CHAPTER 10 BUILDING THERMAL CONTROL

10.2 Transient Building Response Using Numerical Inversion of Laplace Domain Response

The method employed for numerical inversion of Laplace transforms has been developed and applied to circuits by Vlach and Singhal (1983). It was first applied to buildings by Athienitis et al (1990). This method provides an efficient way to calculate the time response of systems without determining poles or residues; that is, transfer functions which are transcendental, and time delays may be included in the system. This permits the use of transfer functions for the building envelope which model the thermal mass as distributed elements. For numerical inversion, s is set equal to z_i/t, where t is the time and z_i are constants determined accurately by Vlach and Singhal. Thus, the method can be easily incorporated in methods dealing with frequency response analysis of systems.

The theory behind the technique is summarized below followed by an example. By definition, the inverse Laplace transforms $v(t)$ of a transfer function $V(s)$ is given by

$$v(t) = \frac{1}{2\cdot\pi\cdot j}\cdot\int_{c-j\cdot\infty}^{c+j\cdot\infty} V(s)\cdot e^{st}\,\mathrm{d}s \qquad (1)$$

The exact inversion, however, is only possible if the poles of $V(s)$ are known. To avoid root finding, substitute *st* with *z*

$$v(t) = \frac{1}{2\cdot\pi\cdot j}\cdot\int_{c'-j\cdot\infty}^{c'+j\cdot\infty} V\left(\frac{z}{t}\right)\cdot e^{z}\,\mathrm{d}z \qquad (2)$$

and subsequently approximate eZ by a Pade rational function

$$R_{N,M} = \frac{P_N(z)}{Q_M(z)} = \frac{\sum_i \left(M+N-i!\cdot\begin{bmatrix}N\\ i\end{bmatrix}\cdot z^i\right)}{\sum_i \left((-1)^i\cdot M+N-i!\cdot\begin{bmatrix}M\\ i\end{bmatrix}\cdot z^i\right)} \qquad (3)$$

where $P_N(z)$ and $Q_M(z)$ are polynomials of orders N and M, respectively. The first M+N+1 terms of this function's Taylor expansion are equal to those of the Taylor expansion of eZ.

When (3) is inserted in (2), the approximation *v'(t)* to *v(t)* becomes

$$v'(t) = \frac{1}{2 \cdot \pi \cdot j \cdot t} \cdot \int_{c'-j\cdot\infty}^{c'+j\cdot\infty} V\left(\frac{z}{t}\right) \cdot R_{M,N} \cdot (z)\, dz \qquad (4)$$

Using residue calculus, it can be shown that

$$R_{N,M} = \sum_i \frac{K_i}{z - z_i} \qquad i=1...M \text{ and } N<M \qquad (5)$$

where K_i are residues and z_i are poles. Closing the path of integration around the poles of $R_{N,M}$ in the right half plane, it can then be shown that

$$v'(t) = -\frac{1}{t} \cdot \sum_i \left(K_i \cdot V\left(\frac{z_i}{t}\right)\right) \qquad i=1..M \qquad (6)$$

which is the basic inversion formula. Real-time functions can now be evaluated by using only the poles z_i in the upper half plane, thus reducing the computations to one half. If *M* is even and the bar denotes complex conjugate, then

$$v'(t) = -\frac{1}{t} \cdot \sum_i \left(K_i \cdot V\left(\frac{z_i}{t}\right) \cdot -\left(\left(\frac{1}{-t}\right) \cdot \sum_i \left(\overline{K_i} \cdot V \cdot \left(\frac{\overline{z_i}}{t}\right)\right)\right)\right)$$

i=1. . . M' (7)

$$v'(t) = \frac{1}{-t} \cdot \sum_i \left(Re \cdot \left(K'_i \cdot V\left(\frac{z_i}{t}\right)\right)\right) \qquad i=1...M'$$

where $M' = M/2$ and $K'_i = 2 \cdot K_i$, and z_i for different *M* and *N* have been calculated with a high precision and tabulated by Vlach and Singhal (1983).

Poles z_i and residues K'_i for inversion are given below for $M = 10$ and $N = 8$.

$$j := \sqrt{-1} \qquad M' := 5 \qquad i := 1, 2 .. M'$$

$$z_1 := 11.83009373916819 + 1.593753005885813j$$

$$z_2 := 11.22085377939519 + 4.792964167565670j$$

$$z_3 := 9.933383722175002 + 8.033106334266296j$$

$$z_4 := 7.781146264464616 + 11.36889164904993j$$

$$z_5 := 4.234522494797000 + 14.95704378128156j$$

$$K'_1 := 16286.62368050479 - 139074.7115516051j$$

$$K'_2 := -28178.11171305163 + 74357.58237274176j$$

$$K'_3 := 14629.74025233142 - 19181.80818501836j$$

$$K'_4 := -2870.418161032078 + 1674.109484084304j$$

$$K'_5 := 132.1659412474876 + 17.47674798877164j$$

Example: We will determine the response of room temperature to a step change in outside temperature for the simple building model considered in **Section 10.1**. The only building parameter that we need is the time constant. For most residential buildings, the overall time constant varies between 12 hours and 48 hours depending on thermal capacity.

$n := 1, 2 .. 36$ $\qquad \Delta t := 1 \cdot hr$

$t_n := n \cdot \Delta t$ $\qquad$ time array

$\tau := 12 \cdot hr$ $\qquad$ time constant

$$s_{i,n} := \frac{z_i}{t_n}$$

The building transfer function is given by

$$G(s) := \frac{1}{\tau \cdot s + 1}$$

The Laplace transfer function of a step input is 1/*s*:

$$T_o(s) := \frac{1}{s}$$

Therefore, the Laplace domain response is

$$T_R(s) := G(s) \cdot T_o(s)$$

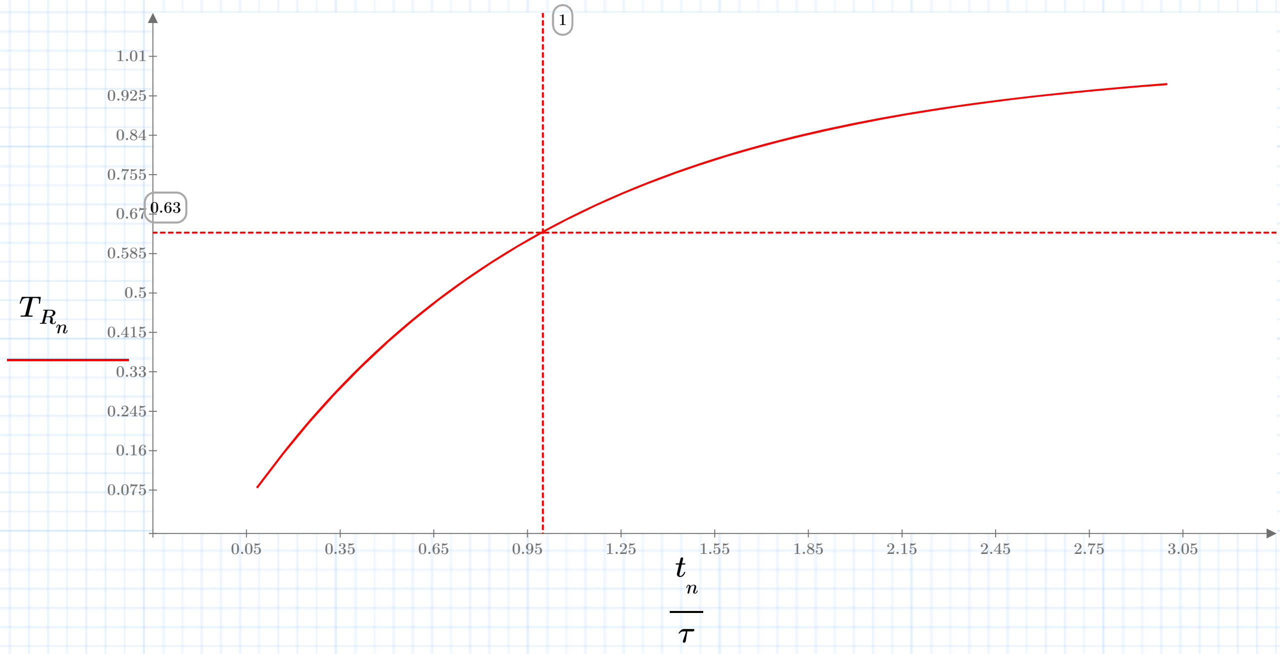
The Laplace transfer function to be inverted is thus

$$V_{i,n} := T_R\left(s_{i,n}\right)$$

You may replace this function with a different one for inversion if you wish.

The time domain response is

$$T_{R_n} := \frac{1}{-t_n} \cdot \left(\sum_i \mathrm{Re}\left(K'_i \cdot V_{i,n}\right)\right)$$



The graph shows the room temperature change due to outside temperature change of 1 degC. Since this is a first order system, it is convenient to show this change as a function of the number of time constants. As we found in **Section 2.1**, 63% of the change occurs after one time constant.

Therefore, since n=12 corresponds to one time constant:

$$T_{R_{12}} = 0.632$$

References

Athienitis, A. K., M. Stylianou and J. Shou. 1990. "A Methodology for Building Thermal Dynamics Studies and Control Applications." *ASHRAE Transactions.* Vol. 96, Pt. 2, pp. 839-48.

Stephanopoulos, G. 1984. *Chemical Process Control, an Introduction to Theory and Practice.* Englewood Cliffs, N.J.: Prentice-Hall.

Vlach, J., and K. Singhal. 1983. *Computer Methods for Circuit Analysis and Design.* Van Nostrand Reinhold Co.