2.3 3-D Resonators

This document calculates the resonant frequency and Q-factor for a 3-D cavity resonator. You provide the following values:

- **a**, **b** and **d**, the resonator dimensions
- **m**, **n** and **p**, the mode numbers

Background

Refer to the following diagram for dimensions and orientation:

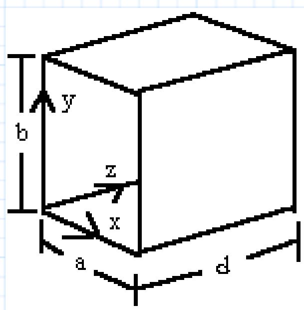


Fig. 2.3.1 3-D resonator

Resonant Frequency

A three-dimensional waveguide or resonator will absorb energy introduced at one of the **resonant frequencies** of the cavity. These frequencies are a function of the cavity dimensions and the oscillating mode (see equation below). The fields in the resonator form standing wave solutions (forward propagating and reflected waves interfering with each other). The resonant cavity, unlike 1- and 2-D waveguides, cannot sustain waves at frequencies other than the resonant (cutoff) frequencies.

$$f_r(m,n,p) = \frac{1}{2\cdot\sqrt{\mu 0_0 \cdot \varepsilon 0_0}} \cdot \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{d}\right)^2}$$

Valid Modes

The electric field in a 3-D waveguide will have terms in x, y and z. In the case of TMmnp modes, any one of the three indices **m**, **n** or **p** may be zero, but not more than one simultaneously, or there is no field. For TEmnp modes, **n** may be zero, but **m** and **p** must always be one or greater for a field solution to exist.

Q-Factor

The efficiency with which the cavity will retain energy can be expressed as the Q-factor. This is the ratio of energy absorbed to energy dissipated; it is most easily derived for the TM110 mode, which is the dominant cavity mode.

Mathcad Implementation

The equations that follow define resonance characteristics for a three-dimensional waveguide:

$\varepsilon_0 := 8.854 \cdot 10^{-12} \, \frac{F}{m}$ $\mu_0 := 4 \cdot \pi \cdot 10^{-7} \, \frac{H}{m}$

First, specify cavity dimensions along the x, y and z axes, and mode numbers:

$a := 3 \, cm$ $b := 1.5 \, cm$ $d := 3 \, cm$

$m := 1$ $n := 1$ $p := 0$

The equation for the resonant frequency (derived from the dispersion relation and the guidance conditions) is:

$$f_r(m, n, p) := \frac{1}{2 \cdot \sqrt{\mu_0 \cdot \varepsilon_0}} \cdot \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{d}\right)^2}$$

If there is no variation in a given direction, the corresponding dimension does not affect the calculation. For example, in the calculation shown below, the mode is TM110 and the z dimension is no longer of consequence.

$f_r(1, 1, 0) = 11.173 \, GHz$

Q-Factor

It is possible to define the Quality Factor, **Q**, for the cavity in terms of the cavity dimensions. The expression below is derived for an air-filled cavity resonating in the TEM110 mode only, which is the dominant cavity mode.

Enter the material-dependent conductivity (shown for copper):

$$\sigma := 6.7 \cdot 10^{7}\ \Omega^{-1} \cdot m^{-1}$$

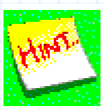
$$\omega_{r110}(a,b) := \frac{\pi}{\sqrt{\mu_0 \cdot \varepsilon_0}} \cdot \sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2}$$

$$\omega_{r110}(a,b) = 70.2\ GHz$$

$$Q_{110}(a,b) := \sqrt{\frac{2 \cdot \sigma}{\omega_{r110}(a,b) \cdot \varepsilon_0}} \cdot \left(\frac{\pi \cdot d \cdot \left(a^2 + b^2\right)^{\frac{3}{2}}}{2 \cdot \left(a \cdot b \cdot \left(a^2 + b^2\right) + 2 \cdot d \cdot \left(a^3 + b^3\right)\right)} \right)$$

For the dimensions above, and assuming copper walls,

$$Q_{110}(a,b) = 1.121 \cdot 10^{4}$$



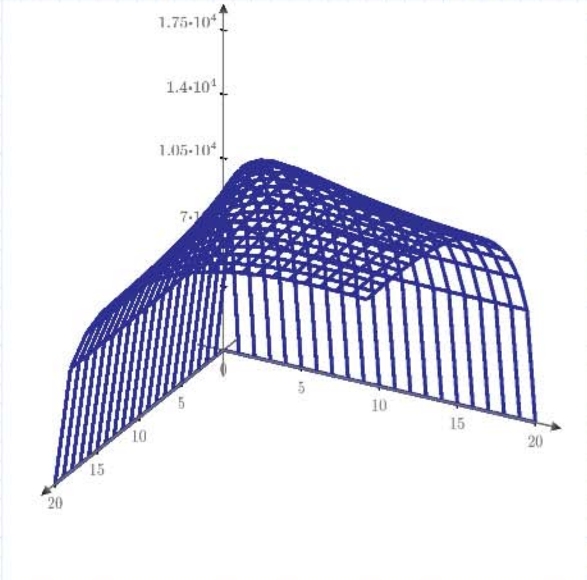
You may want to maximize the expression for **Q** to choose cavity dimensions for the resonator. You can do this graphically or numerically in Mathcad. You can also define the various lengths in terms of each other, to examine particular cases of resonance.

Number of points to calculate: $i := 0 .. 20$ $\quad j := 0 .. 20$

Define ranges for lengths: $a_i := 0.000001\ cm + .5 \cdot i \cdot cm$

$b_j := 0.000001\ cm + .5 \cdot j \cdot cm$

Define matrix to maximize and plot: $M_{i,j} := Q_{110}\left(a_i, b_j\right)$



M

To find the exact location of the maximum for the function **Q110**, take the partial derivatives in the variables **a** and **b**, set them to zero, and use a solve block to solve for the dimensions of the resonator.

Guesses of **a** and **b** are made to start the solve block.

$a := 5 \cdot cm \qquad b := 5 \cdot cm$

$$\frac{d}{da} Q_{110}(a, b) = 0 \cdot cm^{-1} \qquad \frac{d}{db} Q_{110}(a, b) = 0 \cdot cm^{-1}$$

Note: A derivative of **Q** with respect to distance has units of 1/length.

The dimensions which maximize **Q** are:

$$\begin{bmatrix} a \\ b \end{bmatrix} := \mathbf{Minerr}(a, b)$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix} cm$$

The maximum of **Q** is

$$Q_{110}(a, b) = 1.45 \cdot 10^4$$