# Section 3.1a Review of System Transients

## Section 3.1.1 Introduction: State Equations

In RLC circuits, the inductors and capacitors of the circuit store energy in the form of magnetic and electric field, respectively. The capacitor and inductor energies are given respectively by Equations (3.1.1) and (3.1.2).

$$W_C = \frac{1}{2} \cdot C \cdot V_C^{\,2} \qquad \textbf{(3.1.1)}$$

$$W_L = \frac{1}{2} \cdot L \cdot I_L^{\,2} \qquad \textbf{(3.1.2)}$$

where C is the capacitance, VC is the voltage across the capacitor, L is the inductance, and IL is the current through the inductor.

Since inductor currents and capacitor voltages describe the total energy storage in the circuit at any time, these variables are the **state variables** of the circuit. That is, all properties of the circuit can be derived by knowing the values of the circuit state variables and the circuit excitation (inputs).

If a circuit is excited by a dc source, the energy stored in capacitors and inductors is constant at steady state. When the circuit excitation is ac, electric and magnetic energy are exchanged between circuit capacitors and inductors, and between capacitors and inductors and the excitation source. The magnitude of the energy interchange, at steady state, is constant for each circuit storage component and it is described by the reactive power.

A special case occurs under ac excitation, when energy is exchanged only between circuit inductors and capacitors and the net reactive power exchange with the excitation source is zero. This situation is known as **resonance**. Resonance is a system property that causes the energy exchange between the storage components of the circuit to occur at a particular frequency, the circuit **natural frequency**. Resonance or "ringing" is always present in transient periods. Resonance can also manifest at steady state, if an excitation frequency coincides with one of the natural frequencies of the circuit.

If the circuit is perturbed from its steady state, the balance of the energy storage in the circuit will be disturbed and the system will enter a transient period. During this period, the circuit energy is redistributed among the energy storage elements in a manner determined by the circuit natural response. Given positive damping in the circuit, the stored energy will attain a new balance and the circuit will operate at a new steady state. Here, a distinction must be made between normal operating condition of the system and steady state. Normal operating conditions are power system steady states. However, a steady state does not necessarily represent a normal system operating condition.

To obtain the transient response of the circuit, the rate with which inductor currents and capacitor voltages vary must be described. Consequently, a set of differential equations in terms of the state variables of the circuit (**state equations**) must be obtained. The solution of these equations provides the circuit transient response and it depends on the circuit excitation and the circuit conditions at the moment the transient was initiated, i.e. the initial conditions.

The procedure for obtaining the state equations of a general RLC circuit involves the following steps: (For a comprehensive discussion of the topic refer to R.A. Rohrer: *Circuit theory: An Introduction to the State Variable Approach* , McGraw-Hill Co., New York, 1970).

**Step 1:** Obtain a normal tree of the circuit by including all voltage sources and as many capacitors as possible as tree branches. The tree links include as many inductors as possible. The voltage of the tree capacitors and the current of the link inductors are the circuit state variables.

**Step 2:** Write the cutset equations (i.e. KCL equations) for all cutsets defined by the tree capacitors.

$$\sum_{n_c} I = 0 \qquad \textbf{(3.1.3)}$$

where nc is the number of elements incident to the cth capacitive cutset.

**Step 3:** Write the fundamental loop equations (i.e. KVL equations) defined by the link inductors.

$$\sum_{n_L} V = 0 \qquad \textbf{(3.1.4)}$$

where nL is the number of elements incident to the fundamental loop defined by the Lth inductor.

**Step 4:** Express the voltage of all link capacitors in terms of the voltage of voltage sources and tree capacitors. Also, express all tree inductor currents in terms of the link inductor currents.

**Step 5:** Replace all inductor voltages in Equation (3.1.4) and all capacitor currents in Equation (3.1.5) using the terminal equations.

$$V_L = L \cdot \frac{\mathrm{d}}{\mathrm{d}t} I_L$$

$$I_C = C \cdot \frac{\mathrm{d}}{\mathrm{d}t} V_C \qquad \textbf{(3.1.5)}$$

where the currents of tree inductors and the voltages of link capacitors are expressed according to Step 4.

**Step 6:** This step involves elimination of all resistor voltages and currents from Equations (3.1.3) and (3.1.4). The only terms retained are the voltages, currents and their derivatives for the voltage sources, tree capacitors and link inductors. To eliminate the resistors, the fundamental cutset and loop equations defined by the resistors are written. After this elimination, the circuit state equations result in an ordinary differential equation.

$$\frac{d}{dt}x = A \cdot x + B \cdot u + D \cdot \frac{d}{dt}u \qquad \textbf{(3.1.6)}$$

where x is the state vector defined as

$$x = \left[ I_L \;\; V_C \right]^{T}$$

and u is a vector containing the circuit voltage sources.

The eigenvalues of the state matrix, A, provide the circuit natural modes, i.e. the natural frequencies and the corresponding damping of the circuit. The solution of Equation (3.1.6) takes the general form.

$$x(t) = \sum_n \left( K_n \cdot e^{\alpha_n \cdot t} \cdot cos\left( \omega_n \cdot t + \phi_n \right) \right) + x_{ss} \qquad \textbf{(3.1.7)}$$

where

$\alpha$n is the damping of the nth natural mode, i.e. the real part of the nth eigenvalue of A;
$\omega$n is the frequency of the nth natural mode, i.e. the imaginary part of the nth eigenvalue of A;
Kn, $\phi$n are constants determined by the circuit initial conditions; and
xss is the steady state response.

The first part of Equation (3.1.7) represents the transient part of the response of the system. This part decays to zero if the damping of all system modes is positive. The second term in Equation (3.1.7) represents the steady state response of the system. This part can also be obtained using impedance and phasor representation of the circuit for sinusoidal excitations.

Although Equation (3.1.7) provides a closed form solution of Equation (3.1.6), it is only practical for circuits of small order. For circuits of high order or for circuits that include non-linear components (e.g. transformer saturation, arrester v-i characteristic), it is preferable to obtain a numerical solution of Equations (3.1.6).

## Section 3.1.2 Review of Second Order Circuits

In most preliminary studies of power system transients, a second order RLC circuit (one with a resistor, inductor and capacitor) representation of the system suffices for obtaining the essential parts of the system response.

To demonstrate the effect of the different parts of the system response, consider the second order circuit of Figure 3.1.1.

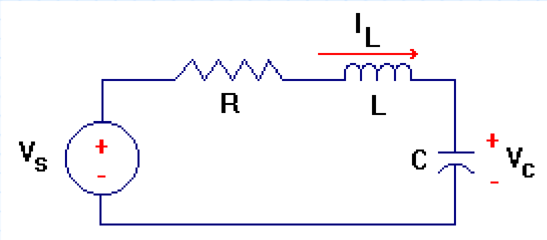


**Fig. 3.1.1 A second order circuit**

Choosing the inductor current and the capacitor voltage as state variables, the circuit state equations are

$$\begin{bmatrix} \frac{d}{dt} I_L \\ \frac{d}{dt} V_C \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \cdot \begin{bmatrix} I_L \\ V_C \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot V_s(t)$$

The characteristic equation of the circuit describes its resonant behavior, and is given by

$$s^2 + b \cdot s + \omega_o^2 = 0$$

where

$$b := \frac{R}{L} \qquad \omega_o := \frac{1}{\sqrt{L \cdot C}}$$

Suppose we wish to solve for the voltage across the capacitor over all time. This voltage will have an exponential form, and will depend on the roots of the characteristic equation,

$$\lambda 1 := \frac{-b + \sqrt{b^2 - 4 \cdot \omega_o^2}}{2} \qquad \lambda 2 := \frac{-b - \sqrt{b^2 - 4 \cdot \omega_o^2}}{2}$$

With reference to Equation (3.1.7), the circuit damping is given by

$$\alpha := -\frac{b}{2} \qquad \alpha = -40 \ s^{-1}$$

which is the real part of the roots. The circuit's natural frequency is

$$\omega_n := \frac{\sqrt{\left| b^2 - 4 \cdot \omega_o^2 \right|}}{2} \qquad \omega_n = \left(1.24 \cdot 10^3\right) s^{-1}$$

The general solution of the capacitor voltage is

$$V_C(t) = k_1 \cdot e^{\lambda 1 \cdot t} + k_2 \cdot e^{\lambda 2 \cdot t} + V_{Css}$$

Consider the following values for R, L, and C,

$$R \equiv 4\ \Omega \qquad L \equiv 50\ mH \qquad C \equiv 13\ \mu F$$

A dc source, with magnitude

$$V_o \equiv 100\ V$$

is applied at time t = 0. After charging to its steady state value, the voltage across the capacitor is

$$V_{Css} \equiv V_o$$

Given these parameters, the system damping and natural frequencies are

$$\alpha = -40\ \frac{rad}{s} \qquad \omega_n = \left(1.24 \cdot 10^3\right)\ \frac{rad}{s}$$

The question remains, what happens to the voltage across the capacitor before it reaches its steady state value? We know the exponential rates of charging, λ1 and λ2, by solving for the roots of the characteristic equation. We can solve for the constants, k1 and k2, using the initial state of the circuit, as follows.

$$V_{Co} \equiv 0\ V \qquad I_{Lo} \equiv 0\ A$$

Therefore, from the system state equations, the derivative of the capacitor voltage at t = 0 is

$$DV_{Co} := \frac{I_{Lo}}{C}$$

Thus, we have

$$V_C(0) = V_{Co} = k_1 + k_2 + V_o \qquad \frac{d}{dt} V_C(0) = DV_{Co} = \lambda 1 \cdot k_1 + \lambda 2 \cdot k_2$$

Solving the above equation for k1 and k2, we obtain

$$\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} := \begin{bmatrix} 1 & 1 \\ \lambda 1 \cdot sec & \lambda 2 \cdot sec \end{bmatrix}^{-1} \cdot \begin{bmatrix} V_{Co} - V_o \\ DV_{Co} \cdot sec \end{bmatrix}$$

$$k_1 = (-50 + 1.613i)\ V \qquad k_2 = (-50 - 1.613i)\ V$$

That is, k2 is the complex conjugate of k1. The capacitor voltage is

$$V_C(t) := k_1 \cdot e^{\lambda 1 \cdot t} + k_2 \cdot e^{\lambda 2 \cdot t} + V_{Css}$$

$$\lambda 1 = \left(-40 + 1.24i \cdot 10^3\right)\ s^{-1} \qquad \lambda 2 = \left(-40 - 1.24i \cdot 10^3\right)\ s^{-1}$$

So the voltage will be a decaying envelope, with a 40 second decay constant, around an oscillating voltage of frequency ωn; The oscillations will be centered at the steady state voltage for the capacitor, VCss.

Using the values for the damping and natural frequency calculated before, the envelopes of the response are given by:

$$E1(t) := \left|k_1\right| \cdot 2 \cdot e^{\alpha \cdot t} + V_{Css}$$

$$E2(t) := V_{Css} - \left|k_1\right| \cdot 2 \cdot e^{\alpha \cdot t}$$

The system response for 60 msec is shown below.

$$t := 0 \cdot ms, .1 \cdot ms .. 60 \cdot ms$$

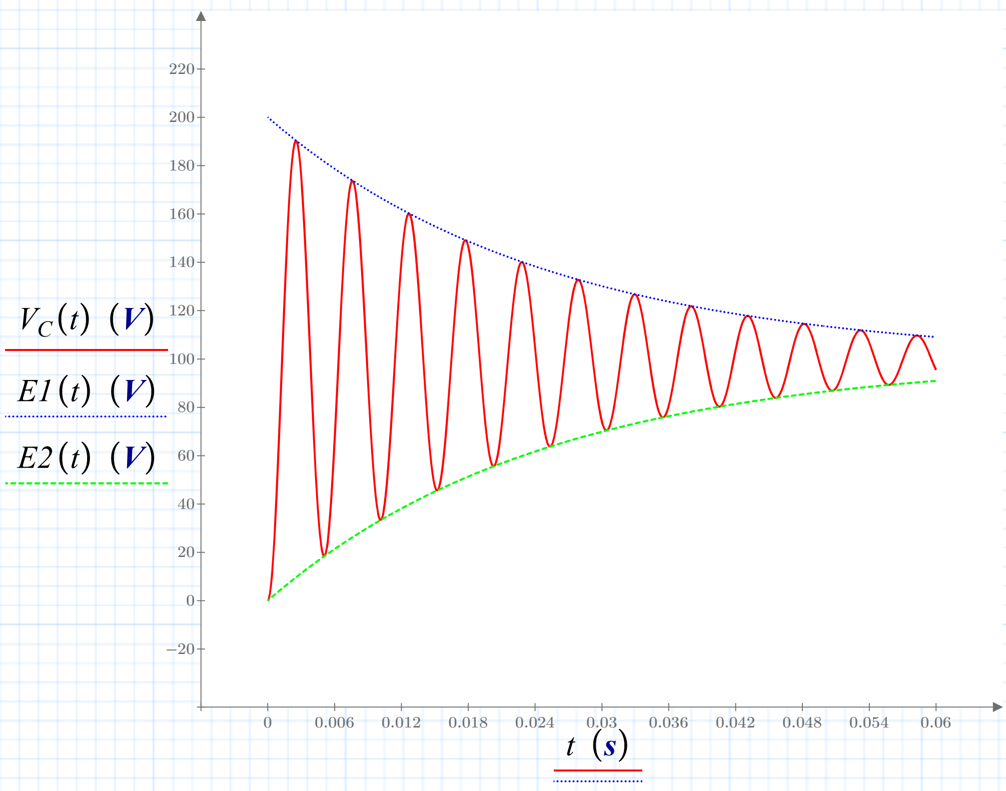


**Fig. 3.1.2 Capacitor voltage response**

The envelopes of the system response describe the rate at which the "ringing" due to resonance decays. These envelopes are functions of the system damping, that is, the dissipation of energy through resistive loss.