

CHAPTER 11 z-TRANSFORM APPLICATIONS

11.2 Linear Constant-Coefficient Difference Equations

This section demonstrates sequence generation and closed-form solutions to a second-order difference equation. The difference equation is solved by **z**-transforming the equation and solving for the transform of the solution sequence. You provide:

- **a** and **b**, the coefficients of the difference equation
- **s0** and **s1**, the initial values of the sequence
- **n**, the number of samples

Background

For difference equations with constant coefficients, you can obtain the solution in closed form by transforming the equation, and solving for the transform of the solution sequence. Then find the inverse transform of the solution by expanding in partial fractions, as shown in **Section 11.1: z-Transform and Inverse Transform**.

For example, the general second-order difference equation

$$a \cdot s_n + b \cdot s_{n+1} = s_{n+2}$$

can be expressed as a sum of sequences of the form

$$a \cdot r^n \quad \text{and} \quad b \cdot n \cdot r^n$$

Mathcad Implementation

Iterative Sequence Generation

Given constants **a**, **b** and initial values **s0**, **s1**, Mathcad can generate the beginning of the difference sequence by iteration. The range variable **m** starts at 2, so that **m - 1** and **m - 2** will be defined as index variables. Fig. 11.2.1 shows this calculated solution sequence, which has a sinusoidal appearance.

$$\begin{aligned} n &:= 2..100 & i &:= 0..100 \\ a &:= -1 & b &:= 1.9 \\ s_0 &:= 1 & s_1 &:= -1 & s_n &:= b \cdot s_{n-1} + a \cdot s_{n-2} \end{aligned}$$

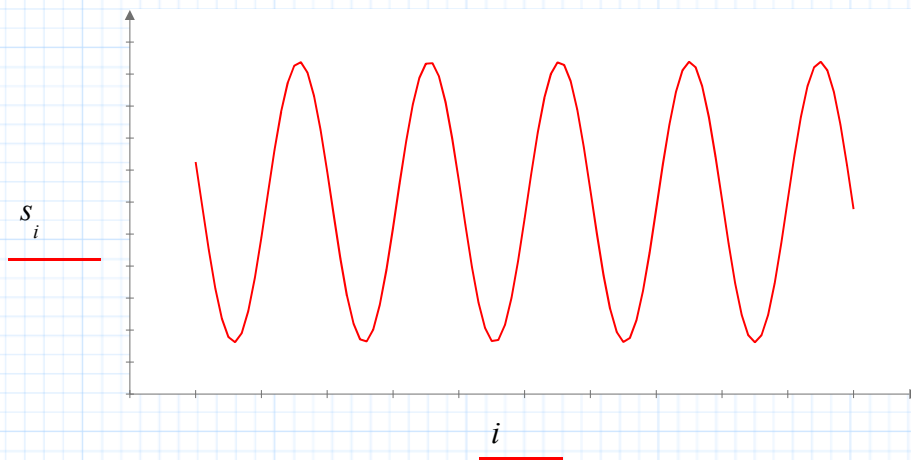


Fig. 11.2.1 Generated sequence

Example 11.2.2 z-Transform Closed-Form Solution

You can use **z**-transforms to obtain a solution in closed form. A closed form solution is often more useful, since it allows you to find the amplitude at large **n** without calculating all the intermediate values, and shows how the solution behaves over all time. First, transform the equation; shifting the sequence forward corresponds to subtracting the first term of the sequence from the transform and multiplying by **z**. Thus **Y**, the solution of the transform, satisfies the following equation:

$$a \cdot Y + b \cdot (Y - s_0) \cdot z = ((Y - s_0) \cdot z - s_1) \cdot z \quad b = 1.9$$

Using the symbolic processor, we can solve for **Y** **symbolically**. Copy the above equation. Place the cursor on the right side of the equation, and insert the symbolic keyword **solve**, then enter a **comma**, then enter **Y**. **Mathcad Ribbon: Symbolics** panel: Keyword **solve**. Assign the **solution for Y** to a **function of z**.

$$Y(z) := a \cdot Y + b \cdot (Y - s_0) \cdot z = ((Y - s_0) \cdot z - s_1) \cdot z \xrightarrow{\text{solve, Y}} \frac{1.9 \cdot z - z \cdot (z - 0.1)}{z^2 - 1.9 \cdot z + 1}$$

The numerical inverse transform is the sequence given by:

$$r := 2$$

$$x(n, X) := \frac{r^n}{2 \cdot \pi} \cdot \int_0^{2 \cdot \pi} X(r \cdot \exp(1j \cdot \theta)) \cdot \exp(n \cdot \theta \cdot 1j) d\theta$$

where \mathbf{r} is the radius of a circle in the complex plane that includes all the poles of \mathbf{F} .

The inverse transform of \mathbf{Y} gives the solution sequence. For example, the fifth term of the solution is

$$x(5, Y) = -3.379 + 6.432j \cdot 10^{-15} \quad s_5 = -3.379$$

To find the corresponding time sequence in closed form, expand \mathbf{Y} in partial fractions. The partial-fraction expansion of \mathbf{Y} depends on the roots of the denominator. If the discriminant

$$b := 1.9$$

$$D := b^2 + 4 \cdot a = -0.39$$

is nonzero, the denominator of \mathbf{Y} has two distinct roots $\mathbf{r1}$ and $\mathbf{r2}$, and \mathbf{Y} can be expanded in partial fractions as follows:

$$r1 := .5 \cdot (b + \mathbf{if}(b > 0, \sqrt{D}, -\sqrt{D})) = 0.95 + 0.312j$$

$$r2 := \frac{-a}{r1} = 0.95 - 0.312j$$

The numerator coefficients are given by

$$\begin{bmatrix} A \\ B \end{bmatrix} := \begin{bmatrix} 1 & 1 \\ r2 & r1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} s_0 \\ -s_1 + b \cdot s_0 \end{bmatrix} = \begin{bmatrix} 0.5 + 1.681j \\ 0.5 - 1.681j \end{bmatrix}$$

The closed-form solution sequence is

$$S(n) := A \cdot r1^n + B \cdot r2^n$$

For example,

$$S(5) = -3.379 \quad S(3000) = 1.65$$

The plotted solution matches the solution found above by direct iteration.

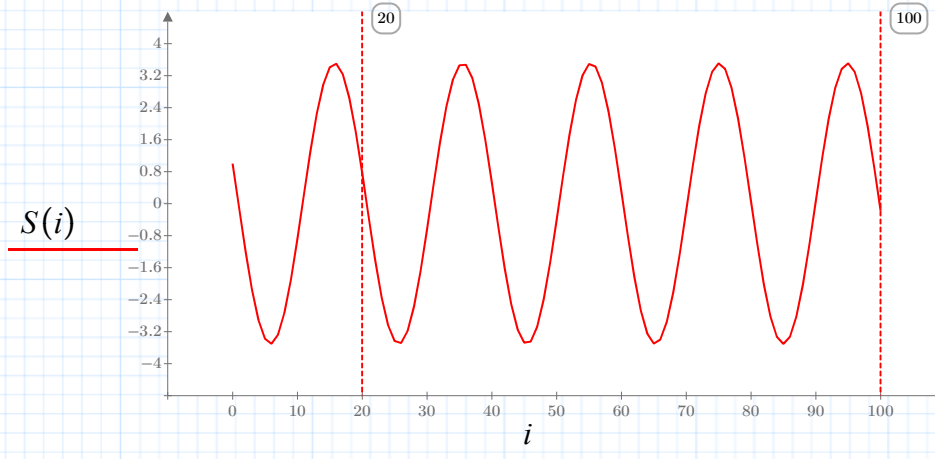


Fig. 11.2.2 Closed-form sequence

In real terms, the solution is

$$S(n) := 2 \cdot |A| \cdot \cos(n \cdot \arg(rI) + \arg(A))$$

so the solution sequence oscillates with amplitude

$$2 \cdot |A| = 3.508$$

and period

$$\frac{2 \cdot \pi}{\arg(rI)} = 19.786$$

To examine other choices for the coefficients and initial values, modify the definitions above and recalculate. The calculations will be correct as long as **Y** has two distinct poles, that is, as long as **D** is nonzero.

Example 11.2.3 Decaying Sequence

In this last example, the coefficients and initial values for the difference equation have been chosen so that the roots of the denominator of the **z**-transform are equal and nonzero (that is, **D** is zero):

$$a := -.81$$

$$b := 1.8$$

$$s_1 := 1$$

$$s_0 := .5$$

$$s_n := b \cdot s_{n-1} + a \cdot s_{n-2}$$

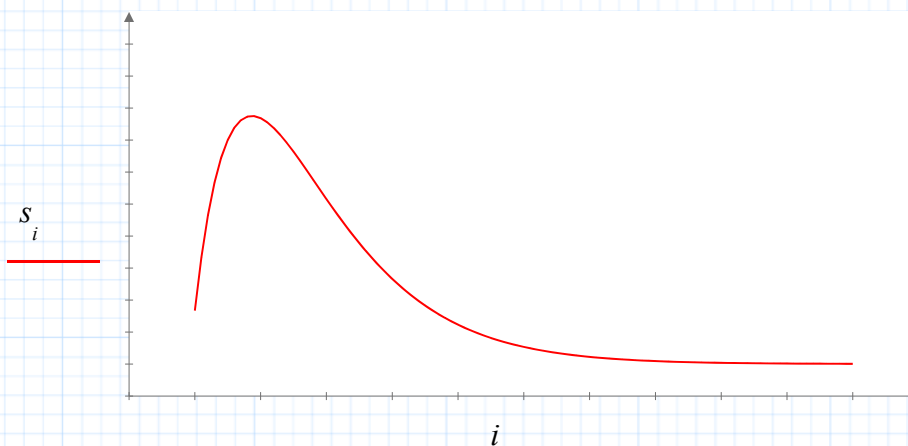


Fig. 11.2.3 Generated sequence

Y can be expanded in partial fractions as

$$Y(z) := \frac{A \cdot z}{z-r} + \frac{B \cdot z}{(z-r)^2}$$

The root is:

$$r := \frac{b}{2} = 0.9$$

The coefficients **A** and **B** are

$$A := s_0 = 0.5$$

$$B := s_1 - \frac{b \cdot s_0}{2} = 0.55$$

The discrete-time series is

$$S(n) := \left(A + \frac{B}{r} \cdot n \right) \cdot r^n$$

As a check:

$$s_5 = 2.1$$

$$S(5) = 2.1$$