CHAPTER 10 BUILDING THERMAL CONTROL

10.4 Z-Transforms and Their Application to Digital Thermal Control and Simulation

Z-transforms provide a simple and elegant method for solving linear difference equations. They are thus suited to simulation of digital control systems, such as those used in HVAC systems, and for energy analysis of buildings with hourly weather data. They play the same role as Laplace transforms for continuous systems. Here we consider their application for thermal control. First we review the basic theory, followed by an example.

Consider a continuous function *y(t)* sampled at uniform intervals *t*. The *Z*-transform of the sequence of sampled values is defined by

$$Z(y(0), y(t), y(2 \cdot t), \text{"..."}) = \sum_n \left(y(n \cdot t) \cdot z^{-n}\right) \qquad n = 0, 1 .. \infty$$

Recall that for a sequence of impulses the Laplace transforms are

$$y(s) = \sum_n (y(n \cdot t) \cdot exp(-n \cdot t \cdot s))$$

Substituting

$$z = exp(t \cdot s)$$

we obtain

$$y(s) = \sum_n \left(y(n \cdot t) \cdot z^{-n}\right) = y(z)$$

Therefore, the *Z*-transform of a sequence of sampled values is a special case of the Laplace transform of the same sequence of impulses when $z = exp(t\ s)$ is substituted.

Example: unit step

$$f(t) := 1 \cdot (t > 0)$$

$$z(f(t)) = 1 + 1 \cdot z^{-1} + 1 \cdot z^{-2} .. \sum_{n} z^{-n} = \frac{1}{1 - z^{-1}}$$

Exponential Function:

$$Z\left(e^{a \cdot t}\right) = \frac{z}{z - e^{-a \cdot t}}$$

Time delay:

$$t_d = k \cdot t$$

Given that $Z(f(\tau)) = f(z)$ then $Z\left(f\left(\tau - t_d\right)\right) = f(z) \cdot z^{-k}$

Integral:

$$y(\tau) = \int_0^{n \cdot t} f(\tau) \, d\tau \qquad y(z) = \frac{t}{2} \cdot \frac{1 + z^{-1}}{1 - z^{-1}} \cdot f(z)$$

Derivative:

$$y(z) = \frac{1}{t} \cdot \left(1 - z^{-1}\right) \cdot f(z)$$

Final-Value Theorem:

This can be used to find the steady-state value of a system's response.

$lim(y(t))$ as as t--> ∞ is equal to

$lim\left(1 - z^{-1}\right) \cdot y(z)$ as z --> ∞

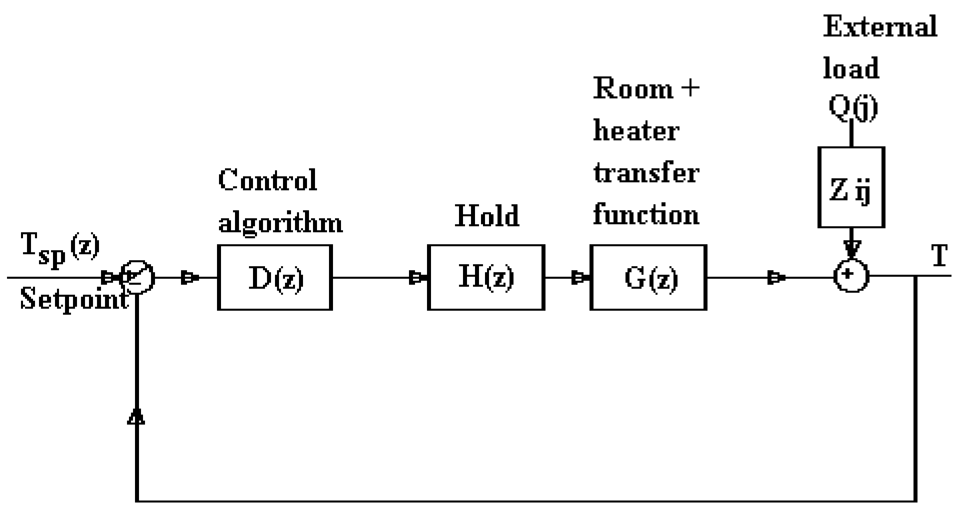
PID Algorithm:

There are two forms of the digital PID algorithm: the position form and the velocity form. The velocity form has the advantage that it does not need initialization in that it outputs only the required change in the controlled variable. The PID controller transfer function in velocity form is given by

$$G_c(z) = K_p\left(\left(1+\frac{t}{\tau_I}-\frac{\tau_D}{t}\right)-\left(1+\frac{2\cdot\tau_D}{t}\right)\cdot z^{-1}+\frac{\tau_D}{t}\cdot z^{-2}\right)$$

where K_P is the proportional gain, τ_I is the integral time and τ_D is the derivative time (t is sampling time).
A continuous function needs to be constructed from the discrete signals when a device is to be controlled by a computer. A zero-order hold element is usually used for this purpose.

Example: We will analyze a simple digital feedback control loop and determine the system response to a step change in the setpoint. The loop consists of the control algorithm, a zero-order hold which converts the digital sampled values to short pulses (analog output) and the room-heating system transfer function.



$Tsp(z) := \frac{1}{1-z^{-1}}$ unit step change of the setpoint

$K_c := \frac{1}{\Delta°C}$

$D(z) := K_c$ proportional control constant

We will consider a first-order transfer function for *G(z)*:

$\tau_h := 6\ hr$ time constant

$K_h := 1$ normalized system capacity

$t := 1800\ s$ sampling period

It is recommended that the sampling period be between 0.1 and 0.2 times the system time constant to ensure stability (Stephanopoulos 1984).

$$H(z) \cdot G(z) = Z\left(\left(1 - \frac{e^{-t \cdot s}}{s}\right) \cdot \frac{K_h}{\tau_h \cdot s + 1}\right)$$

pulse transfer function -- heating system combination (in Laplace domain)

$$HG(z) := K_h \cdot \frac{\left(1 - \exp\left(-\frac{t}{\tau_h}\right)\right) \cdot z^{-1}}{1 - \exp\left(-\frac{t}{\tau_h}\right) \cdot z^{-1}}$$

HG(z) is known as the pulse z-transfer function (the zero-order hold times the process transfer function).

Closed-loop response to setpoint changes:

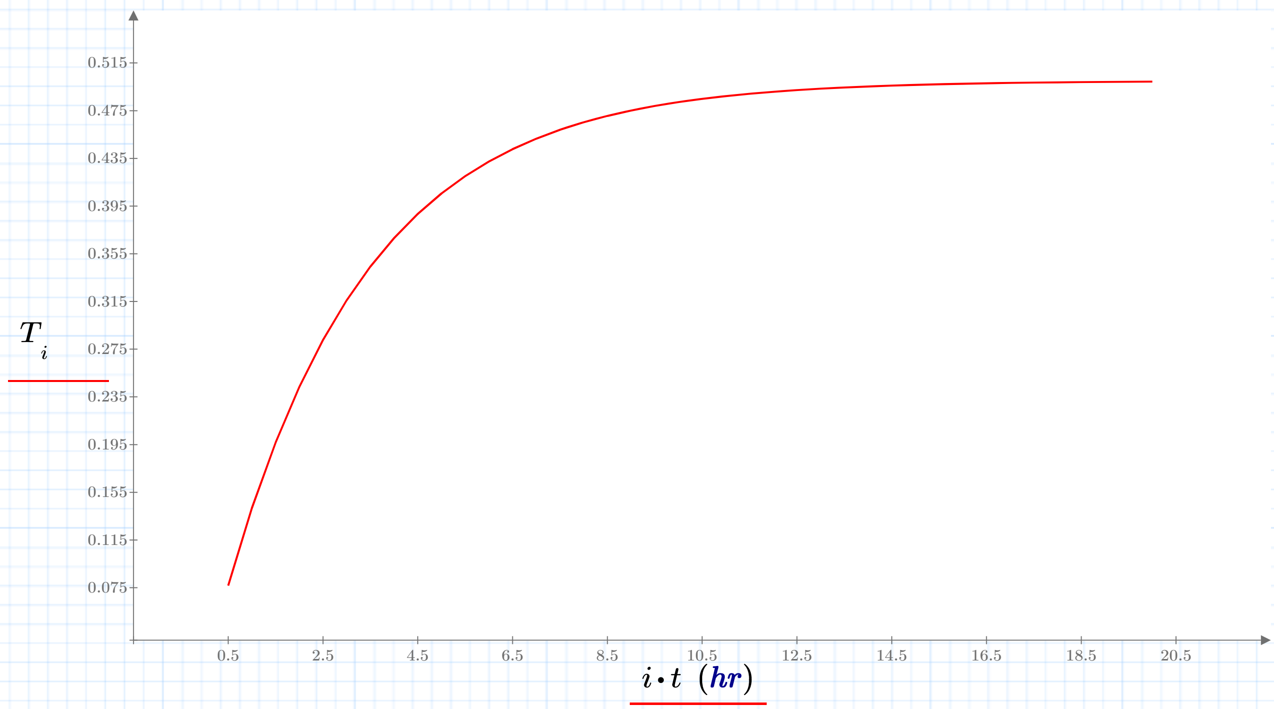
$$\frac{T(z)}{T_{sp}(z)} = \frac{HG(z) \cdot D(z)}{1 + HG(z) \cdot D(z)}$$

The system response T in the time domain is determined as follows:

$$T(z) := \frac{HG(z) \cdot D(z)}{1 + HG(z) \cdot D(z)} \cdot Tsp(z)$$

$i := 1, 2 .. 40$ time array

$$T_i := \frac{K_c \cdot K_h}{\frac{1}{K} + K_c \cdot K_h} \cdot \left(1 - \exp\left(\frac{-i \cdot t}{\frac{\tau_h}{1 + K_c \cdot K_h \cdot K}}\right)\right)$$



The steady state response is equal to 0.5 degC, that is, there is an offset of 0.5 degC at steady state (offset is the steady-state difference between the desired setpoint and the actual temperature).

Stability

If the system closed-loop transfer function can be expressed as a ratio of two polynomials in z^{-1} (as is the case now) then the system is stable if the roots of the denominator lie in the unit circle.

Guess roots:

$j := -\sqrt{1}$ $\qquad$ $z := 0.9$

$r1 := \mathbf{root}(1 + HG(z) \cdot D(z) \cdot K, z)$ $\qquad$ $r1 = 0.84$

(only one root)

Therefore the system is stable. Note that in this case the root can be determined analytically as

$$\exp\left(-\frac{t}{\tau_h}\right) \cdot 2 - 1 = 0.84$$

As can be seen, if the sampling interval *t* is not appropriate, the sampling process may introduce instabilities even for proportional control of a heating system.

References

Stephanopoulos, G. 1984. *Chemical process control, an introduction to theory and practice*. Englewood Cliffs, N.J.: Prentice-Hall.