

## CHAPTER 5 FEEDBACK CIRCUITS AND STABILITY CRITERIA

### 5.3 Polar Plots and Nyquist Plots

This document constructs polar plots and Nyquist stability plots for system functions, and examines how the plots can be used to design stable systems. The polar plot example includes the method for drawing a contour of constant closed-loop magnitude. The Nyquist plot example shows a transfer function with one pole on the imaginary axis. You provide:

- $G(s)$ , the transfer function which describes your system,
- $f$ , the frequency range to plot.

#### Background

Nyquist plots show the path of the open-loop transfer function in the complex plane as  $\omega$  varies. The Nyquist stability criterion states the following: two conditions of the Nyquist path must be met for a system to be stable. First, there must be no clockwise encirclement of the point  $(-1, 0)$  in the complex plane for a system to have closed-loop stability. Second, the number of negative, or counterclockwise encirclements, of the point  $(-1, 0)$  is equal to or greater than the number of open-loop poles in the right-half complex plane. This test is useful in the case where the transfer function is not known explicitly.

This technique can also be used to determine the maximum value of gain for a system based on the location of the real axis crossings; physically, the principles are similar to those discussed in **Section 5.2 Root-Locus Technique**.

#### Mathcad Implementation

##### Polar Stability Plot

This example makes a polar plot of a transfer function and draws one contour of constant closed-loop magnitude. To draw the plot, enter a definition for the transfer function  $G(s)$ :

$$G(s) := \frac{45000}{s \cdot (s+2) \cdot (s+30)}$$

The frequency range defined by the next two equations provides a logarithmic frequency scale running from 1 to 100. You can change this range by editing the definitions for  $m$  and  $\omega_m$ :

$$m := 0 .. 100 \quad \omega_m := 10^{.02 \cdot m}$$

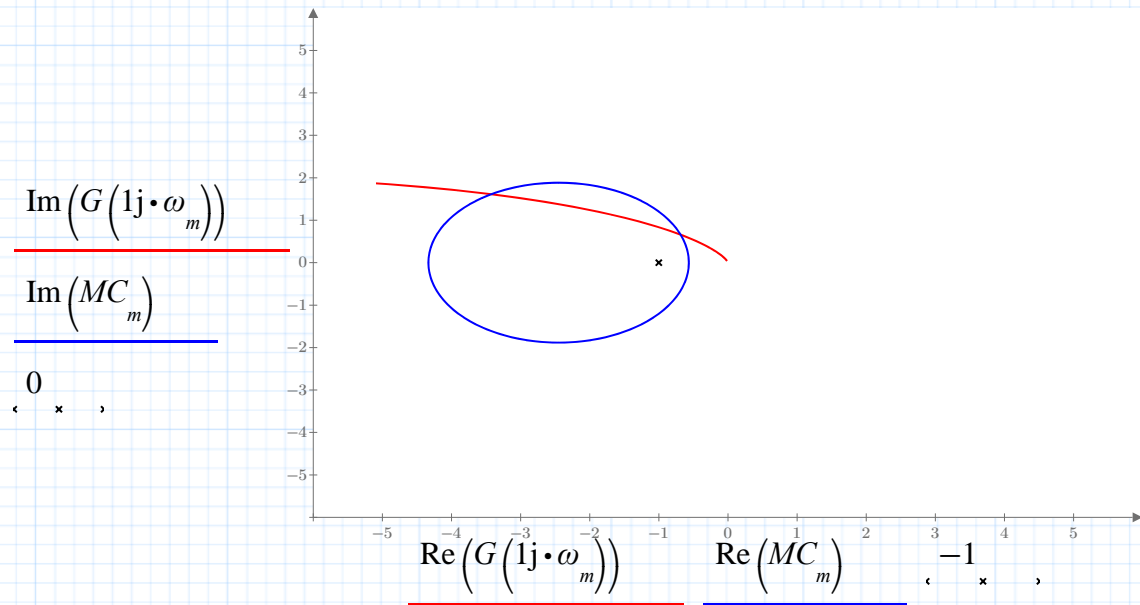
Now enter a value for  $M$  to define the closed-loop magnitude contour that will be plotted.

$$M := 1.3$$

Calculate the points on the M-circle:

$$MC_m := \left( \frac{-M^2}{M^2 - 1} + j \frac{M}{M^2 - 1} \cdot \exp(2 \cdot \pi \cdot 1j \cdot .01 \cdot m) \right)$$

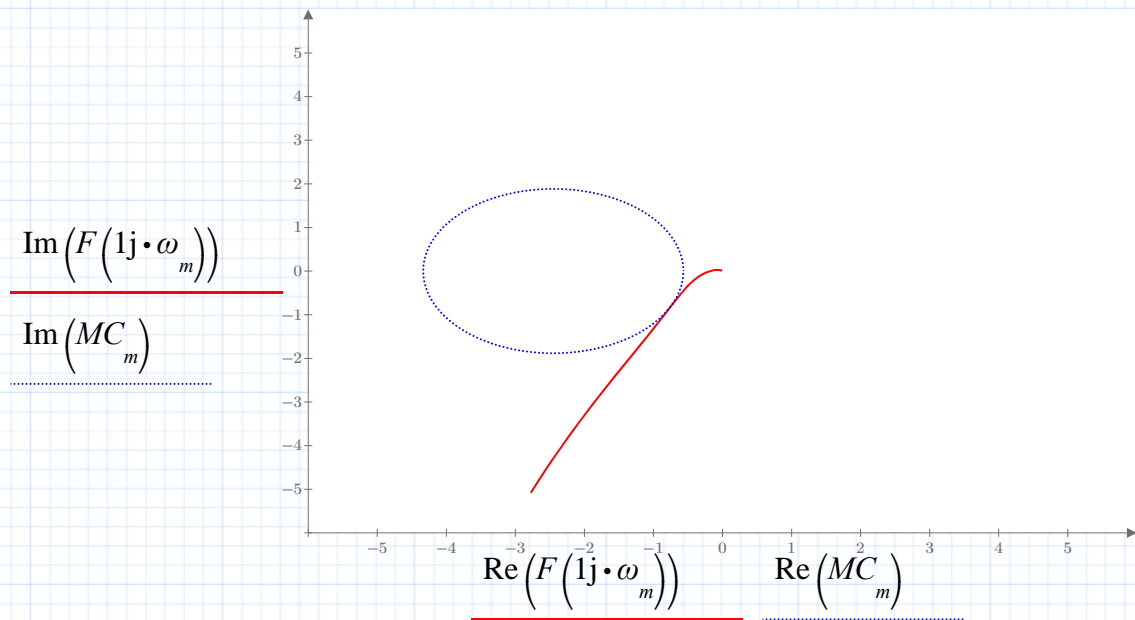
The first plot shows  $\mathbf{G}$ , the contour of constant closed-loop magnitude,  $\mathbf{M}$ , and an  $x$  at the point  $(-1, 0)$  on the real axis.



**Fig. 5.3.1 Polar plot with contour of constant closed-loop magnitude**

This plot indicates closed-loop instability. The next plot shows a compensated system which is stabilized with additional elements (described as a multiplicative transfer function). This additional function causes the open-loop system function to be tangential to the contour of constant closed-loop magnitude.

$$F(s) := .03 \cdot \left( \frac{1 + .25 \cdot s}{1 + .025 \cdot s} \right) \cdot G(s)$$

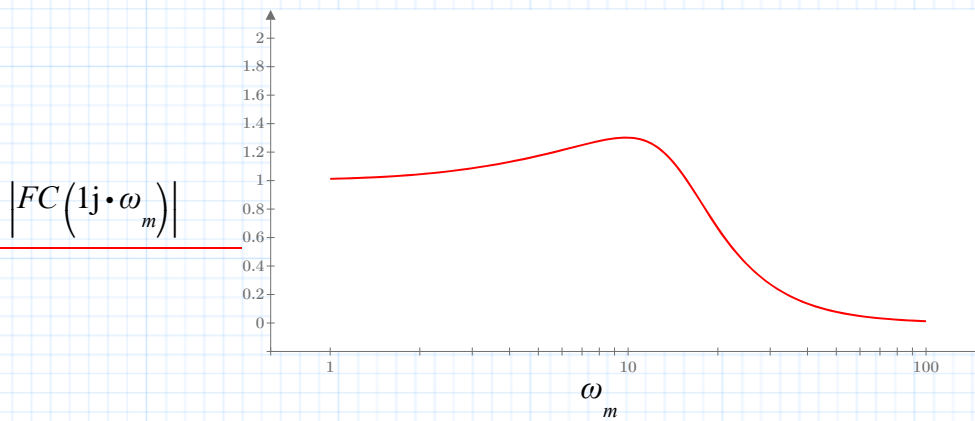


**Fig. 5.3.2 Polar plot for compensated system**

The plot indicates that the compensated system is stable. Notice that reducing the gain of the original function  $G(s)$  would have a similar effect.

Since the system plot is approximately tangent to the  $M$ -circle for  $M = 1.3$ , the magnitude of the resonance peak will be about 1.3. To find the location of the resonance peak, plot the closed loop gain, using a log scale on the frequency axis:

$$FC(s) := \frac{F(s)}{1 + F(s)}$$



**Fig. 5.3.3 Magnitude vs. frequency plot**

Make a guess for the peak location based on the graph, then solve:

$$x := 10 \quad \omega\pi_\pi := \text{root} \left( \frac{d}{dx} |FC(1j \cdot x)|, x \right)$$

The peak is at

$$\omega\pi_\pi = 9.844$$

The peak height is

$$|FC(1j \cdot \omega\pi_\pi)| = 1.302$$

## Nyquist Stability Plot

This example draws a Nyquist plot for a transfer function with a single simple pole on the imaginary axis and all other poles in the left half-plane. Define the transfer function  $G(s)$  and enter the imaginary part of the imaginary pole as the value for  $c$ .

$$G(s) := \frac{45000}{s \cdot (s + 30) \cdot (s + 2)}$$

The pole on the  $j$ -axis is at  $cj$ . The Nyquist path must go around this point. Enter  $c$  for the transfer function above:

$$c := 0$$

Mathcad will assemble the Nyquist path from segments and semicircular arcs. (The values given for the radius  $R$  of the large arc and the radius  $r$  of the small arc will work well for most functions  $G(s)$ , but you may need to adjust one or both of these values to obtain a good plot.) Each segment is plotted with 50 points (given by the range variable  $n$ ), and these pieces of the path are assembled into a single path which is stored in the vector  $P$ .

First parametrize the Nyquist path for a transfer function with a single pole on the  $j$ -axis at  $cj$  and the remaining poles in the left half plane:

$$n := 0 .. 100 \quad t := .01 \cdot n \quad m := 0 .. 400$$

Segment on  $j$ -axis:

$$L(t, a, b, c) := c \cdot 1j + a \cdot 1j + t \cdot (b - a) \cdot 1j$$

Semicircular arc with center at  $cj$  and radius  $r$ ; clockwise if  $d = 1$ , counterclockwise if  $d = -1$ :

$$A(t, c, r, d) := c \cdot 1j + r \cdot \exp(1j \cdot (.5 - t) \cdot d \cdot \pi)$$

$$\text{radius of outer arc: } R := 100 \quad \text{radius of inner arc: } r := 1$$

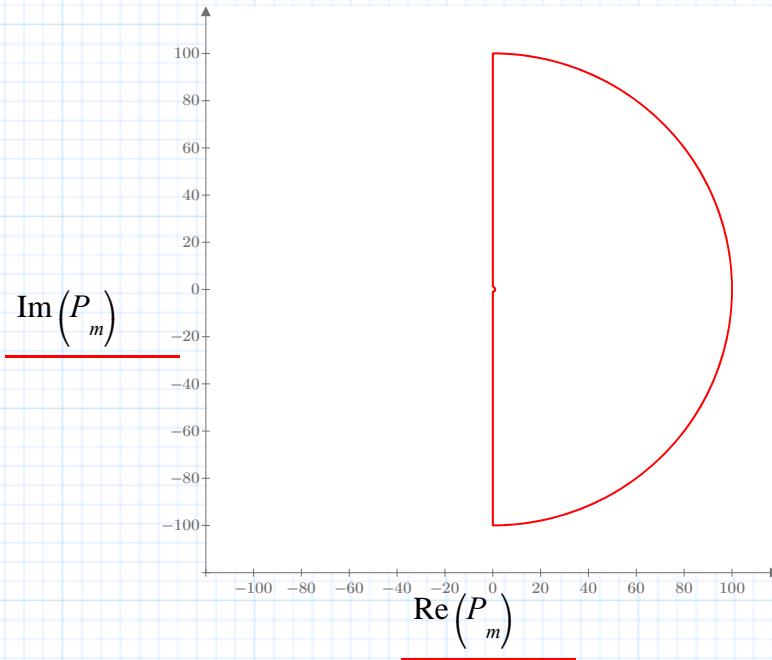
The four pieces of the Nyquist path are stored in the array  $P$ :

$$P_n := L(t_n, r, R, c)$$

$$P_{n+100} := A(t_n, c, R, 1)$$

$$P_{n+200} := L(t_n, -R, -r, c)$$

$$P_{n+300} := A(t_n, c, r, -1)$$



**Fig. 5.3.4 Nyquist path in the complex plane**

For the Nyquist plot to give accurate information, the original Nyquist path in the  $s$ -plane must be large enough to include the poles we're interested in. Let's find and plot the two poles of the closed-loop transfer function corresponding to the open-loop function  $G$  defined above (we'll let the symbolic processor do the solving):

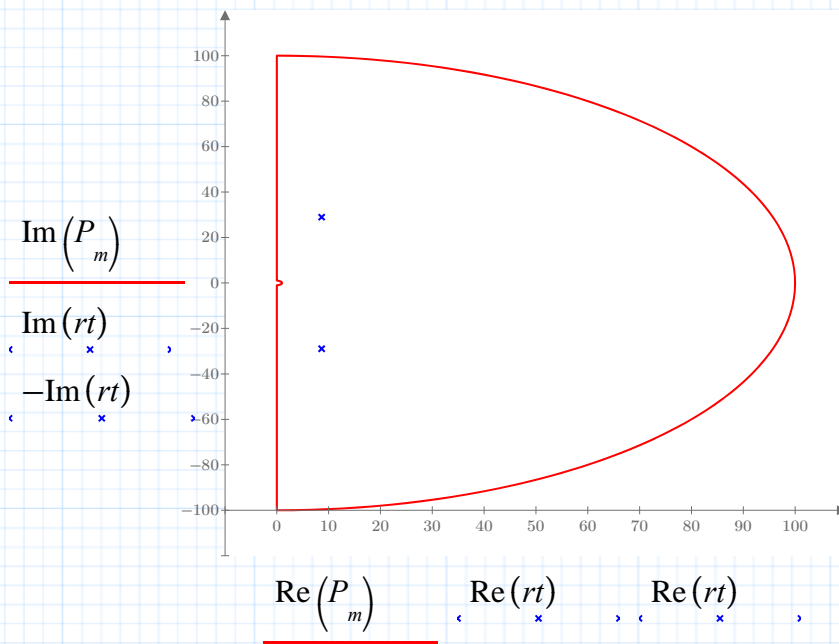
$$s \cdot (s + 30) \cdot (s + 2) + 45000.0$$

$$\begin{bmatrix} -49.298694746178114858 \\ 8.649347373089057428 + 28.948089087516422929 \cdot 1j \\ 8.649347373089057428 - 28.948089087516422929 \cdot 1j \end{bmatrix}$$

The complex poles of the closed-loop transfer function are

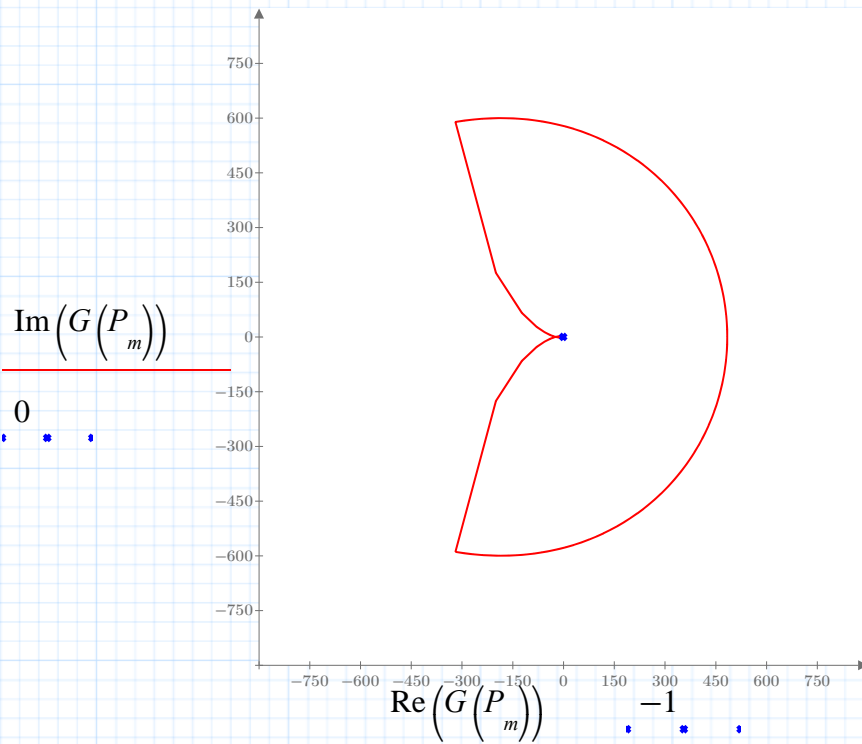
$$rt := 8.649347373089057428 + 28.948089087516422929 \cdot 1j$$

and its conjugate.



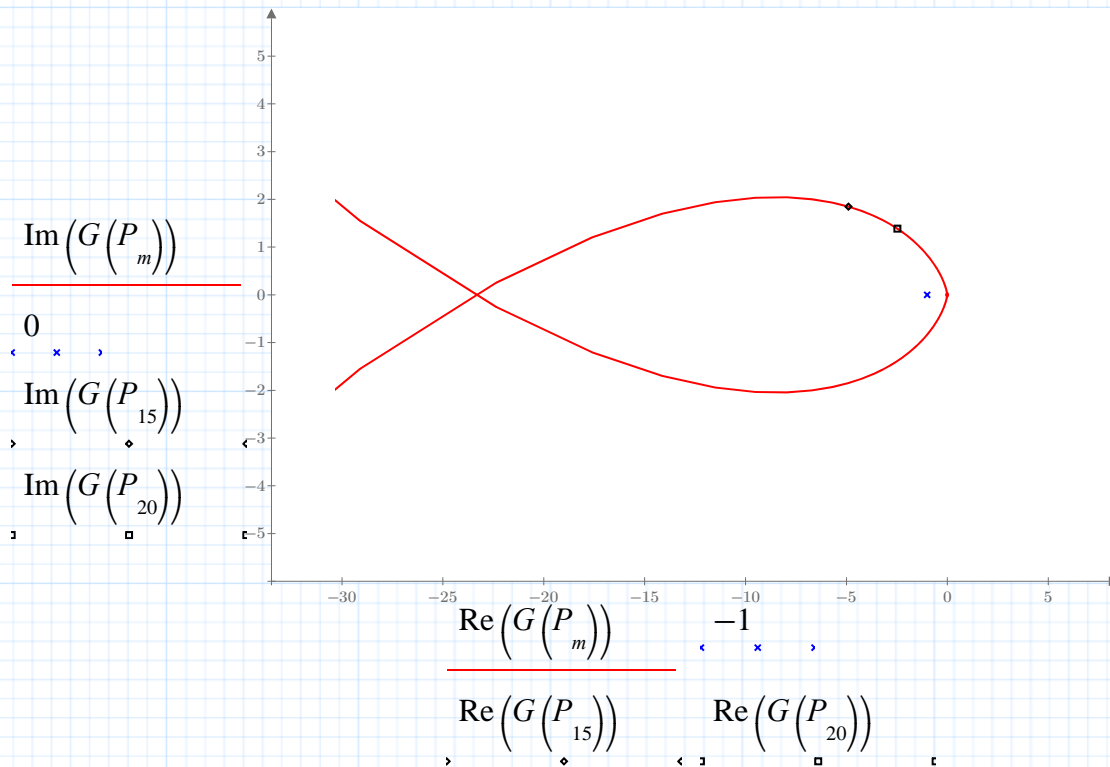
**Fig. 5.3.5 Nyquist path and complex poles of closed-loop transfer function**

So with  $R = 100$  these poles are well inside the  $s$ -plane contour. Now plot the Nyquist plot, which is the contour of the Nyquist path,  $P_m$ , in the  $G(s)$  plane.



**Fig. 5.3.6 Nyquist stability plot**

We'll magnify the region around the origin from **Fig. 5.3.6**. The diamond and box markers indicate the direction of the path: it moves from the diamond toward the box. The upper piece of this loop is the path we plotted in **Fig. 5.3.1**. The x shows the point  $(-1, 0)$ .



Note that this contour circles the  $(-1, 0)$  point twice, corresponding to the two poles in the right half-plane. If you change  $\mathbf{R}$  to 10, the poles now lie outside the Nyquist path, and you'll see that the image path makes no counter-clockwise circuits of the  $(-1, 0)$  point. Instead it makes a backwards loop to avoid the  $(-1, 0)$  point entirely. (The direction markers are out of sight, but the direction is in from the lower right, counterclockwise around  $-1$ , and back out to the left.)

Another interesting experiment is to change  $\mathbf{G}$  to the compensated function  $\mathbf{F}$  in the axis arguments for **Fig. 5.3.4** (go back to  $\mathbf{R} = 100$ ). Again the Nyquist plot avoids the  $(-1, 0)$  point, because the poles have moved into the left half-plane. Using the symbolic processor, find the roots of:

$$1350 \cdot (1 + .25 \cdot s) + (1 + .025 \cdot s) \cdot s \cdot (s + 30) \cdot (s + 2)$$

The four roots are

$$\begin{bmatrix} -7.7184214651757497 + 12.321085839499558445 \cdot 1j \\ -7.7184214651757497 - 12.321085839499558445 \cdot 1j \\ -4.949466715942553026 \\ -51.613690353705947574 \end{bmatrix}$$



To draw a Nyquist plot for a transfer function with two or more poles on the  $\mathbf{j}$  axis, add the necessary pieces to the Nyquist path: the path should make a semicircular arc in the right half-plane around each  $\mathbf{j}$ -axis pole. You will need to define additional coefficients  $\mathbf{c}'$ ,  $\mathbf{c}''$ , and so on, giving the imaginary parts of the imaginary poles. For each additional root, add one more segment and one more arc to  $\mathbf{P}$  and add 100 to the upper limit of the range variable  $\mathbf{m}$ .