11.1 z-Transform and Inverse Transform

This section manipulates the z-transform and its inverse both numerically and algebraically. You provide:

- **s**, the sequence to transform
- **H(z)**, a system function

Background

The unilateral **z**-transform describes a mapping between a linear time-invariant sequence in discrete time, and a circle in the complex plane. The mapping is given by the function:

$$X(z) = \sum_n \left(x(n) \cdot z^{-n} \right)$$

which is really just a generalization of the discrete-time Fourier Transform. See **Section 8: Convolution and Deconvolution** for more information on Fourier Transforms. The **z**-transform effectively wraps the linear frequency axis around a circle in the complex **z**-plane with a radius **r.**

Inverse Transformation by Contour Integral

For any **z**-transform, the inverse transform is given by

$$x(n,F) = \frac{r^n}{2 \cdot \pi} \cdot \int_0^{2 \cdot \pi} F(r \cdot exp(1j \cdot \theta)) \cdot exp(n \cdot \theta \cdot 1j) \, d\theta$$

where **r** is the radius of a circle in the complex plane that includes all the poles of **F**; that is, a circle that lies in the region of convergence.

Inverse Transformation by Partial Fractions

Since **z**-transformations are linear, any sequence of summed components can be transformed by transforming each component. Conversely, a closed-form formula for the inverse of a complicated transform can be determined by finding the poles of the function, and expressing it as partial fractions, i.e.:

$$H(z) = \frac{A}{z - root1} + \frac{B}{z - root2} + \frac{C}{z - root3} + \text{“...”}$$

Each fraction can then be inverted and the resulting sequence of terms summed to yield the inverse transform of **H(z)**. Use the following transform pair to construct the time expression:

$$X(z) = \frac{\alpha}{z - \beta} \qquad \longleftrightarrow \qquad x_0 = 0 \qquad x_n = \alpha \cdot \beta^{n-1}$$

Mathcad Implementation

To begin, define a sequence to be transformed:

$$n := 0 .. 15 \qquad s_n := 2^{-n}$$

Note: The signal can be read in from a data file.

The **z**-transform of the sequence **s** is defined by:

$$X(z) := \sum_n \frac{s_n}{z^n}$$

The inverse transform is the sequence given by

$$r := 2 \qquad x(n, X) := \frac{r^n}{2 \cdot \pi} \cdot \int_0^{2 \cdot \pi} X(r \cdot \exp(1j \cdot \theta)) \cdot \exp(n \cdot \theta \cdot 1j) \, d\theta$$

where **r** is the radius of a circle in the complex plane that includes all the poles of **X**.

The inverse transform recovers the original sequence from its transform. For example, the third term of the sequence defined above is

$$s_3 = 0.125 \qquad\qquad x(3, X) = 0.125 + 1.020495i \cdot 10^{-16}$$

When the **z**-transform of a signal is explicitly given, you can use this inversion formula to look at the first few terms of the original sequence. Note that the result might contain a small imaginary part that results from truncation error in the integral.

Step Input Response by Integration

In the **z**-plane, the response of a system to an input signal is given by the product of the signal transform and the system transfer function. Inversion of this product gives the response in discrete time. This procedure is easier than convolution in time for determining a system response. If a discrete-time system has the transfer function:

$$H(z) := \frac{z + .5}{z^2 - z + .5}$$

then its response to a step input has transform:

$$Y(z) := \frac{z + .5}{z^2 - z + .5} \cdot \frac{z}{z - 1}$$

The response in discrete time is the sequence defined by

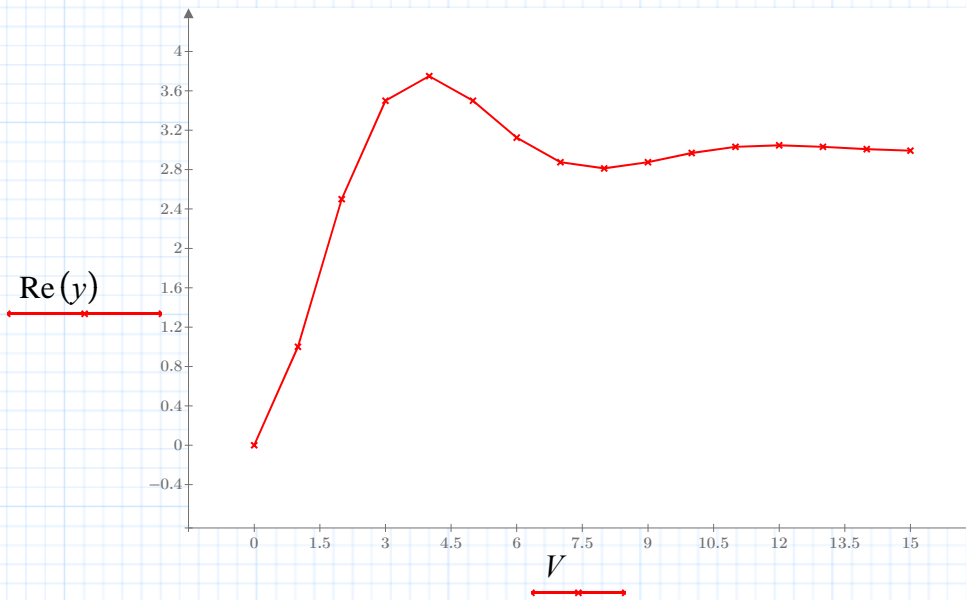$$y_n := x(n, Y) \qquad V_n := n$$

$\mathrm{Re}\,(y)$

V

Fig. 11.1.1 System response to a unit step input

Step Input Response by Partial Fraction Expansion

This time, find the inverse transform of a system response **Y(z)** algebraically by using the roots of the denominator. Write **Y** as a sum of fractions and then apply a standard inverse transform to each fraction.

For the **Y(z)** defined above, one root is:

$r1 := 1$

The others can be found with Mathcad's **root** function (these roots will be complex, so use a complex guess):

$z := 1 \cdot 1\mathrm{j}$

$r2 := \mathbf{root}\left(z^2 - z + .5, z\right)$ $\qquad\qquad r2 = 0.5 + 0.5\mathrm{j}$

$r3 := \overline{r2}$ $\qquad\qquad r3 = 0.5 - 0.5\mathrm{j}$

The transform can now be expressed as follows:

$$H(z) = \frac{A}{z-r1} + \frac{B}{z-r2} + \frac{C}{z-r3}$$

$A + B + C = 1$

$A \cdot (r2 + r3) + B \cdot (r1 + r3) + C \cdot (r1 + r2) = -.5$

$A \cdot (r2 \cdot r3) + B \cdot (r1 \cdot r3) + C \cdot (r1 \cdot r2) = 0$

You can solve for the numerators of each fraction by inverting the coefficient matrix:

$$
\begin{bmatrix} A \\ B \\ C \end{bmatrix} := \begin{bmatrix} 1 & 1 & 1 \\ r2+r3 & r1+r3 & r1+r2 \\ r2 \cdot r3 & r1 \cdot r3 & r1 \cdot r2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 \\ -.5 \\ 0 \end{bmatrix}
$$

$$
\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 3 - 5.551j \cdot 10^{-17} \\ -1 - 0.5j \\ -1 + 0.5j \end{bmatrix}
$$

Now write the inverse transform of **Y(z)** as the sum of the individual transforms:

$$
p_0 := 0 \qquad\qquad p_n := A \cdot r1^{n-1} + B \cdot r2^{n-1} + C \cdot r3^{n-1}
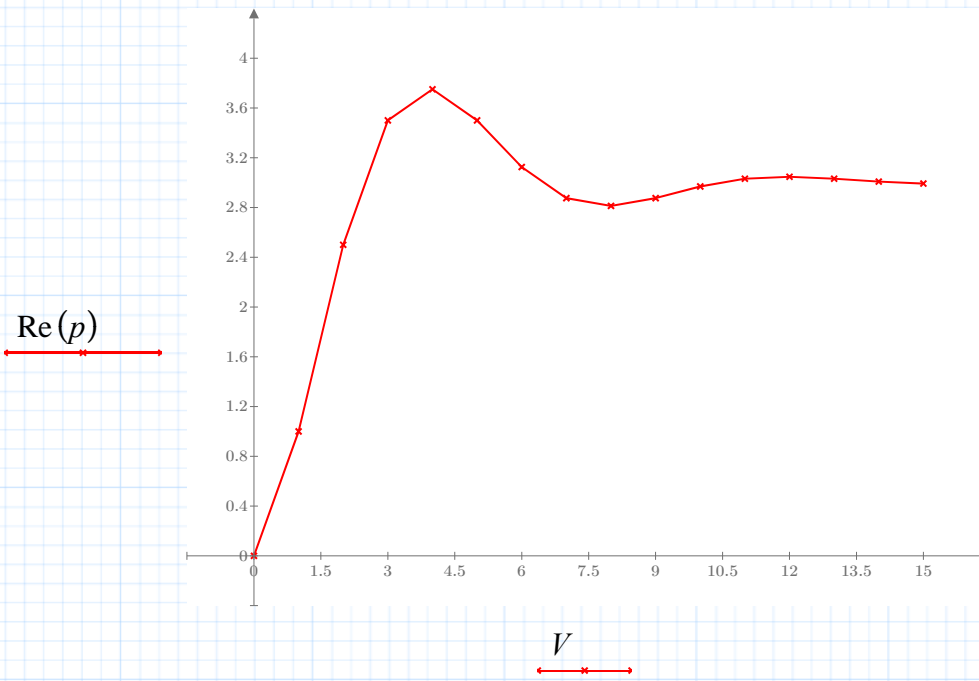$$

$$\mathrm{Re}\,(p)$$

$$V$$

Fig. 11.1.2 System response to a unit step

Which is identical to the answer we found by integrating, but the calculation is much quicker.

Region of Convergence

For the transform to converge, that is, for the frequency response to be defined in the complex **z**-plane, the following relationship must hold:

$$\sum_{n} \left(|x(n)| \cdot (|r \cdot z|)^{-n} \right) < \infty$$

For example, the transform of the unit step function

$$H(z) = \frac{z}{z-1}$$

will only converge for values of |**z**| or |**r**| > 1(greater than the value of the poles). Remember that if you represent the **z**-transform of an *infinite* sequence by a *finite* sum in Mathcad, the sum will behave differently from the true **z**-transform. For example, it may converge at a pole of the transform, when the closed-form expression would yield a singularity.

$$x(n) := 1 \qquad n := 1 \,..\, 15 \qquad X(z) := \sum_{n} \left(x(n) \cdot z^{-n} \right)$$

Notice that this sum is explicit only over the range 1 to 15, but the ideal step function sequence is infinite. Examine the effect of the variable **r** on the graph below.

$$\omega := 1 \,..\, 21 \qquad r := 2 \qquad z(\omega) := r \cdot e^{1j \cdot \omega \cdot .1 \cdot \pi}$$
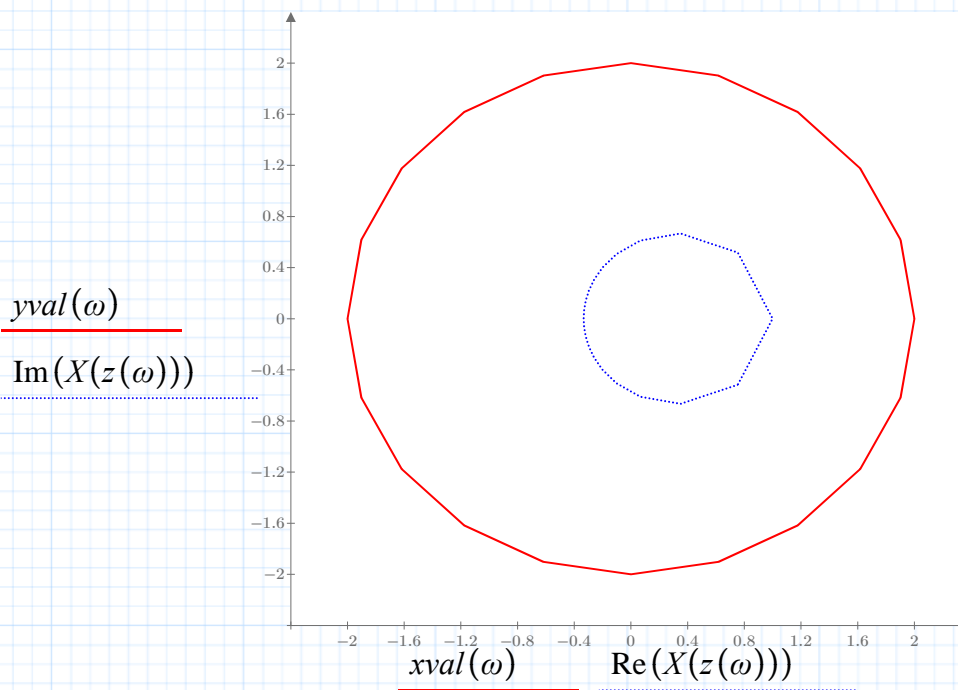
$$xval(\omega) := \mathrm{Re}\,(z(\omega)) \qquad yval(\omega) := \mathrm{Im}\,(z(\omega))$$

Fig. 11.1.3 The complex z-plane