

CHAPTER 5 FEEDBACK CIRCUITS AND STABILITY CRITERIA

5.2 Root-Locus Technique

The following examples illustrate some of the calculations associated with root-locus design. Mathcad's **root** function is used to find closed-loop poles for a given gain factor, to find the gain and phase margins, and to calculate the damping ratio and find the gain factor corresponding to a given damping ratio. You supply:

- **D(s)**, the denominator which describes your transfer function,.
- **K**, the gain.

For calculation of RLC system functions, please see **Section 4: Network Analysis Using an Admittance Matrix**.

Background

Root-locus analysis examines the movement of poles in the complex plane for a variable-gain closed-loop transfer function. A closed-loop system can be designed in this way by determining an optimal value for the gain, **K**. This is a useful technique, because it shows graphically the extremes of the circuit and helps to define regions of stability for gain in the range between

$$K = 0 \quad \text{and} \quad K = \infty$$

Root-locus calculations also allow us to find the gain and phase margins and the damping ratio in graphical or analytical fashion. The gain margin will be given by the point where the root locus crosses the imaginary axis in the complex plane. The phase margin is associated with the place where the root locus has a magnitude of one, and the damping ratio is equivalent to the cosine of the angle of the poles.

The root-locus technique is applicable only for known, rational transform functions. If the system function is not known analytically, use the Nyquist criterion.

See **Section 5.3: Polar Plots and Nyquist Plots** for information on this technique.

Mathcad Implementation

This document demonstrates the root-locus method of analysis of feedback systems. You will enter an open-loop transfer function in terms of a variable gain, \mathbf{K} , and possibly a guess value to be used by the **root** function.

Define the open-loop transfer function in terms of the gain factor \mathbf{K} and the denominator $\mathbf{D}(s)$:

$$K := 10 \quad D(s) := s \cdot (s + 3) \cdot (s + 4)$$

The design value is

$$H(K, s) := \frac{K}{D(s)}$$

Closed-Loop Poles

For a given \mathbf{K} , the poles of the closed-loop transfer function are roots of $\mathbf{H}(s)$. To find these poles, find roots of $\mathbf{D}(s) = -\mathbf{K}$:

$$s := 1j \quad (\text{a complex guess})$$

$$s1 := \text{root}(D(s) + K, s) = -1 + 1j$$

The conjugate is also a root.

$$s2 := \text{Re}(s1) - \text{Im}(s1) \cdot 1j = -1 - 1j$$

To find the real root, use a real guess:

$$s3 := -4 \quad s3 := \text{root}(D(s3) + K, s3) = -5$$

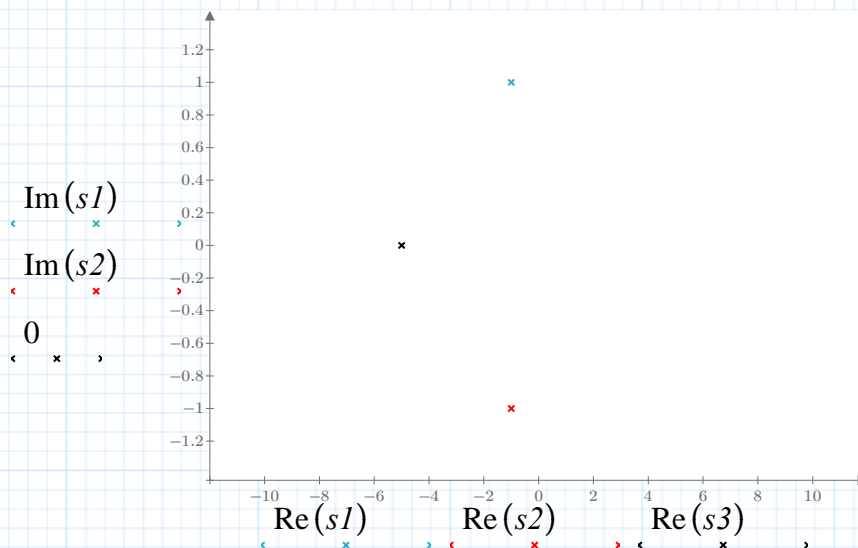


Fig. 5.2.1 Roots of the transfer function in the complex plane

Phase Margin

The phase margin is found by solving for the frequency at which the closed-loop gain satisfies $|\mathbf{H}| = 1$ (0 dB). Continue to define \mathbf{K} as the design value for the system, and enter a guess for the frequency at which $|\mathbf{H}| = 1$ (the unity gain crossover frequency). Enter the exact value for the frequency back into the expression for the gain, and find the phase.

$$f := 1 \quad (\text{a guess in Hz}) \quad f := \text{root}(|H(K, 1j \cdot f)| - 1, f) = 0.791$$

The phase margin in degrees is

$$\arg(H(K, 1j \cdot f)) \cdot \frac{180}{\pi} + 180 = 64.058$$

Damping Ratio and Gain

To find the damping ratio for a given gain, use the **root** function to find one of the complex conjugate roots of the closed-loop transfer function for the design value of \mathbf{K} . Then use **arg** and **cos** to calculate the ratio:

$$s := 1j \quad (\text{a guess})$$

$$\text{pole} := \text{root}(D(s) + K, s) = -1 + 1j$$

The damping ratio is

$$\cos(\pi - \arg(\text{pole})) = 0.707$$

Notice that as the root locus moves towards the imaginary axis, the damping ratio goes to zero. A value of \mathbf{K} which yields roots in the right-hand side of the complex plane results in a system with negative damping, i.e., an unstable system.



In some cases a damping ratio may be available from measured data, but not the transfer function. To find the gain \mathbf{K} which produces a given damping ratio, solve for the \mathbf{K} that produces the required arguments for the complex pole \mathbf{s} . Here it is convenient to use a Mathcad solve block. Enter the measured damping ratio \mathbf{dr} . Then enter a guess for the pole that corresponds to this damping ratio and for the gain that will yield this pole. The solve block will attempt to find the exact values of the pole and the gain together.

Guess Values	$s := -1 + 1j$	$K := 5$	$dr := .55$
Constraints	$\arg(s) = \pi - \arccos(dr)$	$D(s) + K = 0$	
Solver	$\begin{bmatrix} s \\ K \end{bmatrix} := \text{Find}(s, K)$		

$$\begin{bmatrix} s \\ K \end{bmatrix} = \begin{bmatrix} -0.711 + 1.079j \\ 11.02 - 2.592j \end{bmatrix}$$

Note: For a system of order higher than 2, this method will yield only the dominant pair of poles.