

CHAPTER 3 TRANSMISSION LINE CALCULATIONS

3.3 The Smith Chart

This document provides a Smith chart which maps normalized impedance, $Z_n(z)$, as a function of the reflection coefficient, Γ . The chart is used to plot a reflection coefficient corresponding to a given normalized load impedance, and subsequently to solve an impedance matching example. Graphical methodology and results are compared with Mathcad's complex arithmetic capability.

You provide:

- Z_L , the complex load impedance of the line

Background

The Smith chart shows, on the Γ plane, contours of constant $\text{Re}(Z)$ and $\text{Im}(Z)$, that is, contours of constant resistance and reactance. Pivoting around the center of the Smith chart on a circle of constant radius $|\Gamma|$ is equivalent to examining the impedance along the transmission line, and will provide a graphical solution for $Z_n(z)$.

Typically, $\Gamma(z)$ and $Z(z)$ can be easily calculated in Mathcad without using a Smith chart (see **Section 3.2 Reflection Coefficient Calculations**), but the chart gives a convenient way of visualizing these relationships. Plots can also provide a first guess for numerical solutions of impedance matching problems (see definition of the **root** function in the *Mathcad User's Guide*.)

Smith Chart Restrictions

The plot covers the portion of the complex Γ plane with real and imaginary parts less than 1 in absolute value, since not more than 100% of the power can be reflected. The curves on the Smith chart represent loci of constant **normalized** resistance and reactance; you should plot only normalized impedances on the Smith Chart.

Mathcad Implementation

This document describes Mathcad plotting of transmission line impedances on a Smith chart, and shows example of impedance matching using the chart.

Constants

$$\epsilon_0 := 8.854 \cdot 10^{-12} \frac{F}{m} \quad \mu_0 := 4 \cdot \pi \cdot 10^{-7} \frac{H}{m}$$

The data for the Smith Chart is stored in the Excel Component below. We set up the plot by using the *Outputs* section to map data from the Excel object to variables in the Mathcad worksheet.

SR	SI
1	0
0.998	0.06279
0.9921	0.1253
0.9823	0.18747
0.9686	0.2487
0.9511	0.309
0.9298	0.3681
0.9048	0.4258

Outputs

```
SR := excel "Sheet1!A2:A875"  
SI := excel "Sheet1!B2:B875"
```

These instructions plot the first column of numbers in the data file on the horizontal axis against the second column on the vertical axis, creating a simplified Smith chart, with the following gridlines:

Circles of constant resistance **r** for **r** = 0, .2, .5, 1, 2, and 5.

Contours of constant reactance **x** are at **x** = 0, .2, .5, 1, 2, 5, -.2, -.5, -1, -2, and -5.

To plot values of the reflection coefficient on the chart, plot **Re(Γ)** against **Im(Γ)**. As an example, the sequence of instructions below shows the procedure for mapping a reflection coefficient onto the Smith chart. On the same chart, the circle representing all other **Γ(z)** is plotted.

Given a complex normalized load impedance

$$Z_{nL} := 2 + 2j$$

Calculate the reflection coefficient

$$\Gamma_L := \frac{Z_{nL} - 1}{Z_{nL} + 1} = 0.538 + 0.308i \quad |\Gamma_L| = 0.62$$

To draw a circle at radius $|\Gamma|$, define

$$f(\rho, \theta) := \rho \cdot \exp(1j \cdot \theta) \quad \text{step} := 0, .01 \dots 1$$

$$\text{real}\Gamma(\text{step}) := \text{Re}(f(|\Gamma_L|, 2 \cdot \pi \cdot \text{step})) \quad \text{imag}\Gamma(\text{step}) := \text{Im}(f(|\Gamma_L|, 2 \cdot \pi \cdot \text{step}))$$

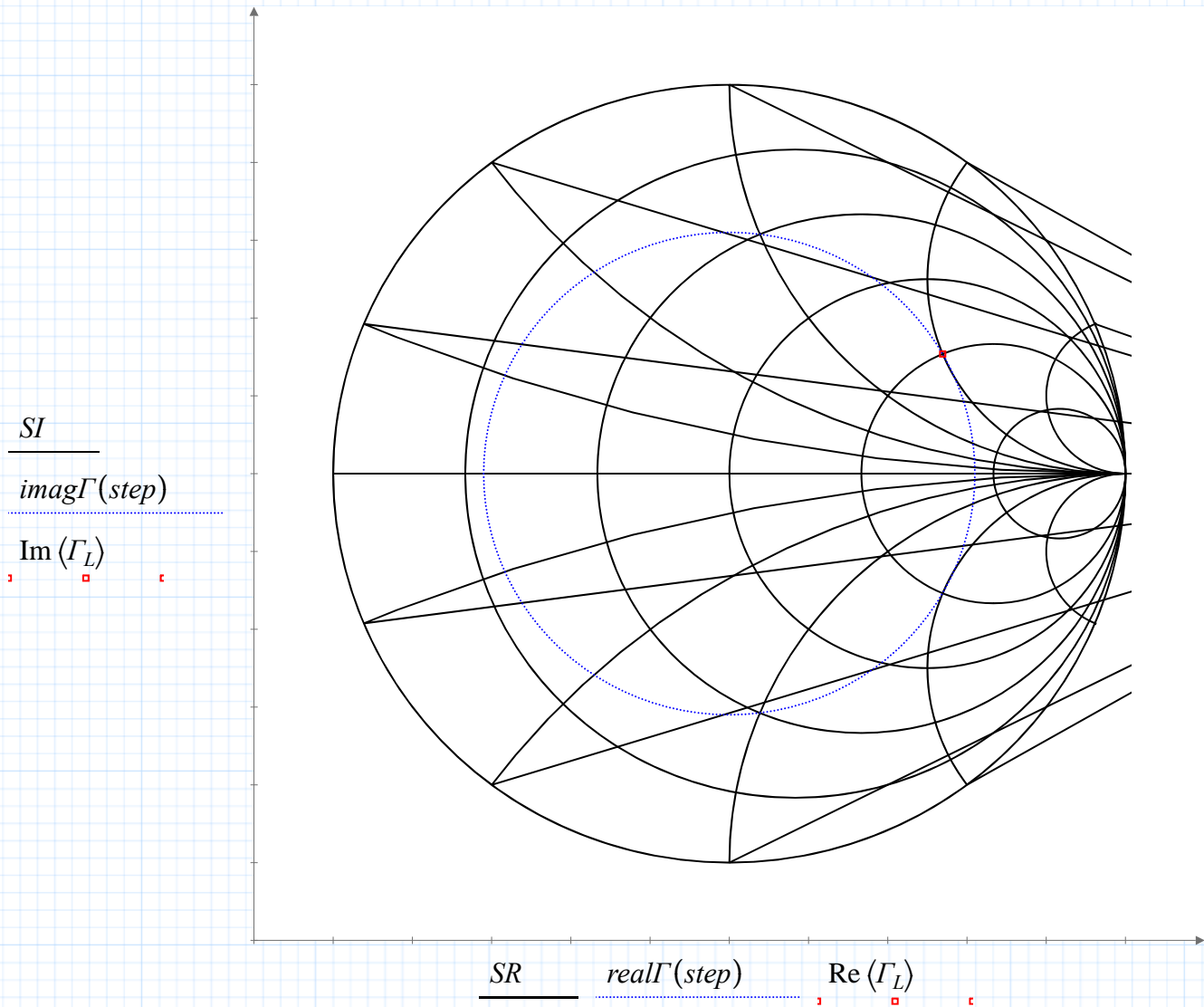


Fig. 3.3.1 Smith chart with circle of constant magnitude $|\Gamma|$.

Circles of constant resistance for $\mathbf{r} = 0, .2, .5, 1, 2,$ and 5 .

Contours of constant reactance for $\mathbf{x} = 0, .2, .5, 1, 2, 5, -2, -.5, -1, -2,$ and -5 .

Note that the constant $|\Gamma|$ circle intersects the $\mathbf{r} = 2$ and $\mathbf{x} = 2$ contours that represent the impedance \mathbf{Z}_n . The intersection is marked with a box.

Impedance Matching

To use the diagram for impedance matching, find the admittance at the point where the $|\Gamma|$ circle crosses the $r = 1$ contour, then match the imaginary part of the admittance at that point.

The chart is helpful in providing a guess value for the imaginary part x of the admittance. Mathcad can find x exactly using the **root** function, and determine how far (in wavelengths) the parallel shunt admittance should be placed from the load.

To illustrate this idea, use a guess value for x by looking at the Smith chart above, and calculate a matching admittance, starting with the normalized admittance

$$Y_n(x) := 1 + 1j \cdot x$$

The corresponding normalized impedance is $Z_n(x) := \frac{1}{Y_n(x)}$

and the reflection coefficient is $\Gamma(x) := \frac{Z_n(x) - 1}{Z_n(x) + 1}$

For the case of a load impedance $Z_n L$, as given above, the plot above indicates that the constant $|\Gamma|$ circle crosses the $r = 1$ contour between $x = 1$ and $x = 2$. So we'll guess

$$x := 1.5 \quad x := \text{root}(|\Gamma(x)| - |\Gamma_L|, x)$$

The actual value is

$$x = 1.581$$

The distance from the load where the shunt should be added is calculated from the phases of the reflection coefficients. The expression for this will be different, depending on whether our impedances are in adjacent or opposite quadrants of the Γ plane.

$$\text{angle}(x) := (\arg(\Gamma_L) - \arg(\Gamma(x)))$$

$$z(x) := \text{if} \left(\text{angle}(x) > 0, \frac{1}{4 \cdot \pi} \cdot \text{angle}(x), \frac{1}{4 \cdot \pi} \cdot (2 \cdot \pi + \text{angle}(x)) \right)$$

$$z(x) = 0.22 \quad \text{wavelengths}$$

The matching admittance at this position should be

$$\text{shunt} := -x \cdot 1j = -1.581i$$

Graphically, the impedance matching calculation would appear as follows on a Smith chart:

Graphing Parameters

Index for drawing radius $|\Gamma|$: $i := 0..1$ $m := 0..1$

Real and imaginary Γ starting and ending points:

$$\Gamma r_0 := 0 \quad \Gamma i_0 := 0$$

$$\Gamma r_1 := \text{Re}(\Gamma_L) \quad \Gamma i_1 := \text{Im}(\Gamma_L)$$

Real and imaginary Admittance reflected through the origin to show impedance:

$$Yr_0 := -\text{Re}(\Gamma(x)) \quad Yi_0 := -\text{Im}(\Gamma(x))$$

$$Yr_1 := \text{Re}(\Gamma(x)) \quad Yi_1 := \text{Im}(\Gamma(x))$$

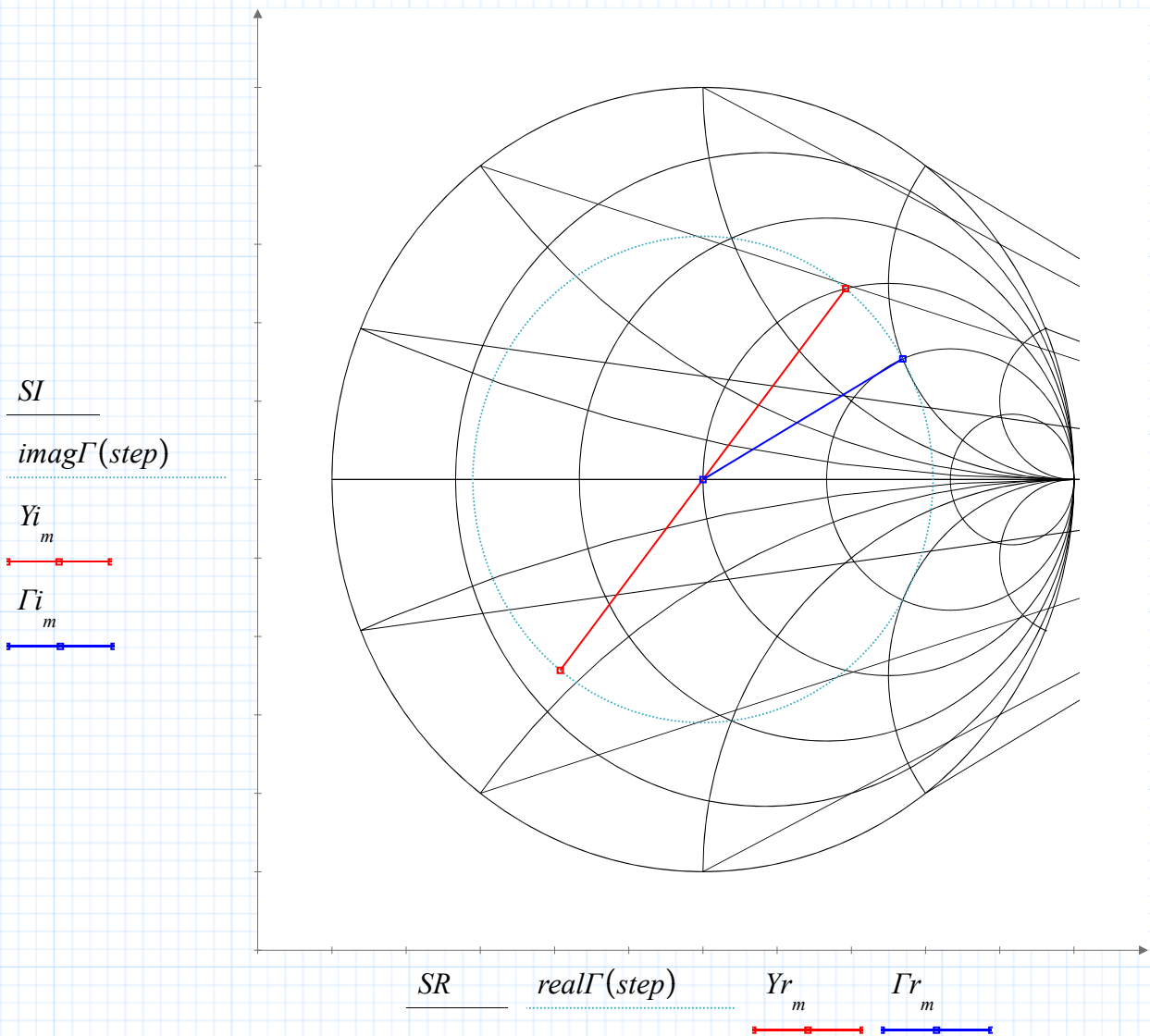


Fig. 3.3.2 Smith chart for impedance matching



It is possible to match the load from the other point where the constant $|\Gamma|$ circle crosses the $r = 1$ contour, i.e., between $x = -1$ and $x = -2$. It turns out that this second admittance has the same susceptance magnitude, but with the opposite sign. The shunt will also need to be placed at a different location with respect to the load in this case.

$$x := -1.5 \quad x := \mathbf{root}(|\Gamma(x)| - |\Gamma_L|, x)$$

The actual value is

$$x = -1.581$$

The distance from the load where this admittance occurs:

$$z(x) = 0.363 \quad \text{wavelengths}$$

The matching admittance at this position should be

$$\mathit{shunt} := -x \cdot 1j \quad \mathit{shunt} = 1.581i$$

See **3.2 Reflection Coefficient Calculations** for examples of using complex variables and Mathcad's **root** function to solve problems of this kind directly without the aid of a Smith chart.