Sensitivity

About Sensitivity

Sensitivity based on the node Admittance Matrix **Y**

.$\mathbf{R_{im}}$ is the **R**educed incidence matrix [(n_o-1)xb]

x is a parameter that changes of a small amount δx,

$$Y = R_{im} \cdot Y_b \cdot R_{im}^T \Rightarrow \delta Y = R_{im} \cdot \delta Y_b \cdot R_{im}^T \Rightarrow \boxed{\frac{\partial}{\partial x} Y = R_{im} \cdot \left(\frac{\partial}{\partial x} Y_b\right) \cdot R_{im}^T}$$

$$\delta E = -Y^{-1} \cdot \delta Y \cdot E = -Y^{-1} \cdot R_{im} \cdot \delta Y_b \cdot R_{im}^T \cdot E$$

$$\delta E = -Y^{-1} \cdot R_{im} \cdot \delta Y_b \cdot V$$

$$\frac{\partial}{\partial x} E = -Y^{-1} \cdot R_{im} \cdot \left(\frac{\partial}{\partial x} Y_b\right) \cdot V \quad \Rightarrow$$

$$R_{im} \cdot \left(\frac{\partial}{\partial x} Y_b\right) \cdot V = R_{im} \cdot \left(\frac{\partial}{\partial x} Y_b\right) \cdot R_{im}^T \cdot E = \left(\frac{\partial}{\partial x} Y\right) \cdot E$$

$$\boxed{\frac{\partial}{\partial x} E = -Y^{-1} \cdot R_{im} \cdot \left(\frac{\partial}{\partial x} Y_b\right) \cdot V = -Y^{-1} \cdot \left(\frac{\partial}{\partial x} Y\right) \cdot E}$$

Relation which links the partial derivative of the node voltage phasor **E** to the partial derivative of the branch admittance matrix $\mathbf{Y_b}$. It lets to calculate the sensitivity of E in respone to a variation of the parameter x, as will be seen immediately below.

About Sensitivity

Sensitivity

Sensitivity definition given the performance $\mathcal{P}$ and the parameter x_i:

$$S_{x.i} = \frac{x_i}{\mathcal{P}} \cdot \frac{\partial}{\partial x_i} \mathcal{P} = \frac{\partial}{\partial \ln x_i} \ln(\mathcal{P})$$

The performance $\mathcal{P}$ be the node voltage phasor $\mathbf{E_i}$, than the sensitivity of $\mathcal{P}$ vs the variations of the parameter x is given by:

$$S_{xE_i} = -\frac{x}{E_i} \cdot \left(Y^{-1} \cdot \frac{\partial}{\partial x_i} Y \cdot E\right)_i \qquad i := 0 .. n_b - 1$$

where **Y** is the network admittance matrix.

5.1.1) Calculation of the sensitivity of the circuit performance A_0 with respect to R3 and R1:

The filter project could be developed even considering the desired sensitivity of the various parameters.

In this particular case, the performance is the voltage gain: $\mathcal{P} = A_0 = \frac{-R3}{R1}$

Sensitivity Calculation

$R1 := R1 \qquad R3 := R3$

$$S_{R1} := \frac{R1}{-\frac{R3}{R1}} \cdot \frac{\partial}{\partial R1}\left(-\frac{R3}{R1}\right) \text{ simplify } \rightarrow -1$$

$$S_{R3A0} := \frac{R3}{-\frac{R3}{R1}} \cdot \frac{\partial}{\partial R3}\left(-\frac{R3}{R1}\right) \text{ simplify } \rightarrow 1$$

Sensitivity Calculation

$$\left|\frac{\Delta A_0}{A_0}\right| = \left|S_{R1}\right| \cdot \frac{|\Delta R1|}{R1} + \left|S_{R3A0}\right| \cdot \frac{|\Delta R3|}{R3} = \frac{|\Delta R1|}{R1} + \frac{|\Delta R3|}{R3}$$

$$\boxed{S_{A.0} = \left|\frac{\Delta A_0}{A_0}\right| = \frac{|\Delta R1|}{R1} + \frac{|\Delta R3|}{R3}}$$

5.1.2) Calculation of the sensitivity of the circuit performance ω_0 with respect to C1, C2, R2, R3

$$S_{x.i} = \frac{x_i}{\mathcal{P}} \cdot \frac{\partial}{\partial x_i} \mathcal{P} = \frac{\partial}{\partial \ln x_i} \ln(\mathcal{P})$$

$$\mathcal{P} = \omega_0 = \frac{1}{\sqrt{R2 \cdot R3 \cdot C1 \cdot C2}}$$

Sensitivity Calculation

$$\boxed{S_{\omega.o} = \frac{1}{2} \cdot \left(\frac{|\Delta C1|}{C1} + \frac{|\Delta C2|}{C2} + \frac{|\Delta R1|}{R1} + \frac{|\Delta R2|}{R2} \right)}$$

$$\frac{|\Delta C1|}{C1} + \frac{|\Delta C2|}{C2} = 2 \cdot S_{\omega.o} - \left(\frac{|\Delta R1|}{R1} + \frac{|\Delta R2|}{R2} \right)$$

5.1.3) Calculation of the sensitivity of the circuit performance Q with respect to C1, C2, R1, R2, R3

$$S_{x.i} = \frac{x_i}{\mathcal{P}} \cdot \frac{\partial}{\partial x_i}\mathcal{P} = \frac{\partial}{\partial \ln x_i}\ln(\mathcal{P})$$

$$Q = \frac{1}{\sqrt{\frac{C2}{C1}}\cdot\left(\frac{\sqrt{R2\cdot R3}}{R1}+\sqrt{\frac{R3}{R2}}+\sqrt{\frac{R2}{R3}}\right)}$$

Sensitivity Calculation

$R1 := R1 \qquad R2 := R2 \qquad R3 := R3 \qquad (C1 := C1) \qquad (C2 := C2)$

$$S_{QC1} := \frac{C1}{\frac{1}{\sqrt{\frac{C2}{C1}}\cdot\left(\frac{\sqrt{R2\cdot R3}}{R1}+\sqrt{\frac{R3}{R2}}+\sqrt{\frac{R2}{R3}}\right)}}\cdot\frac{\partial}{\partial C1}\left[\frac{1}{\sqrt{\frac{C2}{C1}}\cdot\left(\frac{\sqrt{R2\cdot R3}}{R1}+\sqrt{\frac{R3}{R2}}+\sqrt{\frac{R2}{R3}}\right)}\right] \text{ simplify } \rightarrow \frac{1}{2}$$

$$S_{QC2} := \frac{C2}{\frac{1}{\sqrt{\frac{C2}{C1}}\cdot\left(\frac{\sqrt{R2\cdot R3}}{R1}+\sqrt{\frac{R3}{R2}}+\sqrt{\frac{R2}{R3}}\right)}}\cdot\frac{\partial}{\partial C2}\left[\frac{1}{\sqrt{\frac{C2}{C1}}\cdot\left(\frac{\sqrt{R2\cdot R3}}{R1}+\sqrt{\frac{R3}{R2}}+\sqrt{\frac{R2}{R3}}\right)}\right] \text{ simplify } \rightarrow -\frac{1}{2}$$

$$S_{QR1} := \frac{R1}{\frac{1}{\sqrt{\frac{C2}{C1}}\cdot\left(\frac{\sqrt{R2\cdot R3}}{R1}+\sqrt{\frac{R3}{R2}}+\sqrt{\frac{R2}{R3}}\right)}}\cdot\frac{\partial}{\partial R1}\left[\frac{1}{\sqrt{\frac{C2}{C1}}\cdot\left(\frac{\sqrt{R2\cdot R3}}{R1}+\sqrt{\frac{R3}{R2}}+\sqrt{\frac{R2}{R3}}\right)}\right] \text{ simplify } \rightarrow$$

$$\boxed{S_{QR1} = \frac{1}{1+R1\cdot\left(\frac{1}{R3}+\frac{1}{R2}\right)}}$$

$$S_{QR2} := \frac{R2}{\frac{1}{\sqrt{\frac{C2}{C1}}\cdot\left(\frac{\sqrt{R2\cdot R3}}{R1}+\sqrt{\frac{R3}{R2}}+\sqrt{\frac{R2}{R3}}\right)}}\cdot\frac{\partial}{\partial R2}\left[\frac{1}{\sqrt{\frac{C2}{C1}}\cdot\left(\frac{\sqrt{R2\cdot R3}}{R1}+\sqrt{\frac{R3}{R2}}+\sqrt{\frac{R2}{R3}}\right)}\right] \text{ simplify } \rightarrow$$

$$\boxed{S_{QR2} = -\frac{1}{2}\cdot\left(\frac{R1\cdot(R2-R3)+R2\cdot R3}{R1\cdot(R2+R3)+R2\cdot R3}\right)}$$

$$S_{QR3} := \frac{R3}{\frac{1}{\sqrt{\frac{C2}{C1}}\cdot\left(\frac{\sqrt{R2\cdot R3}}{R1}+\sqrt{\frac{R3}{R2}}+\sqrt{\frac{R2}{R3}}\right)}}\cdot\frac{\partial}{\partial R3}\left[\frac{1}{\sqrt{\frac{C2}{C1}}\cdot\left(\frac{\sqrt{R2\cdot R3}}{R1}+\sqrt{\frac{R3}{R2}}+\sqrt{\frac{R2}{R3}}\right)}\right] \text{ simplify } \rightarrow$$

$$S_{QR3} = \frac{1}{R3\cdot\left(\frac{1}{R1}+\frac{1}{R2}\right)+1} - \frac{1}{2}$$

$$S_Q = \frac{|\Delta Q|}{|Q|} = |S_{QC1}|\cdot\frac{|\Delta C1|}{C1} + |S_{QC2}|\cdot\frac{|\Delta C2|}{C2} + |S_{QR1}|\cdot\frac{|\Delta R1|}{R1} + |S_{QR2}|\cdot\frac{|\Delta R2|}{R2} \ldots + |S_{QR3}|\cdot\frac{|\Delta R3|}{R3}$$

Sensitivity Calculation

$$S_Q = \frac{1}{2}\cdot\left(\frac{|\Delta C1|}{C1}+\frac{|\Delta C2|}{C2}\right) + \frac{1}{1+R1\cdot\left(\frac{1}{R3}+\frac{1}{R2}\right)}\cdot\frac{|\Delta R1|}{R1} + \frac{1}{2}\cdot\left|\left(\frac{R1\cdot(R2-R3)+R2\cdot R3}{R1\cdot(R2+R3)+R2\cdot R3}\right)\right|\cdot\frac{|\Delta R2|}{R2} \ldots + \left|\frac{1}{R3\cdot\left(\frac{1}{R1}+\frac{1}{R2}\right)+1}-\frac{1}{2}\right|\cdot\frac{|\Delta R3|}{R3}$$

if: $\frac{|\Delta C1|}{C1} = \frac{|\Delta C2|}{C2} = \frac{|\Delta R1|}{R1} = \frac{|\Delta R2|}{R2} = \frac{|\Delta R3|}{R3} = rtol$

$$S_Q = \left(\left[\left|\frac{1}{R3\cdot\left(\frac{1}{R1}+\frac{1}{R2}\right)+1}-\frac{1}{2}\right| + \frac{1}{\left|R1\cdot\left(\frac{1}{R2}+\frac{1}{R3}\right)+1\right|} + \frac{\left|R2\cdot\left(\frac{1}{R1}+\frac{1}{R3}\right)-1\right|}{2\cdot\left|R2\cdot\left(\frac{1}{R1}+\frac{1}{R3}\right)+1\right|} + 1\right]\cdot rtol\right)$$

Summary

$$S_{A.0} = \left|\frac{\Delta A_0}{A_0}\right| = \frac{|\Delta R1|}{R1} + \frac{|\Delta R3|}{R3} = 2\cdot rtol$$

$$S_{\omega.o} = \frac{1}{2}\cdot\left(\frac{|\Delta C1|}{C1}+\frac{|\Delta C2|}{C2}+\frac{|\Delta R1|}{R1}+\frac{|\Delta R2|}{R2}\right) = 2\cdot rtol$$

$$S_Q = \left(\left[\left|\frac{1}{R3\cdot\left(\frac{1}{R1}+\frac{1}{R2}\right)+1}-\frac{1}{2}\right| + \frac{1}{\left|R1\cdot\left(\frac{1}{R2}+\frac{1}{R3}\right)+1\right|} + \frac{\left|R2\cdot\left(\frac{1}{R1}+\frac{1}{R3}\right)-1\right|}{2\cdot\left|R2\cdot\left(\frac{1}{R1}+\frac{1}{R3}\right)+1\right|} + 1\right]\cdot rtol\right)$$

Example

are given: rtol := 5.0% ,

R1 := 0.047·kΩ, or R1 := 179.5·Ω

C1 := 47.0·nF ,

from the definition of A_0 is: $R3 := -A_0 \cdot R1$,

from the definition of damping factor, is derived: $R2 := \dfrac{1}{2 \cdot C1 \cdot \zeta - \left(\dfrac{1}{R1} + \dfrac{1}{R3}\right)}$,

therefore, for R2_>0, must be: $\left(C1 > \dfrac{1}{2 \cdot \zeta} \cdot \left(\dfrac{1}{R1} + \dfrac{1}{R3}\right)\right)$, $\quad \dfrac{1}{2 \cdot \zeta} \cdot \left(\dfrac{1}{R1} + \dfrac{1}{R3}\right) = 0.975 \cdot nF$

namely: $R2 = 3.458 \cdot \Omega$.

From the definition of ω_0 , is derived: $C2 := \dfrac{1}{C1 \cdot R2 \cdot R3 \cdot \omega_0^{\,2}}$,

$$C2 = 8.683 \times 10^3 \cdot pF ,$$

$$\frac{\omega_0}{2 \cdot \zeta} = 0.1 \qquad \left(\frac{\omega_0}{2 \cdot Q}\right) = 3.142 \times 10^{-3} \cdot \frac{Grads}{sec}$$

$$R1 = 0.18 \cdot k\Omega$$

$$R2 = 3.458\Omega \qquad 4 \cdot Q^2 \cdot \left(\left|A_0\right| + 1\right) \cdot C2 = 3.82 \cdot nF$$

$$R3 = 1.795 \cdot k\Omega \qquad C1 = 47 \cdot nF$$

Voltage gain: $-\dfrac{R3}{R1} = -10$

Quality factor Q: $\dfrac{1}{\sqrt{\dfrac{C2}{C1}} \cdot \left(\dfrac{\sqrt{R2 \cdot R3}}{R1} + \sqrt{\dfrac{R3}{R2}} + \sqrt{\dfrac{R2}{R3}}\right)} = 0.1$

frequency resulting from the Poles' geometric mean:

$$\frac{1}{\sqrt{R2 \cdot R3 \cdot C1 \cdot C2}} = 6.283 \times 10^{-4} \cdot \frac{Grads}{sec} \qquad \omega_0 = 6.283 \times 10^{-4} \cdot \frac{Grads}{sec}$$

The previous findings, after a slight modification of one or more of them, may be used as a guess for a new solution (not always found, also after many attempts) to the system, as follows:

Given

$$\omega_0 = \frac{1}{\sqrt{R2 \cdot R3 \cdot C1 \cdot C2}}$$

$$A_0 = -\frac{R3}{R1}$$

$$Q = \frac{1}{\sqrt{\dfrac{C2}{C1}} \cdot \left(\dfrac{\sqrt{R2 \cdot R3}}{R1} + \sqrt{\dfrac{R3}{R2}} + \sqrt{\dfrac{R2}{R3}}\right)}$$

$$\zeta = \frac{1}{2}\cdot\left[\frac{1}{C1}\cdot\left(\frac{1}{R1}+\frac{1}{R2}+\frac{1}{R3}\right)\right]$$

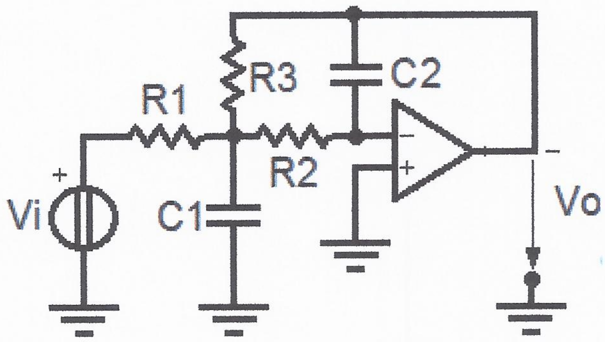
$$R1 > 0.0\cdot\Omega$$

$$R2 > 0.0\cdot\Omega$$

$$R3 > 0.0\cdot\Omega$$

$$Rx := \text{Find}(R1, R2, R3)$$

$$Rx = \begin{pmatrix} 179.5 \\ 3.458 \\ 1.795\times 10^{3} \end{pmatrix}\cdot\Omega \qquad C1 = 47\cdot nF \qquad C2 = 8.683\times 10^{3}\cdot pF$$



$$A_0 = -10$$

$$\omega_0 = 6.283\times 10^{-4}\cdot\frac{Grads}{sec}$$

$$2\cdot\zeta = 6.283\times 10^{-3}\cdot\frac{Grads}{sec}$$

Sensitivity for this example

$$S_{A.0} = \left|\frac{\Delta A_0}{A_0}\right| \qquad S_{A.0} := (rtol + rtol) \qquad rtol = 5\cdot\%$$

$$S_{\omega.0} = \frac{|\Delta\omega_o|}{|\omega_o|} \qquad S_{\omega.0} := \frac{1}{2}\cdot(rtol + rtol + rtol + rtol)$$

$$S_Q := \left(\left[\left|\frac{1}{R3\cdot\left(\frac{1}{R1}+\frac{1}{R2}\right)+1} - \frac{1}{2}\right| + \frac{1}{\left|R1\cdot\left(\frac{1}{R2}+\frac{1}{R3}\right)+1\right|} + \frac{\left|R2\cdot\left(\frac{1}{R1}+\frac{1}{R3}\right)-1\right|}{2\cdot\left|R2\cdot\left(\frac{1}{R1}+\frac{1}{R3}\right)+1\right|} + 1\right]\cdot rtol\right)$$

$S_{A.0} = 10\cdot\%$ $\qquad$ $S_{\omega.0} = 10\cdot\%$ $\qquad$ $S_Q = 9.981\cdot\%$