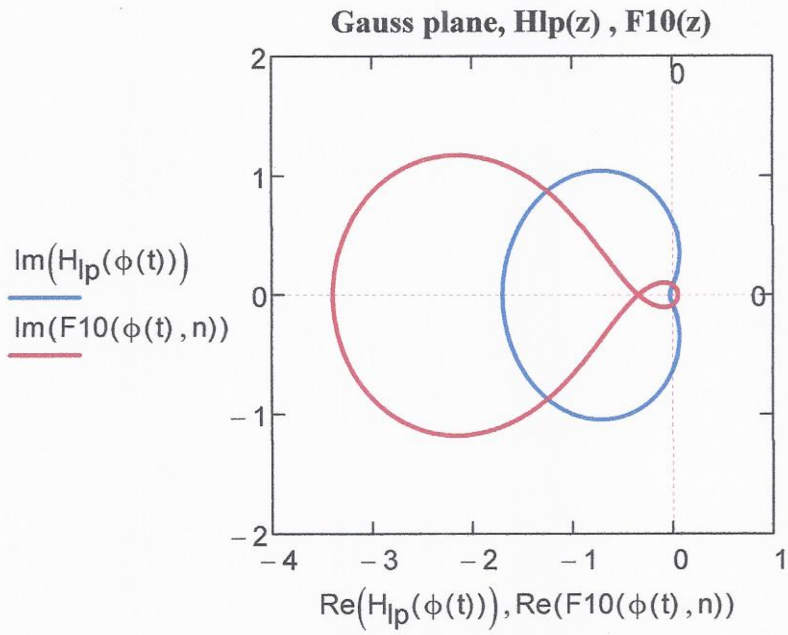


fig.:5.8.2

Knowing the poles of the function F10(z,n), it can be written too:

$$F10(z,n) = \frac{\alpha_2}{\gamma_2} \cdot \frac{z^n \cdot (z+1)^2}{(z-p0)\cdot(z-p1)\cdot(z-p2)} \quad (5.8.5)$$

The corresponding sequence is given by:

$$h10_n = \sum_{j=0}^{2} \lim_{z \to p_j} \left[(z-p_j) \cdot F10(z) \right] \quad (5.8.6)$$

hence: $$h10_k = \frac{\alpha_2}{\gamma_2} \cdot \sum_{j=0}^{2} \lim_{z \to p_j} \left[(z-p_j) \cdot \frac{z^k \cdot (z+1)^2}{(z-p0)\cdot(z-p1)\cdot(z-p2)} \right] \quad (5.8.7)$$

or: $$h10_k = \frac{\alpha_2}{\gamma_2} \cdot \left[\lim_{z \to p0} \left[(z-p0) \cdot \frac{z^k \cdot (z+1)^2}{(z-p0)\cdot(z-p1)\cdot(z-p2)} \right] \ldots + \lim_{z \to p1} \left[(z-p1) \cdot \frac{z^k \cdot (z+1)^2}{(z-p0)\cdot(z-p1)\cdot(z-p2)} \right] \ldots + \lim_{z \to p2} \left[(z-p2) \cdot \frac{z^k \cdot (z+1)^2}{(z-p0)\cdot(z-p1)\cdot(z-p2)} \right] \right] \quad (5.8.8)$$

simplifying: $$h10_k = \frac{\alpha_2}{\gamma_2} \cdot \left[\lim_{z \to p0} \left[\frac{z^k \cdot (z+1)^2}{(z-p1)\cdot(z-p2)} \right] \ldots + \lim_{z \to p1} \left[\frac{z^k \cdot (z+1)^2}{(z-p0)\cdot(z-p2)} \right] \ldots + \lim_{z \to p2} \left[\frac{z^k \cdot (z+1)^2}{(z-p0)\cdot(z-p1)} \right] \right] \quad (5.8.9)$$

(1) $p0 = 0$

Calculation of $\lim_{z \to 0} \left[\frac{z^k \cdot (z+1)^2}{(z-p1)\cdot(z-p2)} \right]$ If k=0 $\lim_{z \to 0} \left[\frac{z^0 \cdot (z+1)^2}{(z-p1)\cdot(z-p2)} \right] = \frac{1}{p1 \cdot p2}$

$z := z$ $\beta_2 := \beta_2$ $\gamma_2 := \gamma_2$

for $k > 0$ $$\lim_{z \to 0} \left[\frac{z^k \cdot (z+1)^2}{(z-p1)\cdot(z-p2)} \right] = 0 \quad (5.8.10)$$

$$\boxed{I_0 = \frac{1}{p1 \cdot p2} \cdot \delta(k,0)} \qquad \frac{1}{p1 \cdot p2} = \frac{1}{\gamma_2} \quad (5.8.11)$$

(2) $p0 = 0$

Calculation of $\lim_{z \to p1} \left[\frac{z^k \cdot (z+1)^2}{(z-p0)\cdot(z-p2)} \right] = \frac{p1^k \cdot (p1+1)^2}{p1 \cdot (p1-p2)}$

$$\boxed{I_1 = \frac{p1^k \cdot (p1+1)^2}{p1 \cdot (p1-p2)}} \quad (5.8.12)$$

(3) $p0 = 0$

Calculation of $\lim_{z \to p2} \left[\frac{z^k \cdot (z+1)^2}{(z-p0)\cdot(z-p1)} \right] = \frac{p2^k \cdot (p2+1)^2}{p2 \cdot (p2-p1)}$

$$\boxed{I_2 = \frac{p2^k \cdot (p2+1)^2}{p2 \cdot (p2-p1)}} \qquad (5.8.13)$$

The sequence is:

$$\boxed{h10_k := \frac{\alpha_2}{\gamma_2} \cdot \left[\frac{1}{p1 \cdot p2} \cdot \delta(k,0) + \frac{p1^k \cdot (p1+1)^2}{p1 \cdot (p1-p2)} \dots + \frac{p2^k \cdot (p2+1)^2}{p2 \cdot (p2-p1)} \right]} \qquad (5.8.14)$$

Proof (5.8.14)

Proof of (5.8.14):

the z transform of (5.8.14) should be the given z transfer function here rewritten:

$$H_{lp}(z) = \alpha_2 \cdot \frac{1 + 2 \cdot z^{-1} + z^{-2}}{z^{-2} + \beta_2 \cdot z^{-1} + \gamma_2} = \frac{\alpha_2}{\gamma_2} \cdot \frac{(z+1)^2}{z^2 + \frac{\beta_2}{\gamma_2} \cdot z + \frac{1}{\gamma_2}} = \frac{\alpha_2}{\gamma_2} \cdot \frac{(z+1)^2}{(z-p1)\cdot(z-p2)}$$

z transform of each term of (5.8.14)

$$p1 := p1 \qquad p2 := p2$$

$$\frac{1}{p1 \cdot p2} \cdot \delta(k,0) \text{ ztrans}, k \rightarrow \frac{1}{p1 \cdot p2}$$

$$\frac{p1^k \cdot (p1+1)^2}{p1 \cdot (p1-p2)} \text{ ztrans}, k \rightarrow -\frac{z \cdot (p1+1)^2}{p1 \cdot (p1-p2) \cdot (p1-z)}$$

$$\frac{p2^k \cdot (p2+1)^2}{p2 \cdot (p2-p1)} \text{ ztrans}, k \rightarrow \frac{z \cdot (p2+1)^2}{p2 \cdot (p1-p2) \cdot (p2-z)}$$

$$\frac{\alpha_2}{\gamma_2} \cdot \mathcal{Z} \left[\frac{1}{p1 \cdot p2} \cdot \delta(k,0) + \frac{p1^k \cdot (p1+1)^2}{p1 \cdot (p1-p2)} \dots + \frac{p2^k \cdot (p2+1)^2}{p2 \cdot (p2-p1)} \right] = \frac{\alpha_2}{\gamma_2} \cdot \left[\frac{1}{p1 \cdot p2} - \frac{z \cdot (p1+1)^2}{p1 \cdot (p1-p2) \cdot (p1-z)} + \frac{z \cdot (p2+1)^2}{p2 \cdot (p1-p2) \cdot (p2-z)} \right]$$

$$\frac{1}{p1 \cdot p2} - \frac{z \cdot (p1+1)^2}{p1 \cdot (p1-p2) \cdot (p1-z)} + \frac{z \cdot (p2+1)^2}{p2 \cdot (p1-p2) \cdot (p2-z)} = \frac{(z+1)^2}{(p1-z) \cdot (p2-z)}$$

q.e.d.

Proof (5.8.14)

Second case: $\boxed{\zeta = \omega_5}$ $T_s := 2 \cdot T_{sstp}$

$$\alpha_{21} := \frac{A_5 \cdot T_s^2 \cdot \omega_5^2}{(T_s \cdot \omega_5 - 2)^2} \qquad \beta_{21} := \frac{T_s \cdot \omega_5 + 2}{(T_s \cdot \omega_5 - 2)}$$

$$H4(z) := \alpha_{21} \cdot \frac{\left(z^{-1}+1\right)^2}{\left(z^{-1}+\beta_{21}\right)^2} \qquad (5.8.15)$$

$\alpha_{21} = -2.0982304898$ $\qquad \beta_{21} = -1.9161289188$

$z := z \qquad \alpha_{21} := \alpha_{21} \qquad \beta_{21} := \beta_{21}$

Define the function: $F20(z,k) = z^{k-1} \cdot H4(z)$ (5.8.16)

$$\boxed{F20(z,k) := \frac{\alpha_{21} \cdot z^k \cdot (z+1)^2}{z \cdot \left(\beta_{21} \cdot z + 1\right)^2}} \qquad (5.8.17)$$

$$\text{Poles of } F20(z,k) = \frac{\alpha_{21} \cdot z^k \cdot (z+1)^2}{z \cdot \left(\beta_{21} \cdot z + 1\right)^2}$$

(5.8.18)

$\zeta = 0.018 \cdot \frac{\text{Grads}}{\text{sec}}$ $\qquad \omega_5 = 0.192 \cdot \frac{\text{Grads}}{\text{sec}}$

$\zeta = \omega_5 \qquad z := z \qquad \alpha21 := \alpha21 \qquad \beta_{21} := \beta_{21} \qquad \zeta := \zeta$

$$\text{poles} := z \cdot \left(\beta_{21} \cdot z + 1\right)^2 \text{ solve}, z \rightarrow \begin{pmatrix} 0 \\ -\frac{1}{\beta_{21}} \\ -\frac{1}{\beta_{21}} \end{pmatrix}$$

$$\text{poles} = \begin{pmatrix} 0 \\ 0.522 \\ 0.522 \end{pmatrix} \qquad p00 := \text{poles}_0 \qquad p01 := \text{poles}_1$$

First order pole: $p00 := 0$ $\qquad$ Second order pole: $p01 := -\frac{1}{\beta_{21}}$

$r := \text{ceil}(\max(|\text{poles}|)) \cdot 1.0 \qquad r = 1$

$\xi(t) := r \cdot \cos(t)$

$\psi(t) := r \cdot \sin(t)$

$\phi(t) := \xi(t) + j \cdot \psi(t)$

$t_0 := 0 \quad t_{fin} := 2\cdot\pi \qquad t := t_0, t_0 + \frac{t_{fin} - t_0}{100} .. t_{fin}$

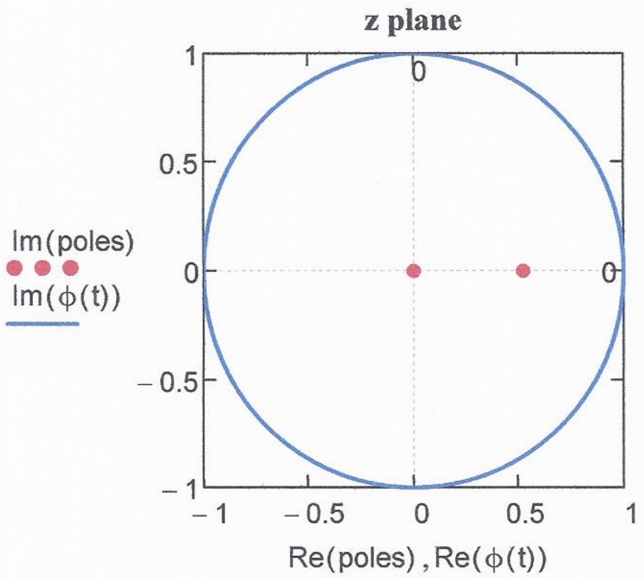


fig.:5.8.3

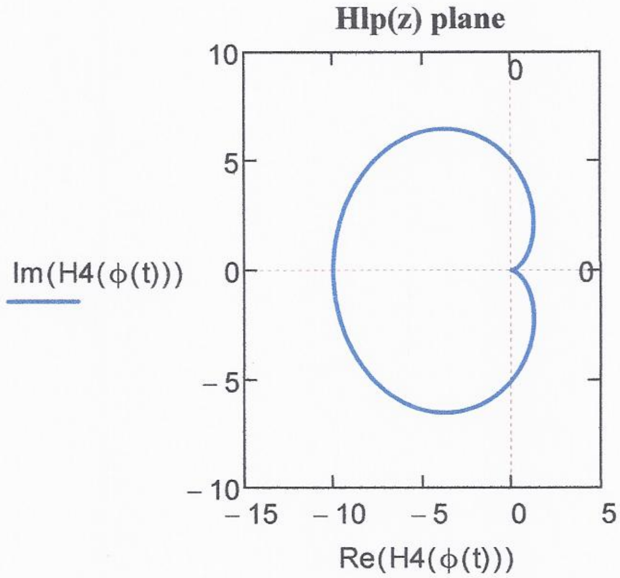


fig.:5.8.4

$$F20(z,k) = \frac{\alpha_{21} \cdot z^k \cdot (z+1)^2}{\beta_{21} \cdot (z - p00) \cdot (z - p01)^2} \qquad (5.8.19)$$

$$h20_k = \mathrm{Res}(F20(z,k), p00) + \mathrm{Res}(F20(z,k), p01) \qquad (5.8.20)$$

$p00 = 0$

$$h20_k = \frac{\alpha_{21}}{\beta_{21}} \cdot \left[\lim_{z \to p00} \left[(z - p00) \cdot \frac{z^k \cdot (z+1)^2}{(z - p00) \cdot (z - p01)^2} \right] \cdots + \lim_{z \to p01} \left[\frac{\partial}{\partial z} \left[(z - p01)^2 \cdot \frac{z^k \cdot (z+1)^2}{(z - p00) \cdot (z - p01)^2} \right] \right] \right] \qquad (5.8.21)$$

namely, simplifying:

$$\boxed{h20_k = \frac{\alpha_{21}}{\beta_{21}} \cdot \left[\lim_{z \to 0} \left[\frac{z^k \cdot (z+1)^2}{(z - p01)^2} \right] \cdots + \lim_{z \to p01} \left[\frac{\partial}{\partial z} \left[\frac{z^k \cdot (z+1)^2}{(z - p00)} \right] \right] \right]} \qquad (5.8.22)$$

(1) $\mathrm{Res}(F20(z,k), p00)$ $\quad p00 = 0$

For k=0 and $p01 \neq 0$

$$\lim_{z \to 0} \left[\frac{z^0 \cdot (z+1)^2}{(z - p01)^2} \right] = \frac{\delta(k,0)}{p01^2} \qquad (5.8.23)$$

For k>0

$$\lim_{z \to 0} \left[\frac{z^k \cdot (z+1)^2}{(z - p01)^2} \right] = \lim_{z \to 0} \left[\frac{z^k \cdot (z+1)^2}{(z - p01)^2} \right] = 0 \qquad (5.8.24)$$

$$\mathrm{Res}(F20(z,k), p00) = \frac{\alpha_{21}}{\beta_{21}} \cdot \frac{\delta(k,0)}{p01^2}$$

(2) $\mathrm{Res}(F20(z,k), p01)$

$$\lim_{z \to p01} \left[\frac{\partial}{\partial z} \left[\frac{z^k \cdot (z+1)^2}{(z - p00)} \right] \right] \qquad (5.8.25)$$

$$p01 = -\frac{1}{\beta_{22}} \qquad \frac{\partial}{\partial z} \left[\frac{z^k \cdot (z+1)^2}{z} \right] = z^{k-2} \cdot (z+1) \cdot (k + z + k \cdot z - 1) \qquad (5.8.26)$$

For k=0 and $p01 \neq 0$

$$\lim_{z \to p01} \left[\frac{\partial}{\partial z} \left[\frac{z^0 \cdot (z+1)^2}{z} \right] \right] = \lim_{z \to p01} \left[\frac{\partial}{\partial z} \left[\frac{(z+1)^2}{z} \right] \right]$$

$$\frac{\partial}{\partial z} \left[\frac{(z+1)^2}{z} \right] \text{ simplify, max } \to 1 - \frac{1}{z^2}$$

$$\lim_{z \to p01} \left[\frac{\partial}{\partial z}\left[\frac{(z+1)^2}{z}\right]\right] = \left(1 - \frac{1}{p01^2}\right)\cdot\delta(k,0)$$

$$Res(F20(z,k),p01) = \frac{\alpha_{21}}{\beta_{21}}\cdot \lim_{z \to p01} \left[z^{k-2}\cdot(z+1)\cdot(k+z+k\cdot z-1)\right] \quad (5.8.27)$$

$$Res(F20(z,k),p01) = \frac{\alpha_{21}}{\beta_{21}}\cdot p01^{k-2}\cdot(p01+1)\cdot[k+p01\cdot(1+k)-1] \quad (5.8.28)$$

$$h20_k = \frac{\alpha_{21}}{\beta_{21}}\cdot\left[\frac{\delta(k,0)}{p01^2} + \left(1-\frac{1}{p01^2}\right)\cdot\delta(k,0) + p01^{k-2}\cdot(p01+1)\cdot[k+p01\cdot(1+k)-1]\right]$$

$$h20_k = \frac{\alpha_{21}}{\beta_{21}}\cdot\left[\delta(k,0) + p01^{k-2}\cdot(p01+1)\cdot[k+p01\cdot(1+k)-1]\right] \quad (5.8.29)$$

Proof (5.8.29)

Proof of (5.8.29):

the z transform of (5.8.29) should be the given z transfer function here rewritten:

$$H4(z) = \alpha_{21}\cdot\frac{\left(z^{-1}+1\right)^2}{\left(z^{-1}+\beta_{21}\right)^2} = \frac{\alpha_{21}\cdot(z+1)^2}{\left(\beta_{21}\cdot z+1\right)^2}$$

$$\frac{\alpha_{21}\cdot\left(\beta_{21}-1\right)\cdot\left[\beta_{21}\cdot(k-1)-k-1\right]}{\beta_{21}^{\;2}}\cdot\left(-\frac{1}{\beta_{21}}\right)^k + \alpha_{21}\cdot\delta(k,0) \;\Bigg|\; \begin{matrix}\text{ztrans}, k \\ \text{simplify}, \max\end{matrix} \rightarrow \frac{\alpha_{21}\cdot(z+1)^2}{\left(\beta_{21}\cdot z+1\right)^2}$$

Proof (5.8.29)

Finally, the result considering both cases: $\zeta \neq \omega_5$ and $\zeta = \omega_5$, is:

$$h20_k := \begin{cases} \dfrac{\alpha_2}{\gamma_2}\cdot\left[\delta(k,0) + \dfrac{p1^{k-1}\cdot(p1+1)^2}{(p1-p2)} \;\ldots\; + \dfrac{p2^{k-1}\cdot(p2+1)^2}{(p2-p1)}\right] & \text{if } \zeta \neq \omega_5 \\ \dfrac{\alpha_{21}}{\beta_{21}}\cdot\left[\delta(k,0) + p01^{k-2}\cdot(p01+1)\cdot[k+p01\cdot(1+k)-1]\right] & \text{otherwise} \end{cases} \quad (5.8.30)$$

$$\omega_5 = 0.192\cdot\frac{Grads}{sec} \qquad \zeta = 0.018\cdot\frac{Grads}{sec}$$

$$\zeta = 0.018\cdot\frac{Grads}{sec} \qquad \omega_5 = 0.192\cdot\frac{Grads}{sec}$$

$h20^T =$

	0	1	2	3	4	5	6
0	-0.22	-0.902	-1.682	-2.262	-2.598	-2.672	...

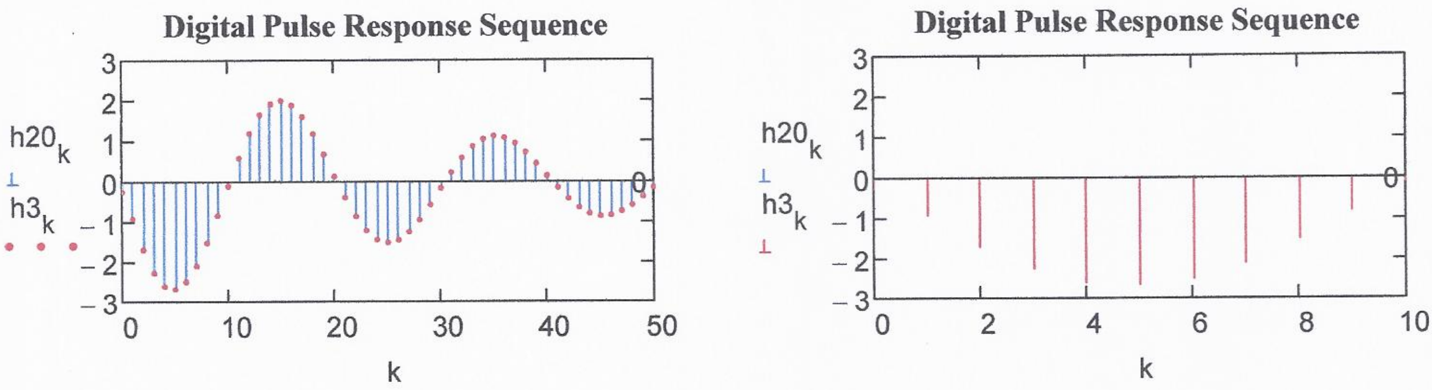


fig.:5.8.5

Stability (S31<∞): $$S31 := \sum_{k=0}^{rows(h1)-1} |h20_k| \qquad S31 = 69.057$$

Energy of the sequence h1: $$E31 := \sum_{k=0}^{rows(h1)-1} \left(|h20_k|\right)^2 \qquad E31 = 82.831$$

$T_s = 3.279 \cdot ns \qquad T_{sstp} = 1.639 \cdot ns$

$$y3_\nu := \sum_{k=0}^{N0-1} \left(if\left(\nu - k \geq 0, h20_k \cdot u1_{\nu-k}, 0\right)\right) \qquad (5.8.31)$$

Redefine the output waveform :

$$y_{sr}(t) := A_5 \cdot V_{pp} \cdot \begin{cases} g_{sr}(t, A_5, \zeta, \omega_5) \cdot \Phi(t) & \text{if } \zeta \neq \omega_5 \\ \left[1 - e^{-t \cdot \omega_5} \cdot (t \cdot \omega_5 + 1)\right] \cdot \Phi(t) & \text{otherwise} \end{cases} \qquad (5.2.1.1)$$

$$t := 0 \cdot T_{test}, 0 \cdot T_{test} + \frac{20 \cdot T_5 - 0 \cdot T_{test}}{1000} .. 20 \cdot T_5 \qquad V_{pp} = 5V$$

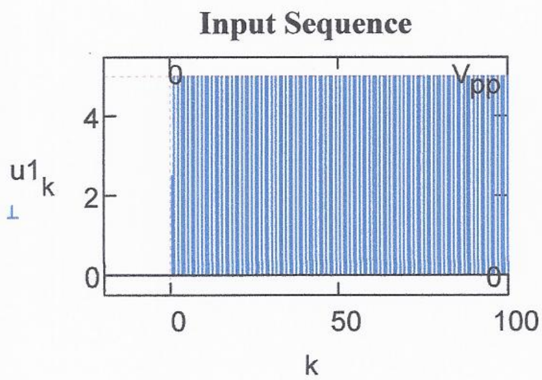


fig.:5.8.6

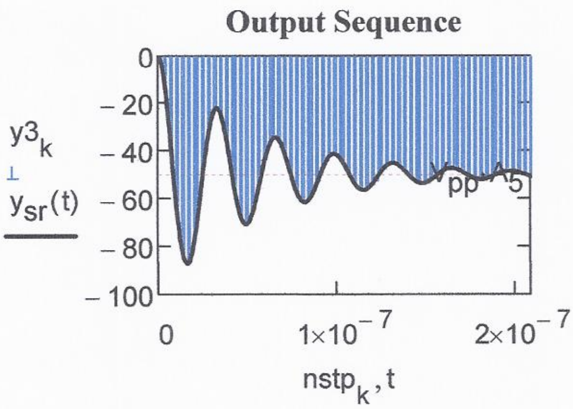
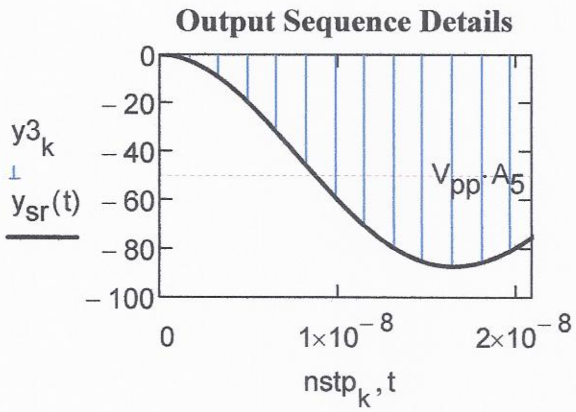


fig.:5.8.7



$y3^T =$

	0	1	2	3	4	5	6	7	8
0	0	-3.357	-9.817	-19.677	-31.827	-45.001	-57.91	-69.359	...

V

Calculation of the filter output as the inverse z transform of $H(z)V_i(z)$

Filter's z transfer function $H_{lp}(z) = \alpha_2 \cdot \frac{1 + 2 \cdot z^{-1} + z^{-2}}{z^{-2} + \beta_2 \cdot z^{-1} + \gamma_2} = \frac{\alpha_2}{\gamma_2} \cdot \frac{(z+1)^2}{z^2 + \frac{\beta_2}{\gamma_2} \cdot z + \frac{1}{\gamma_2}}$ (5.8.32)

$\zeta \neq \omega_5$ $\qquad \zeta = 0.018 \cdot \frac{Grads}{sec}$ $\qquad \omega_5 = 0.192 \cdot \frac{Grads}{sec}$

$z := z \qquad \beta_2 := \beta_2 \qquad \gamma_2 := \gamma_2 \qquad \zeta := \zeta$

$$poles2 := z^2 + \frac{\beta_2}{\gamma_2} \cdot z + \frac{1}{\gamma_2} \;\Big|\; \begin{matrix} solve, z \\ simplify \end{matrix} \rightarrow \begin{pmatrix} -\frac{\beta_2 - \sqrt{\beta_2^2 - 4 \cdot \gamma_2}}{2 \cdot \gamma_2} \\ -\frac{\beta_2 + \sqrt{\beta_2^2 - 4 \cdot \gamma_2}}{2 \cdot \gamma_2} \end{pmatrix}$$

$poles2 = \begin{pmatrix} 0.926 + 0.297j \\ 0.926 - 0.297j \end{pmatrix} \qquad p0 := poles2_0 \qquad p1 := poles2_1$

$$p0 := -\frac{\beta_2 - \sqrt{\beta_2^2 - 4 \cdot \gamma_2}}{2 \cdot \gamma_2} \qquad p1 := -\frac{\beta_2 + \sqrt{\beta_2^2 - 4 \cdot \gamma_2}}{2 \cdot \gamma_2} \qquad (5.8.33)$$

$p0 = 0.926 + 0.297j \qquad p1 = 0.926 - 0.297j$

Step response: $\qquad Y3(z) = H_{lp}(z) \cdot X(z)$ (5.8.34)

of the l. p. filter:

$V_{pp} := V_{pp}$ $\quad V_{pp}$ ztrans $\rightarrow \dfrac{V_{pp} \cdot z}{z-1}$ $\quad$ z transform of the inpu $X(z) := \dfrac{V_{pp} \cdot z}{z-1}$ (5.8.35)

$\alpha 2 := \alpha 2 \quad \beta 2 := \beta 2 \quad \gamma 2 := \gamma 2 \quad p0 := p0 \quad p1 := p1$

$p0 = 0.926 + 0.297j \qquad p1 = 0.926 - 0.297j$

First case $\zeta \neq \omega_5$

$$y330_k := \frac{\alpha 2}{\gamma_2} \cdot \frac{(z+1)^2}{(z-p0)\cdot(z-p1)} \cdot \frac{V_{pp} \cdot z}{z-1} \;\left|\; \begin{array}{l} \text{invztrans}, z, k \\ \text{simplify} \\ \text{factor} \end{array} \right. \rightarrow$$

$$y330_k := V_{pp} \cdot \frac{\alpha 2}{\gamma_2} \cdot \left[\frac{(p0+1)^2 \cdot p0^k}{(p0-p1)\cdot(p0-1)} - \frac{p1^k \cdot (p1+1)^2}{(p0-p1)\cdot(p1-1)} + \frac{4}{(p0-1)\cdot(p1-1)} \right] \quad (5.8.36)$$

Proof

Proof of (5.8.36):

the z transform of (5.8.36) should be the given z transform of the response here rewritten:

$$Y3(z) = V_{pp} \cdot \frac{\alpha 2}{\gamma_2} \cdot \frac{(z+1)^2}{(z-p0)\cdot(z-p1)} \cdot \frac{z}{z-1}$$

$p0 := p0 \qquad p1 := p1$

$$\frac{(p0+1)^2 \cdot p0^k}{(p0-p1)\cdot(p0-1)} \text{ ztrans}, k \rightarrow -\frac{z \cdot (p0+1)^2}{(p0-p1)\cdot(p0-z)\cdot(p0-1)}$$

$$\mathcal{Z}\left[\frac{(p0+1)^2 \cdot p0^k}{(p0-p1)\cdot(p0-1)} \right] = -\frac{z \cdot (p0+1)^2}{(p0-p1)\cdot(p0-z)\cdot(p0-1)}$$

$p0 := p0 \qquad p1 := p1$

$$\frac{p1^k \cdot (p1+1)^2}{(p0-p1)\cdot(p1-1)} \text{ ztrans}, k \rightarrow -\frac{z \cdot (p1+1)^2}{(p0-p1)\cdot(p1-z)\cdot(p1-1)}$$

$$\mathcal{Z}\left[\frac{p1^k \cdot (p1+1)^2}{(p0-p1)\cdot(p1-1)} \right] = -\frac{z \cdot (p1+1)^2}{(p0-p1)\cdot(p1-z)\cdot(p1-1)}$$

$$\mathcal{Z}\left[\frac{4}{(p0-1)\cdot(p1-1)} \right] = \frac{4 \cdot z}{(p0-1)\cdot(p1-1)\cdot(z-1)}$$

$$Y3(z) = V_{pp} \cdot \frac{\alpha 2}{\gamma_2} \cdot \left[-\frac{z \cdot (p0+1)^2}{(p0-p1)\cdot(p0-z)\cdot(p0-1)} + \frac{z \cdot (p1+1)^2}{(p0-p1)\cdot(p1-z)\cdot(p1-1)} + \frac{4 \cdot z}{(p0-1)\cdot(p1-1)\cdot(z-1)} \right]$$

$p0 := p0 \qquad p1 := p1$

$$\left[-\frac{z\cdot(p0+1)^2}{(p0-p1)\cdot(p0-z)\cdot(p0-1)} \cdots + \frac{z\cdot(p1+1)^2}{(p0-p1)\cdot(p1-z)\cdot(p1-1)} \cdots + \left[\frac{4\cdot z}{(p0-1)\cdot(p1-1)\cdot(z-1)}\right]\right] \text{simplify} \rightarrow \frac{z\cdot(z+1)^2}{(p0-z)\cdot(p1-z)\cdot(z-1)}$$

$\alpha_2 = -0.248 \qquad Y3(z) = V_{pp}\cdot\frac{\alpha 2}{\gamma_2}\cdot\frac{z\cdot(z+1)^2}{(p0-z)\cdot(p1-z)\cdot(z-1)} \qquad \gamma_2 = 1.058$

q.e.d.

Proof

Second case: $\zeta = \omega_5$

$A_5 = -10 \qquad \alpha_2 := \frac{A_5\cdot T_s^{\,2}\cdot\omega_5^{\,2}}{(T_s\cdot\omega_5-2)^2} \qquad \gamma_2 := \frac{T_s\cdot\omega_5+2}{(T_s\cdot\omega_5-2)}$

$\alpha_2 = -2.098 \qquad \gamma_2 = -1.916$

$$y331_k := \left[\alpha_2\cdot\frac{(z^{-1}+1)^2}{(z^{-1}+\gamma_2)^2}\right]\cdot\frac{V_{pp}\cdot z}{z-1} \left|\begin{array}{l}\text{invztrans}, z, k\\ \text{simplify}\\ \text{factor}\\ \text{collect}, \left(-\frac{1}{\gamma_2}\right)^k\end{array}\right. \rightarrow$$

$$y331_k := \left[4 + \frac{\left(-\frac{1}{\gamma_2}\right)^k\cdot\left[k\cdot\left[\gamma_2\cdot\left[\gamma_2\cdot(\gamma_2-1)-1\right]+1\right]-\gamma_2\cdot(3\cdot\gamma_2-2)+1\right]}{\gamma_2^{\,2}}\right]\cdot\frac{V_{pp}\cdot\alpha_2}{(\gamma_2+1)^2} \qquad (5.8.37)$$

Proof

Proof of (5.8.37):

the z transform of (5.8.37) should be the given z transform of the response for $\zeta=\omega_5$, here rewritten:

$$y331(z) = \alpha_2\cdot\frac{(z^{-1}+1)^2}{(z^{-1}+\gamma_2)^2}\cdot\frac{V_{pp}\cdot z}{z-1} = \frac{V_{pp}\cdot\alpha_2\cdot z\cdot(z+1)^2}{(\gamma_2\cdot z+1)^2\cdot(z-1)}$$

$\gamma_2 := \gamma_2 \qquad \alpha_2 := \alpha_2 \qquad V_{pp} := V_{pp}$

$$\left[4+\frac{\left(-\frac{1}{\gamma_2}\right)^k \cdot \left[\begin{array}{l} k\cdot[\gamma_2\cdot[\gamma_2\cdot(\gamma_2-1)-1]+1]\ \ldots \\ +1-\gamma_2\cdot(3\cdot\gamma_2-2)\end{array}\right]}{\gamma_2^{\,2}}\right]\cdot\frac{V_{pp}\cdot\alpha_2}{(\gamma_2+1)^2}\ \text{ztrans}, k \rightarrow \frac{V_{pp}\cdot\alpha_2\cdot z\cdot(z+1)^2}{(\gamma_2\cdot z+1)^2\cdot(z-1)}$$

$$\frac{V_{pp}\cdot\alpha_2}{(\gamma_2+1)^2}\cdot\mathcal{Z}\left[4+\frac{\left(-\frac{1}{\gamma_2}\right)^k \cdot \left[\begin{array}{l} k\cdot[\gamma_2\cdot[\gamma_2\cdot(\gamma_2-1)-1]+1]\ \ldots \\ +1-\gamma_2\cdot(3\cdot\gamma_2-2)\end{array}\right]}{\gamma_2^{\,2}}\right] = \frac{V_{pp}\cdot\alpha_2\cdot z\cdot(z+1)^2}{(\gamma_2\cdot z+1)^2\cdot(z-1)}$$

q.e.d.

Proof

Finally, the result, considering both cases: $\zeta \neq \omega_5$ and $\zeta = \omega_5$, is:

$Q_5 = 5.4$ $\qquad \zeta = 0.018\cdot\frac{\text{Grads}}{\text{sec}}$ $\qquad \omega_5 = 0.192\cdot\frac{\text{Grads}}{\text{sec}}$

$$y33_k := V_{pp}\cdot \begin{cases} \frac{\alpha 2}{\gamma_2}\cdot\left[\frac{(p0+1)^2\cdot p0^k}{(p0-p1)\cdot(p0-1)} - \frac{p1^k\cdot(p1+1)^2}{(p0-p1)\cdot(p1-1)} + \frac{4}{(p0-1)\cdot(p1-1)}\right] & \text{if } \zeta \neq \omega_5 \\ \left[4+\frac{\left(-\frac{1}{\gamma_2}\right)^k \cdot \left[\begin{array}{l} k\cdot[\gamma_2\cdot[\gamma_2\cdot(\gamma_2-1)-1]+1]\ \ldots \\ +1-\gamma_2\cdot(3\cdot\gamma_2-2)\end{array}\right]}{\gamma_2^{\,2}}\right]\cdot\frac{\alpha_2}{(\gamma_2+1)^2} & \text{otherwise} \end{cases} \qquad (5.8.37)$$

$V_{pp}\cdot\alpha_2 = -10.491\,\text{V}$ $\qquad \zeta = 0.018\cdot\frac{\text{Grads}}{\text{sec}}$ $\qquad \omega_5 = 0.192\cdot\frac{\text{Grads}}{\text{sec}}$ $\qquad Q_5 = 5.4$

Sequence of the step response.

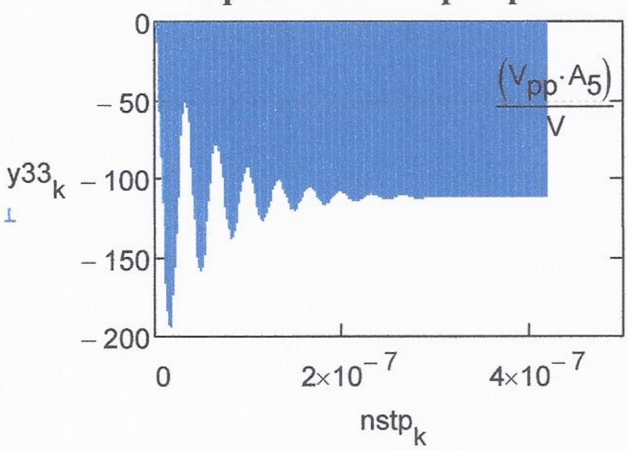


fig.:5.8.8

Sinusoidal response: $Y_{out}(z) = H_{lp}(z) \cdot X(z)$ (5.8.38)

Signal amplitude: $V_m := V_{pp}$,

Signal frequency: $f_{test} = 60.997 \cdot MHz$,

arbitrary sampling frequency: $f_s := 10 \cdot f_{test}$, $f_s = 609.968 \cdot MHz$, (5.8.)

sampling angular frequency: $\omega_s := 2 \cdot \pi \cdot f_s$. $\omega_s = 3.833 \cdot \frac{Grads}{sec}$,

sampling period: $T_s := \frac{1}{f_s}$, $T_s = 1.639 \cdot ns$,

sampling time step: $n_k := \frac{k}{f_s}$, $N0 = 256$ (5.8.)

$\frac{N0}{f_s} \cdot f_5 = 12.8$ $N0 = 256$.

(5.8.)

L. p. filter Input: $x2_k := V_m \cdot \sin(\omega_{test} \cdot n_k)$ (5.8.39)

$n^T =$ (s)

	0	1	2	3	4	5	6
0	0	$1.639 \cdot 10^{-9}$	$3.279 \cdot 10^{-9}$	$4.918 \cdot 10^{-9}$	$6.558 \cdot 10^{-9}$	$8.197 \cdot 10^{-9}$	...

$x2^T =$ (V)

	0	1	2	3	4	5
0	0	2.939	4.755	4.755	2.939	...

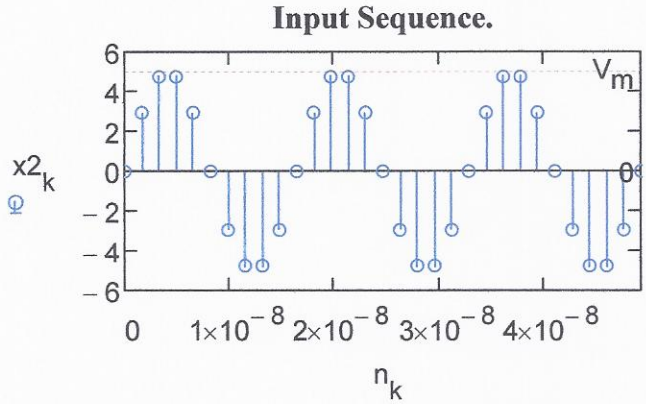


fig.:5.8.9

Z transform of the input signal:

$V_m := V_m$ $\omega_{test} := \omega_{test}$ $T_s := T_s$ $\nu := \nu$

$X(z) := V_m \cdot \sin(\nu \cdot \omega_{test} \cdot T_s)$ ztrans , ν $\rightarrow \frac{V_m \cdot z \cdot \sin(T_s \cdot \omega_{test})}{z^2 - 2 \cdot \cos(T_s \cdot \omega_{test}) \cdot z + 1}$ $N1 = 4$

$$X(z) = \frac{V_m \cdot z \cdot \sin(T_s \cdot \omega_{test})}{z^2 - 2 \cdot \cos(T_s \cdot \omega_{test}) \cdot z + 1} \qquad (5.8.40)$$

$K := \sin(T_s \cdot \omega_{test})$ $\quad \sqrt{1-K^2} = 0.809 \quad$ $K = 0.588$

Transfer function:

$$W_{lp}(s) := \begin{cases} \dfrac{A_5 \cdot \omega_5^2}{s^2 + 2 \cdot \zeta \cdot s + \omega_5^2} & \text{if } \zeta \neq \omega_5 \\ A_5 \cdot \dfrac{\omega_5^2}{(s+\omega_5)^2} & \text{otherwise} \end{cases} \qquad (5.1.7.1)$$

Coefficients of the z transfer function:

$$\alpha_3 := \begin{cases} \dfrac{A_5 \cdot \omega_5^2 \cdot T_s^2}{T_s^2 \cdot \omega_5^2 - 4 \cdot \zeta \cdot T_s + 4} & \text{if } \zeta \neq \omega_5 \\ \dfrac{A_5 \cdot T_s^2 \cdot \omega_5^2}{(T_s \cdot \omega_5 - 2)^2} & \text{otherwise} \end{cases}, \quad \beta_3 := \begin{cases} \dfrac{2 \cdot T_s^2 \cdot \omega_5^2 - 8}{T_s^2 \cdot \omega_5^2 - 4 \cdot \zeta \cdot T_s + 4} & \text{if } \zeta \neq \omega_5 \\ 2 \cdot \dfrac{T_s^2 \cdot \omega_5^2 - 4}{(T_s \cdot \omega_5 - 2)^2} & \text{otherwise} \end{cases},$$

$$\gamma_3 := \begin{cases} \dfrac{T_s^2 \cdot \omega_5^2 + 4 \cdot \zeta \cdot T_s + 4}{T_s^2 \cdot \omega_5^2 - 4 \cdot \zeta \cdot T_s + 4} & \text{if } \zeta \neq \omega_5 \\ \dfrac{(T_s \cdot \omega_5 + 2)^2}{(T_s \cdot \omega_5 - 2)^2} & \text{otherwise} \end{cases}$$

$\alpha_3 = -0.248 \qquad \beta_3 = -1.959 \qquad \gamma_3 = 1.058$

$\mathcal{Z}$ transfer function:

$$H_{lp}(z) := \begin{cases} \left(\alpha_3 \cdot \dfrac{1 + 2 \cdot z^{-1} + z^{-2}}{z^{-2} + \beta_3 \cdot z^{-1} + \gamma_3}\right) & \text{if } \zeta \neq \omega_5 \\ \alpha_3 \cdot \dfrac{(z^{-1}+1)^2}{(z^{-1}+\beta_3)^2} & \text{otherwise} \end{cases}$$

System Output

$$Y_{out}(z) = \alpha_3 \cdot \begin{cases} \dfrac{1 + 2 \cdot z^{-1} + z^{-2}}{z^{-2} + \beta_3 \cdot z^{-1} + \gamma_3} \cdot \dfrac{V_m \cdot z^{-1} \cdot K}{1 - 2 \cdot \sqrt{1-K^2} \cdot z^{-1} + z^{-2}} & \text{if } \zeta \neq \omega_5 \\ \dfrac{(z^{-1}+1)^2}{(z^{-1}+\beta_3)^2} \cdot \left(\dfrac{V_m \cdot z^{-1} \cdot K}{1 - 2 \cdot \sqrt{1-K^2} \cdot z^{-1} + z^{-2}}\right) & \text{otherwise} \end{cases} \qquad (5.8.41)$$

Search of the sequence corresponding to the output:

First case: $\zeta \neq \omega_5$

$$Y_{out}(z) = \frac{\alpha_3 \cdot V_m \cdot z^{-1} \cdot K \cdot \left(1 + 2 \cdot z^{-1} + z^{-2}\right)}{\left(z^{-2} + \beta_3 \cdot z^{-1} + \gamma_3\right) \cdot \left(z^2 - 2 \cdot \sqrt{1 - K^2} \cdot z + 1\right)} \qquad (5.8.42)$$

$$Y_{out}(z) = \frac{K \cdot V_m \cdot z \cdot \alpha_3 \cdot (z+1)^2}{\gamma_3 \cdot z^4 + \left(\beta_3 - 2 \cdot \gamma_3 \cdot \sqrt{1 - K^2}\right) \cdot z^3 + \left(\gamma_3 - 2 \cdot \beta_3 \cdot \sqrt{1 - K^2} + 1\right) \cdot z^2 + \left(\beta_3 - 2 \cdot \sqrt{1 - K^2}\right) \cdot z + 1}$$

$\alpha_3 = -0.248$ $\qquad \beta_3 = -1.959$ $\qquad \gamma_3 = 1.058$

Search of the poles:

$$\text{poles3} := \gamma_3 \cdot z^4 + \left(\beta_3 - 2 \cdot \gamma_3 \cdot \sqrt{1 - K^2}\right) \cdot z^3 \ldots + \left(\gamma_3 - 2 \cdot \beta_3 \cdot \sqrt{1 - K^2} + 1\right) \cdot z^2 \ldots + \left(\beta_3 - 2 \cdot \sqrt{1 - K^2}\right) \cdot z + 1 \text{ solve}, z \rightarrow \quad \text{poles3} := \begin{pmatrix} \sqrt{1-K^2} + K \cdot j \\ \sqrt{1-K^2} - K \cdot j \\ \dfrac{\frac{\beta_3}{2} + \frac{\sqrt{\beta_3^2 - 4 \cdot \gamma_3}}{2}}{\gamma_3} \\ \dfrac{\frac{\beta_3}{2} - \frac{\sqrt{\beta_3^2 - 4 \cdot \gamma_3}}{2}}{\gamma_3} \end{pmatrix}$$

$p_1 := \sqrt{1-K^2} + K \cdot j \qquad p_2 := \sqrt{1-K^2} - K \cdot j \qquad p_3 := -\dfrac{\frac{\beta_3}{2} + \frac{\sqrt{\beta_3^2 - 4 \cdot \gamma_3}}{2}}{\gamma_3} \qquad p_4 := -\dfrac{\frac{\beta_3}{2} - \frac{\sqrt{\beta_3^2 - 4 \cdot \gamma_3}}{2}}{\gamma_3}$

$p_1 = 0.809 + 0.588j \qquad p_2 = 0.809 - 0.588j \qquad p_3 = 0.926 - 0.297j \qquad p_4 = 0.926 + 0.297j$

Thanks to the fundamental theorem of Algebra one can write:

$$Y_{out}(z) = \frac{K \cdot V_m \cdot z \cdot \alpha_3 \cdot (z+1)^2}{\gamma_3 \cdot \left(z - p_1\right) \cdot \left(z - p_2\right) \cdot \left(z - p_3\right) \cdot \left(z - p_4\right)} \qquad (5.8.43)$$

In order to calculate the inverse z transform of the output signal, decompose the (5.8.43) in partial fraction

$$\frac{z\cdot(z+1)^2}{(z-p_1)\cdot(z-p_2)\cdot(z-p_3)\cdot(z-p_4)} \text{ parfrac}, z \rightarrow$$

and rewrite the output signal as a linear combination of terms like $\frac{1}{p_n - z}$:

$$Y_{out}(z) = \frac{\alpha_3\cdot K\cdot V_m}{\gamma_3}\cdot\left[\frac{p_3\cdot(p_3+1)^2}{(p_3-p_1)\cdot(p_2-p_3)\cdot(p_3-p_4)\cdot(p_3-z)} \cdots + \frac{p_1\cdot(p_1+1)^2}{(p_2-p_1)\cdot(p_1-p_3)\cdot(p_1-p_4)\cdot(p_1-z)} \cdots + \frac{p_2\cdot(p_2+1)^2}{(p_1-p_2)\cdot(p_2-p_3)\cdot(p_2-p_4)\cdot(p_2-z)} \cdots + \frac{p_4\cdot(p_4+1)^2}{(p_4-p_1)\cdot(p_2-p_4)\cdot(p_3-p_4)\cdot(p_4-z)}\right] \quad (5.8.44)$$

the poles are:

$p_1 = 0.809 + 0.588j$ $\quad p_2 = 0.809 - 0.588j$ $\quad p_3 = 0.926 - 0.297j$ $\quad p_4 = 0.926 + 0.297j$

$p_1 := p_1$ $\quad p_2 := p_2$ $\quad p_3 := p_3$ $\quad p_4 := p_4$

Furthermore, to simplify calculations, define the following constants:

$$G1 := \begin{cases} \frac{p_3\cdot(p_3+1)^2}{(p_3-p_1)\cdot(p_2-p_3)\cdot(p_3-p_4)} & \text{if } \zeta \neq \omega_5 \\ 0 & \text{otherwise} \end{cases}$$

$$G2 := \begin{cases} \frac{p_1\cdot(p_1+1)^2}{(p_2-p_1)\cdot(p_1-p_3)\cdot(p_1-p_4)} & \text{if } \zeta \neq \omega_5 \\ 0 & \text{otherwise} \end{cases}$$

$$G3 := \begin{cases} \frac{p_2\cdot(p_2+1)^2}{(p_1-p_2)\cdot(p_2-p_3)\cdot(p_2-p_4)} & \text{if } \zeta \neq \omega_5 \\ 0 & \text{otherwise} \end{cases}$$

$$G4 := \begin{cases} \frac{p_4\cdot(p_4+1)^2}{(p_4-p_1)\cdot(p_2-p_4)\cdot(p_3-p_4)} & \text{if } \zeta \neq \omega_5 \\ 0 & \text{otherwise} \end{cases}$$

$$Y_{out}(z) = \frac{\alpha_3 \cdot K \cdot V_m}{\gamma_3} \cdot \left[\frac{G1}{(p_3 - z)} + \frac{G2}{(p_1 - z)} + \frac{G3}{(p_2 - z)} + \frac{G4}{(p_4 - z)} \right] \tag{5.8.44}$$

The inverse z transform $\mathcal{Z}^{-1}\left(\frac{1}{p - z}\right) = \frac{\delta(k,0) - p^k}{p}$

obtaining, after the substitution in (5.8.44):

$$Yout_k = \frac{\alpha_3 \cdot K \cdot V_m}{\gamma_3} \cdot \left[G1 \cdot \left(\frac{\delta(k,0) - {p_3}^k}{p_3}\right) + G2 \cdot \left(\frac{\delta(k,0) - {p_1}^k}{p_1}\right) + G3 \cdot \left(\frac{\delta(k,0) - {p_2}^k}{p_2}\right) + G4 \cdot \left(\frac{\delta(k,0) - {p_4}^k}{p_4}\right) \right]$$

and collecting the Kronecker δ, finally results:

$$\left(\frac{G2}{p_1} + \frac{G1}{p_3} + \frac{G3}{p_2} + \frac{G4}{p_4}\right) \cdot \delta(k,0) - \left(G2 \cdot {p_1}^{k-1} + G1 \cdot {p_3}^{k-1} + G3 \cdot {p_2}^{k-1} + G4 \cdot {p_4}^{k-1}\right)$$

$$G5 := \frac{G2}{p_1} + \frac{G1}{p_3} + \frac{G3}{p_2} + \frac{G4}{p_4}$$

$$\boxed{Yout_k = \frac{\alpha_3 \cdot K \cdot V_m}{\gamma_3} \cdot \left[G5 \cdot \delta(k,0) - \left(G2 \cdot {p_1}^{k-1} + G1 \cdot {p_3}^{k-1} + G3 \cdot {p_2}^{k-1} + G4 \cdot {p_4}^{k-1}\right) \right]}$$

Second case: $\zeta = \omega_5$

$$Yout(z) = \frac{(z^{-1} + 1)^2}{(z^{-1} + \beta_3)^2} \cdot \frac{V_m \cdot z^{-1} \cdot K}{1 - 2 \cdot \sqrt{1 - K^2} \cdot z^{-1} + z^{-2}} = \frac{K \cdot V_m \cdot \alpha_3 \cdot (z + 1)^2}{(\beta_3 \cdot z + 1)^2 \cdot \left(z^2 - 2 \cdot z \cdot \sqrt{1 - K^2} + 1\right)}$$

$$Yout(z) = \frac{K \cdot V_m \cdot \alpha_3 \cdot (z + 1)^2}{(\beta_3 \cdot z + 1)^2 \cdot \left(z^2 - 2 \cdot z \cdot \sqrt{1 - K^2} + 1\right)} \tag{5.8.48}$$

$$Yout(z) = \frac{K \cdot V_m \cdot \alpha_3 \cdot (z+1)^2}{\beta_3{}^2 \cdot z^4 + \left(2 \cdot \beta_3 - 2 \cdot \beta_3{}^2 \cdot \sqrt{1-K^2}\right) \cdot z^3 \ldots + \left(\beta_3{}^2 - 4 \cdot \beta_3 \cdot \sqrt{1-K^2} + 1\right) \cdot z^2 \ldots + \left(2 \cdot \beta_3 - 2 \cdot \sqrt{1-K^2}\right) \cdot z + 1} \qquad (5.8.49)$$

$\alpha_3 = -0.248$ $\qquad \beta_3 = -1.959$ $\qquad \gamma_3 = 1.058$

Search of the poles:

$$poles3 := \beta_3{}^2 \cdot z^4 + \left(2 \cdot \beta_3 - 2 \cdot \beta_3{}^2 \cdot \sqrt{1-K^2}\right) \cdot z^3 \ldots + \left(\beta_3{}^2 - 4 \cdot \beta_3 \cdot \sqrt{1-K^2} + 1\right) \cdot z^2 \ldots + \left(2 \cdot \beta_3 - 2 \cdot \sqrt{1-K^2}\right) \cdot z + 1 \text{ solve}, z \rightarrow$$

$$poles4 := \begin{pmatrix} -\frac{1}{\beta_3} \\ -\frac{1}{\beta_3} \\ \sqrt{1-K^2} + K \cdot j \\ \sqrt{1-K^2} - K \cdot j \end{pmatrix}$$

$p_{11} := -\frac{1}{\beta_3}$ $\qquad p_{22} := -\frac{1}{\beta_3}$ $\qquad p_{33} := \sqrt{1-K^2} + K \cdot j$ $\qquad p_{44} := \sqrt{1-K^2} - K \cdot j$

Thanks to the fundamental theorem of Algebra one can write:

$$Yout(z) = \frac{\alpha_3 \cdot K \cdot V_m \cdot (z+1)^2}{\beta_3{}^2 \cdot \left(z - p_{11}\right)^2 \cdot \left(z - p_{33}\right) \cdot \left(z - p_{44}\right)} \qquad (5.8.50)$$

In order to calculate the inverse z transform of the output signal, decompose the (5.8.50) in partial fraction

$$\frac{(z+1)^2}{\left(z - p_{11}\right)^2 \cdot \left(z - p_{33}\right) \cdot \left(z - p_{44}\right)} \text{ parfrac} \rightarrow$$

and rewrite the z transform of the output signal as a linear combination of terms like $\frac{1}{p_n - z}$ and

$$\frac{1}{(p_{11}}$$

$$Yout(z) = \frac{\alpha_3 \cdot K \cdot V_m}{\beta_3^{\,2}} \cdot \left[\frac{(p_{11}+1)^2}{(p_{11}-p_{33})\cdot(p_{11}-p_{44})\cdot(p_{11}-z)^2} \cdots \right. \\ + \frac{(p_{33}+1)^2}{(p_{11}-p_{33})^2\cdot(p_{44}-p_{33})\cdot(p_{33}-z)} \cdots \\ + \frac{(p_{44}+1)^2}{(p_{11}-p_{44})^2\cdot(p_{33}-p_{44})\cdot(p_{44}-z)} \cdots \\ \left. + \frac{(p_{11}+1)\cdot(p_{33}-2\cdot p_{11}+p_{44})^2\cdot\left[(-p_{33}-p_{44}-2)\cdot p_{11} \cdots + p_{33}+p_{44}\cdot(2\cdot p_{33}+1)\right]}{(p_{11}-p_{33})^2\cdot(p_{11}-p_{44})^2\cdot(z-p_{11})\cdot(2\cdot p_{11}-p_{33}-p_{44})^2} \right] \qquad (5.8.51)$$

Now try to calculate the inverse z transform knowing that:

$$\mathcal{Z}^{-1}\left(\frac{1}{p-z}\right) = \frac{\delta(k,0)-p^k}{p}$$

$$\mathcal{Z}^{-1}\left[\frac{1}{(p-z)^2}\right] = \frac{\delta(k,0)+p^k\cdot(k-1)}{p^2}$$

and defining the following constant coefficients:

$$G11 := \frac{(p_{11}+1)\cdot(p_{33}-2\cdot p_{11}+p_{44})^2\cdot\left[(-p_{33}-p_{44}-2)\cdot p_{11}+p_{33}+p_{44}\cdot(2\cdot p_{33}+1)\right]}{(p_{11}-p_{33})^2\cdot(p_{11}-p_{44})^2\cdot(2\cdot p_{11}-p_{33}-p_{44})^2\cdot p_{11}} \qquad (5.8.52)$$

$$G12 := \frac{(p_{44}+1)^2}{(p_{11}-p_{44})^2\cdot(p_{33}-p_{44})\cdot p_{44}} \qquad (5.8.53$$

$$G22 := \frac{(p_{33}+1)^2}{(p_{11}-p_{33})^2\cdot(p_{44}-p_{33})\cdot p_{33}} \qquad (5.8.54)$$

$$G33 := \frac{(p_{11}+1)^2}{(p_{11}-p_{33})\cdot(p_{11}-p_{44})\cdot p_{11}^{\,2}} \qquad (5.8.55)$$

Substituting the previous definition, the function (5.8.51) takes the form:

$$Yout(z) = \frac{\alpha_3 \cdot K \cdot V_m}{\beta_3^{\,2}} \cdot \left[G33 \cdot \frac{1}{(p_{11} - z)^2} \ldots + G22 \cdot \frac{1}{(p_{33} - z)} \ldots + G12 \cdot \frac{1}{(p_{44} - z)} \ldots + G11 \cdot \frac{1}{(z - p_{11})} \right]$$

Substituting in (5.8.51) the previous coefficients and the inverse z transform found, results:

$$Yout_k = \frac{\alpha_3 \cdot K \cdot V_m}{\beta_3^{\,2}} \cdot \left[G33 \cdot \frac{\left[\delta(k,0) + p_{11}^{\,k} \cdot (k-1)\right]}{p_{11}^{\,2}} \ldots + G22 \cdot \left(\frac{\delta(k,0) - p_{33}^{\,k}}{p_{33}} \right) \ldots + G12 \cdot \frac{\left(\delta(k,0) - p_{44}^{\,k}\right)}{p_{44}} \ldots + G11 \cdot \left(\frac{\delta(k,0) - p_{11}^{\,k}}{p_{11}} \right) \right] \tag{5.8.56}$$

Collecting the Kronecker deltas,

$$Yout_k = \frac{\alpha_3 \cdot K \cdot V_m}{\beta_3^{\,2}} \cdot \left[\left(\frac{G11}{p_{11}} + \frac{G33}{p_{11}^{\,2}} + \frac{G22}{p_{33}} + \frac{G12}{p_{44}} \right) \cdot \delta(k,0) \ldots + \frac{G33 \cdot p_{11}^{\,k} \cdot (k-1)}{p_{11}^{\,2}} - \frac{G22 \cdot p_{33}^{\,k}}{p_{33}} - \frac{G12 \cdot p_{44}^{\,k}}{p_{44}} - \frac{G11 \cdot p_{11}^{\,k}}{p_{11}} \right] \tag{5.8.57}$$

and after having defined the new constant $G44 := \frac{G11}{p_{11}} + \frac{G33}{p_{11}^{\,2}} + \frac{G22}{p_{33}} + \frac{G12}{p_{44}}$

the final result, considering both cases $\zeta \neq \omega_5$ and $\zeta = \omega_5$, is:

$$Yout_k := \alpha_3 \cdot K \cdot V_m \cdot \begin{cases} \frac{1}{\gamma_3} \cdot \left[G5 \cdot \delta(k,0) - \left(G2 \cdot p_1^{\,k-1} + G1 \cdot p_3^{\,k-1} + G3 \cdot p_2^{\,k-1} + G4 \cdot p_4^{\,k-1} \right) \right] & \text{if } \zeta \neq \omega_5 \\ \frac{1}{\beta_3^{\,2}} \cdot \left[G44 \cdot \delta(k,0) \ldots + \frac{G33 \cdot p_{11}^{\,k} \cdot (k-1)}{p_{11}^{\,2}} - \frac{G22 \cdot p_{33}^{\,k}}{p_{33}} - \frac{G12 \cdot p_{44}^{\,k}}{p_{44}} - \frac{G11 \cdot p_{11}^{\,k}}{p_{11}} \right] & \text{otherwise} \end{cases}$$

(5.8.58)

Numerical sequence of the output:

$Yout^T =$

	0	1	2	3
0	$-1.222\cdot10^{-15}$	10.265-28.557j	14.851-23.18j	...

V

Energy of the sequence $Yout_k$: $EYout := \sum_{k=0}^{N0-1} \left(\left|Yout_k\right|\right)^2$ $EYout = 3.747\times10^4 V^2$

$\zeta = 17.743\cdot\frac{Mrads}{sec}$ $\omega_5 = 191.627\cdot\frac{Mrads}{sec}$

$V_{out} := \left|W_{lp}(j\cdot\omega_{test})\right| V_{pp}$ $V_{out} = 16.541V$

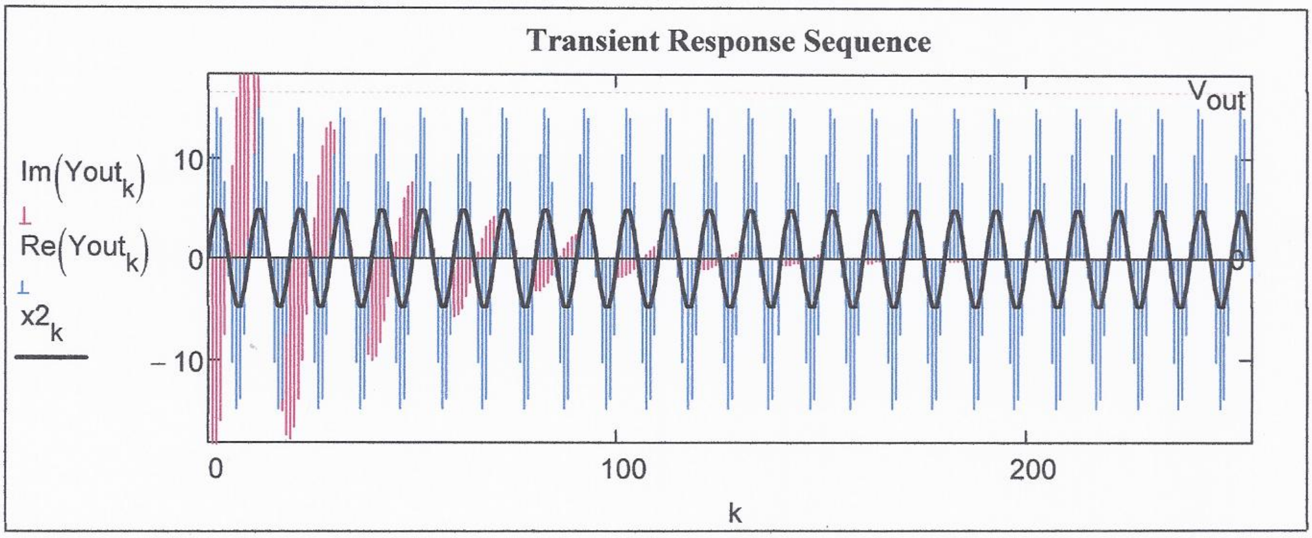


fig.:5.8.10

Pick amplitude of the frequency response if Q_5>0.5 $rpeak := 20\cdot\log\left(\left|\frac{A_5\cdot\omega_5}{2\cdot\zeta}\right|\right)$

$20\cdot\log\left(\left|W_{lp}(j\cdot\omega_{test})\right|\right) = 10.392\cdot dB$ $Q_5 = 5.4$

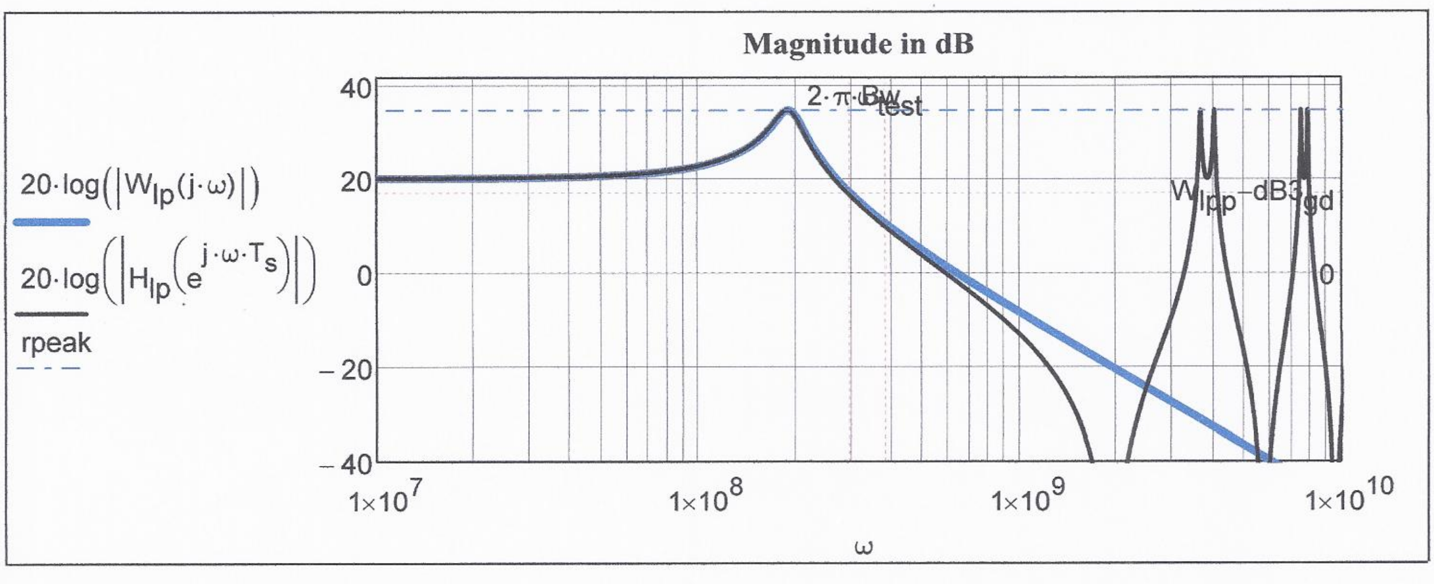


fig.:5.1.8.1

$Bw = 47.1 \cdot MHz$

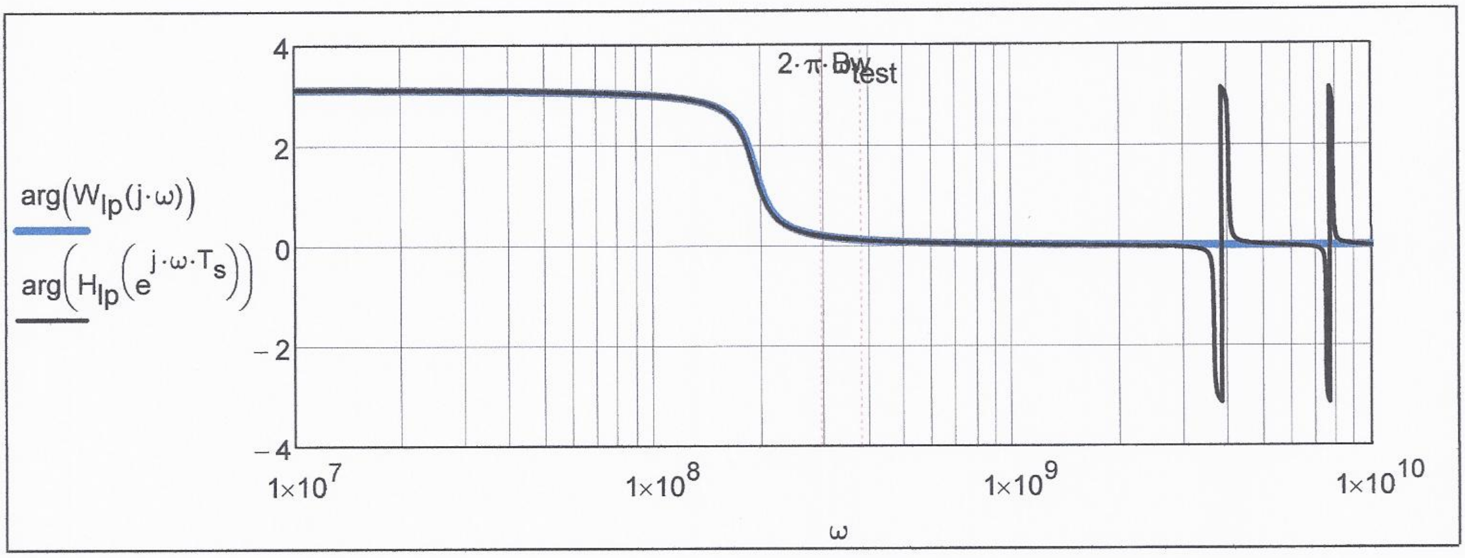


fig.:5.1.8.1'

pick amplitude $WlpdB_{pick} = 34.685$ $\quad \omega_{pick} = 0.19 \cdot \frac{Grads}{sec}$ $\quad W_{lpp} = 20$ $\quad \omega_5 = 0.192 \cdot \frac{Grads}{sec}$

Proof

Proof: **First case:**

$\zeta \neq \omega_5$

$p_1 := p_1 \quad p_2 := p_2 \quad p_3 := p_3 \quad p_4 := p_4 \quad G1 := G1 \quad G2 := G2 \quad G3 := G3 \quad G4 := G4 \quad G5 := G5$

$$G5 = \frac{G2}{p_1} + \frac{G1}{p_3} + \frac{G3}{p_2} + \frac{G4}{p_4}$$

$$\left[\frac{G2}{p_1} + \frac{G1}{p_3} + \frac{G3}{p_2} + \frac{G4}{p_4} \cdot \delta(k,0) \ldots + (-1)\cdot\left(G2 \cdot p_1^{k-1} + G1 \cdot p_3^{k-1} + G3 \cdot p_2^{k-1} + G4 \cdot p_4^{k-1}\right)\right] \begin{vmatrix} \text{ztrans}, k \\ \text{collect}, G1, G2, G3, G4 \rightarrow \\ \text{simplify} \end{vmatrix}$$

results: $$\frac{G2}{p_1 - z} + \frac{G1}{p_3 - z} + \frac{G3}{p_2 - z} + \frac{G4}{p_4 - z}$$

Namely $$Y_{out}(z) = \frac{\alpha_3 \cdot K \cdot V_m}{\gamma_3} \cdot \left(\frac{G2}{p_1 - z} + \frac{G1}{p_3 - z} + \frac{G3}{p_2 - z} + \frac{G4}{p_4 - z}\right)$$

Proof: Second case: $\zeta = \omega_5$

$p_{11} := p_{11} \quad p_{22} := p_{22} \quad p_{33} := p_{33} \quad p_{44} := p_{44}$

$G11 := G11 \quad G22 := G22 \quad G33 := G33 \quad G44 := G44 \quad G55 := G55$

$$\left(\frac{G11}{p_{11}} + \frac{G33}{p_{11}^{2}} + \frac{G22}{p_{33}} + \frac{G12}{p_{44}}\right)\cdot\delta(k,0)\ \ldots$$
$$+\frac{G33\cdot p_{11}^{k}\cdot(k-1)}{p_{11}^{2}} - \frac{G22\cdot p_{33}^{k}}{p_{33}} - \frac{G12\cdot p_{44}^{k}}{p_{44}} - \frac{G11\cdot p_{11}^{k}}{p_{11}}$$

ztrans , k
collect , G11 , G22 , G33 , G12 →
simplify

results: $\frac{G11}{p_{11}-z} + \frac{G33}{(p_{11}-z)^{2}} + \frac{G22}{p_{33}-z} + \frac{G12}{p_{44}-z}$

Namely:

$$Yout_k = \frac{\alpha_3\cdot K\cdot V_m}{\beta_3^{2}}\cdot\left[G33\cdot\frac{1}{(p_{11}-z)^{2}}\ \ldots + G22\cdot\frac{1}{(p_{33}-z)}\ \ldots + G12\cdot\frac{1}{(p_{44}-z)}\ \ldots + G11\cdot\frac{1}{(z-p_{11})}\right]$$

q.e.d.
(Quod Erat Demostrandum)

Proof