

# 5

## ***Analog Equivalent Digital Low Pass II° Order Filter***

### ***Introduction***

A well known in the technical literature second order low pass active filter is here treated. The filter is realized using a common operational amplifier in a negative multiple feedback configuration. After a brief summary of the main results of the circuit analysis, the outputs to several signals among the most common and from an external file (Signal List.xmcd), are produced. This results are compared with the one obtained by the many algorithms implemented applying the z-transform to the output function. Since MATHCAD student edition is devoid of some functions, the program BCSA (Fourier series and signal bandwidth) was written to fill this gap. Four other programs (CANONIC2LP, SYNDIVC, BILINEAR, SYNDIVBL) corresponding to four different algorithms to calculate the filter sequential response, were realized and written in a new worksheet with the purpose to reuse them elsewhere and, last but not the least, to save worksheet's space as well. Thus one will see that, as the analog filter is effective, just is the digital one. The step to the practical application, using a DSP, should be very simple.

*When saving or printing, disable Automatic Calculation (Mathcad 14 s. e.)*

*The subscript "gd" is the acronym of "Global Data.xmcd"*

*The subscript "fs" is the acronym of "Fourier series.xmcd"*

*The subscript "sl" is the acronym of "Signal List.xmcd"*

*The subscript "dp" is the acronym of "Dirac Pulse - formulae.xmcd"*

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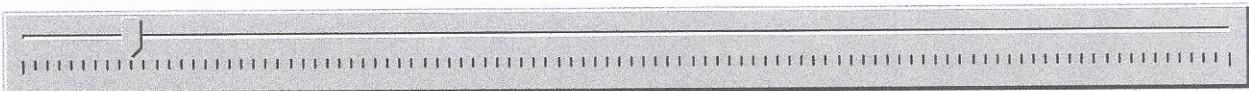
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## References

- Reference:C:\new folder\global data.xmcd
- Reference:C:\new folder\Fourier Series.xmcd
- Reference:C:\new folder\Signals List.xmcd
- Reference:C:\new folder\Nichols Chart.xmcd
- Reference:C:\new folder\Dirac Pulse - formulas.xmcd
- Reference:C:\new folder\Pulse Train Data.xmcd
- Reference:C:\new folder\staircase pulse data.xmcd
- Reference:C:\new folder\staircase pulse train data.xmcd
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- Reference:C:\new folder\staircase 2 pulse train data.xmcd
- Reference:C:\new folder\staircase 3 pulse data.xmcd
- Reference:C:\new folder\staircase 4 pulse data.xmcd
- Reference:C:\new folder\sawtooth pulse data.xmcd
- Reference:C:\new folder\sawtooth pulse train data.xmcd
- Reference:C:\new folder\rf pulse data.xmcd
- Reference:C:\new folder\FM data.xmcd
- Reference:C:\new folder\PM data.xmcd
- Reference:C:\new folder\programs.xmcd

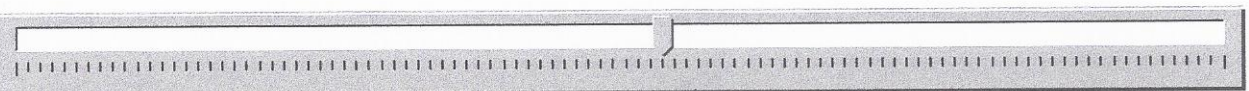
## Definitions and necessary constants

*Scroll the slider, to change the filter's voltage gain*

$k_{vg} :=$  

$A_5 := -k_{vg} \cdot 1$                       Voltage gain:     $A_5 = -10$


*Scroll the slider, to change the filter's pole Q factor*

$k_{qf} :=$  

$Q_5 := k_{qf} \cdot 0.1$                       Pole Q factor:     $Q_5 = 5.4$  ,

For  $Q_5 > 1$  the frequency response of the active filter presents a resonance peak or an overshoot, while for  $Q_5 \leq 1$  it is a flat one.

*Scroll the slider, to change the filter's bandwidth*

$k_{cf} :=$  

$Bw := k_{cf} \cdot 0.10 \cdot \text{MHz}$                       Bandwidth:     $Bw = 47.1 \cdot \text{MHz}$  ,

▶ Calculation of f5

Step amplitude:                       $V_{pp} = 5V$   
Pole frequency:                       $f_5 = 30.498 \cdot \text{MHz}$

Period:  $T_5 := \frac{1}{f_5}, T_5 = 32.789 \cdot \text{ns}$

Pole Angular frequency:  $\omega_5 := 2 \cdot \pi \cdot f_5, \omega_5 = 0.192 \cdot \frac{\text{Grads}}{\text{sec}},$

Filter's bandwidth:  $Bw = 47.1 \cdot \text{MHz}$

time constant:  $\tau := \frac{1}{\omega_5}, \tau = 5.218 \cdot \text{ns}$

damping factor:  $\zeta := \frac{\omega_5}{2 \cdot Q_5}, \zeta = 0.018 \cdot \frac{\text{Grads}}{\text{sec}}, \omega_5 = 2 \cdot \zeta \cdot Q_5,$

If  $Q_5 = 0.5 \Rightarrow \zeta = \omega_5,$

Defined in "global data.xmcd":  $\text{dB3}_{\text{gd}} = 20 \cdot \log(\sqrt{2});$  angular frequency units.

**Chosen test signal period**, to verify the filtering action of the system, it has been selected a test signal frequency outside the passband.

$$T_{\text{test}} := \frac{1}{2} \cdot T_5 \text{ i.e. a sub multiple of the 0 dB Voltage gain period.}$$

$$T_{\text{test}} = 16.394 \cdot \text{ns}$$

Then the signal frequency  $f_{\text{test}} := \frac{1}{T_{\text{test}}},$   $f_{\text{test}} = 0.061 \cdot \text{GHz}$  :  $Bw = 0.047 \cdot \text{GHz} . \omega_{\text{test}} := 2 \cdot \pi \cdot f_{\text{test}},$

$\omega_{\text{test}} = 0.383 \cdot \frac{\text{Grads}}{\text{sec}}$  is higher than the cutoff angular frequency of the filter,  $\frac{\omega_{\text{test}}}{\omega_5} = 2 ,$

As a result, the waveform at the filter output should be strongly attenuated.

Amplifier Gain:  $A_5 = -10 , A_{5\text{dB}} := 20 \cdot \log(|A_5|)$

$$A_{5\text{dB}} = 20 \cdot \text{dB}$$



## 5.1

### II° Order Analog LOW PASS Digital Filter

The analog active filter chosen is the following (the power supply circuit isn't visible):

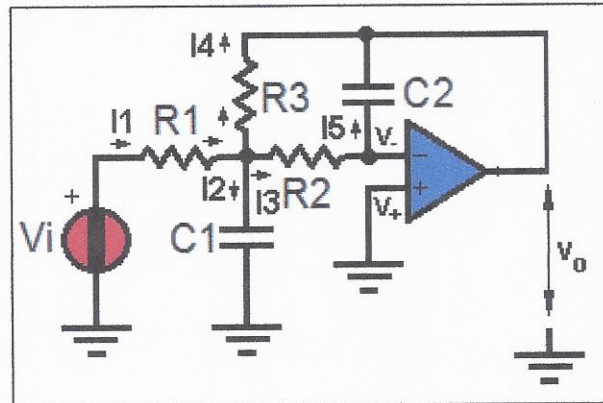


fig.:5.1.1

▢ Network Analysis

Network Analysis:

node admittance matrix:

$$\mathbf{Y} = \begin{pmatrix} \frac{1}{R1} & -\frac{1}{R1} & 0 & 0 \\ -\frac{1}{R1} & \frac{1}{R1} + \frac{1}{R2} + \frac{1}{R3} + C1 \cdot s & -\frac{1}{R2} & -\frac{1}{R3} \\ 0 & -\frac{1}{R2} & \frac{1}{R2} + C2 \cdot s & -C2 \cdot s \\ 0 & -\frac{1}{R3} & -C2 \cdot s & \frac{1}{R3} + C2 \cdot s \end{pmatrix} \quad (5.1.1)$$

▢ L. T. I. System Analysis.

The general transfer function is expressed by the formula:

$$W(s) = \frac{-Y1(s) \cdot Y3(s)}{Y5(s) \cdot (Y1(s) + Y2(s) + Y3(s) + Y4(s)) + Y3(s) \cdot Y4(s)} \quad (5.1.11)$$

**Open loop voltage gain calculation.**

The filter is composed by an ideal op amp in inverting configuration and a further feedback network as here below depicted:

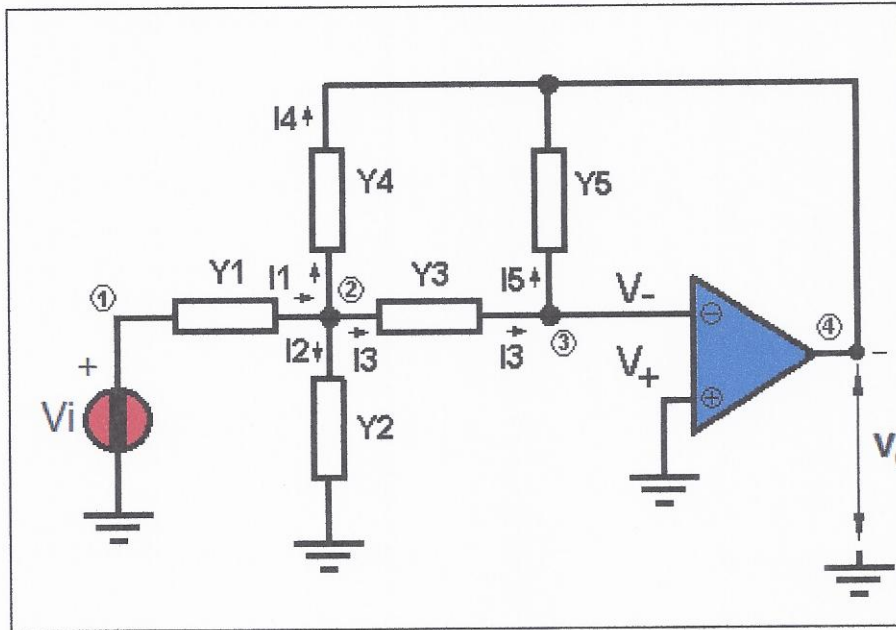


fig.:5.1.5

In terms of a feedback system, a new function could be defined for the direct branch of a new system but with the same t. f. :

$$G(s) = -\frac{Y1(s)}{Y4(s)}$$

and for the feedback branch:  $H(s) = -\frac{Y5(s)}{Y3(s)} \cdot \left( \frac{Y1(s) + Y2(s) + Y3(s) + Y4(s)}{Y1(s)} \right)$ ,

so that:

$$W_{lp}(s) = \frac{G(s)}{1 + G(s) \cdot H(s)}$$

The corresponding open loop gain is:  $GH(s) = \frac{Y5(s) \cdot (Y1(s) + Y2(s) + Y3(s) + Y4(s))}{Y3(s) \cdot Y4(s)}$

Other couple of functions, but describing a new system, can be found, i.e. :

$$G(s) = \frac{-Y1(s) \cdot Y3(s)}{Y5(s) \cdot (Y1(s) + Y2(s) + Y3(s) + Y4(s))}$$

and:

$$H(s) = -\frac{Y4(s)}{Y1(s)}$$

$$GH(s) = \frac{Y3(s) \cdot Y4(s)}{Y5(s) \cdot (Y1(s) + Y2(s) + Y3(s) + Y4(s))}$$

In fact, substituting the two last relations in the generic transfer function of a feedback system, you get:

$$\frac{G(s)}{1 + G(s) \cdot H(s)} = \frac{-Y_1(s) \cdot Y_3(s)}{Y_5(s) \cdot (Y_1(s) + Y_2(s) + Y_3(s) + Y_4(s)) \cdot (1 + GH(s))}$$

and simplifying results:

$$W_{lp}(s) = \frac{G(s)}{1 + G(s) \cdot H(s)} = \frac{-Y_3(s) \cdot Y_1(s)}{(Y_2(s) + Y_3(s) + Y_4(s) + Y_1(s)) \cdot Y_5(s) + Y_3(s) \cdot Y_4(s)}$$

that is the already known transfer function derived earlier.

But now determine the open loop gain of the given filter. Start from its circuit:

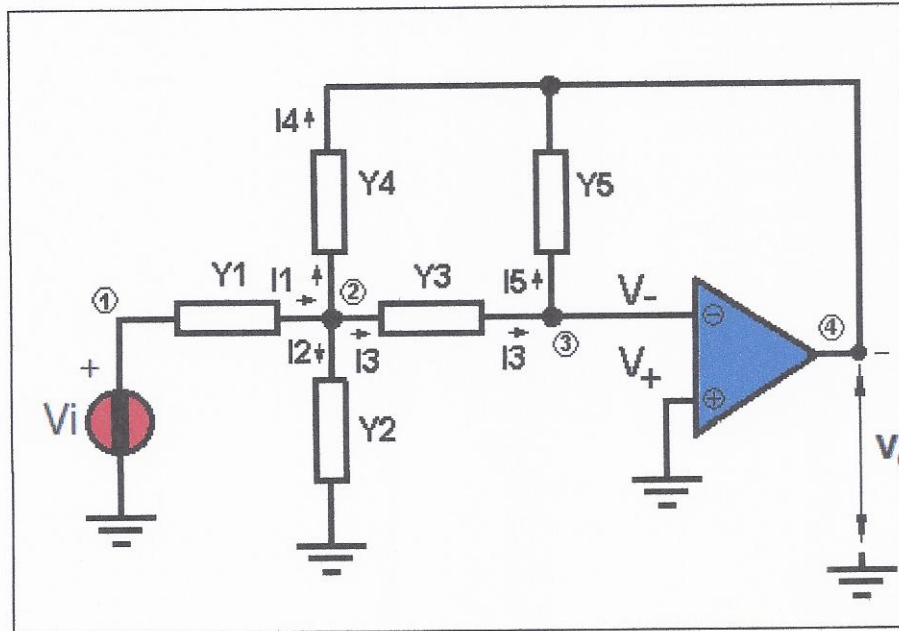


fig.:5.1.5'

The feedback network can be redrawn as follows:

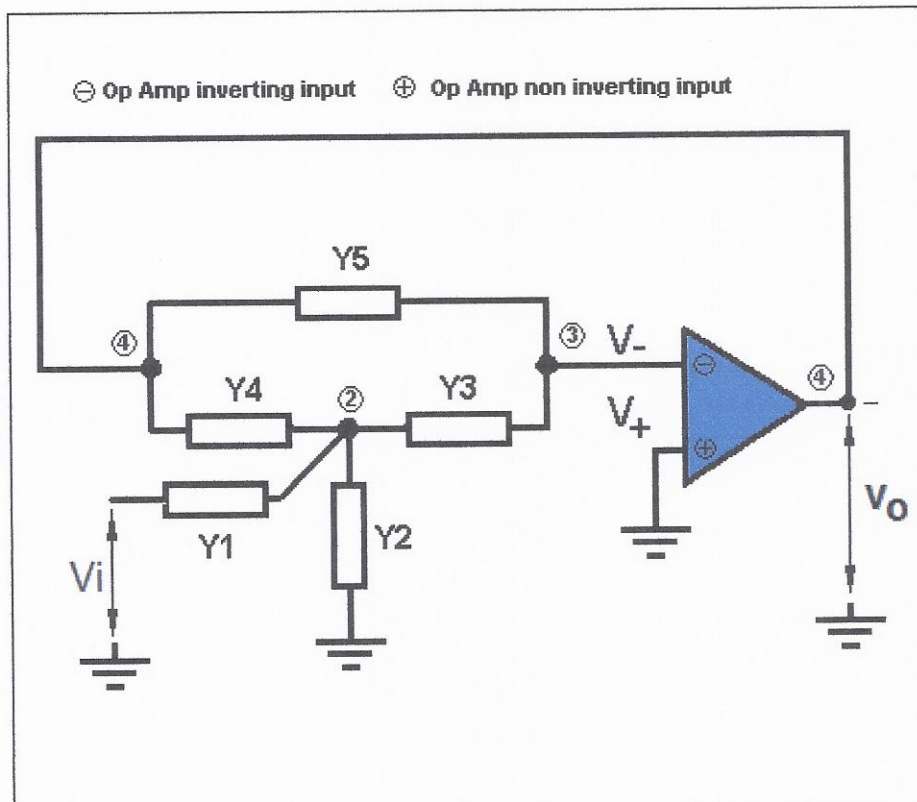


fig.:5.1.6



Let shift the bridged T along the loop, the network assumes the following form, although the topology of the network is unchanged:

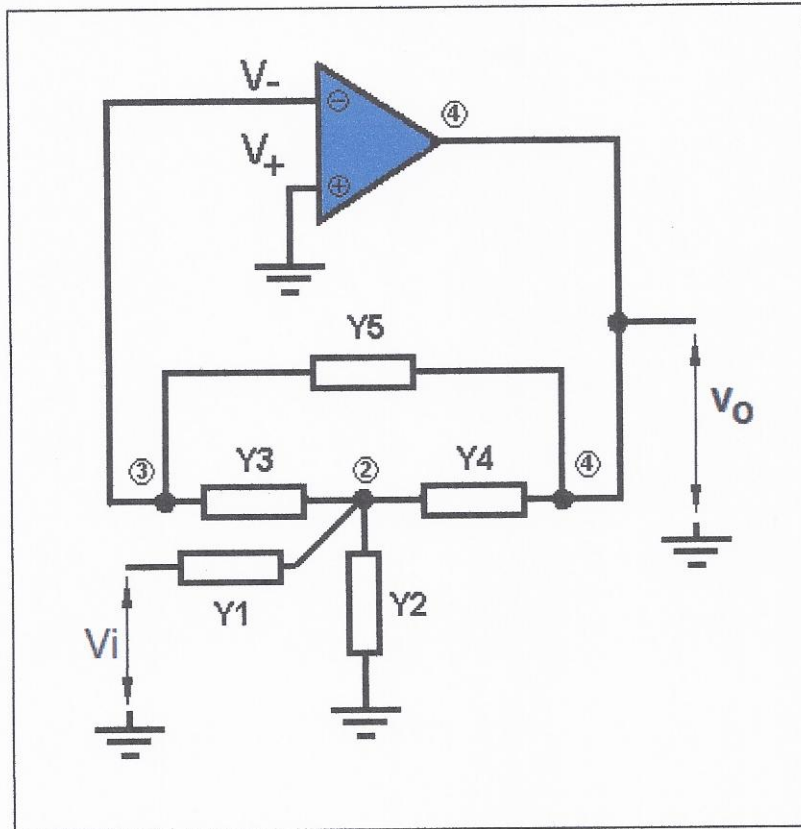


fig.:5.1.7

Now apply the known procedure to determine the open loop gain, namely: remove the effect of the input generator ( $V_i=0$ = input shorted), choose a cut on the loop and apply an independent voltage test source  $V_T$  the right side of the cut . It has been chosen the following cut:

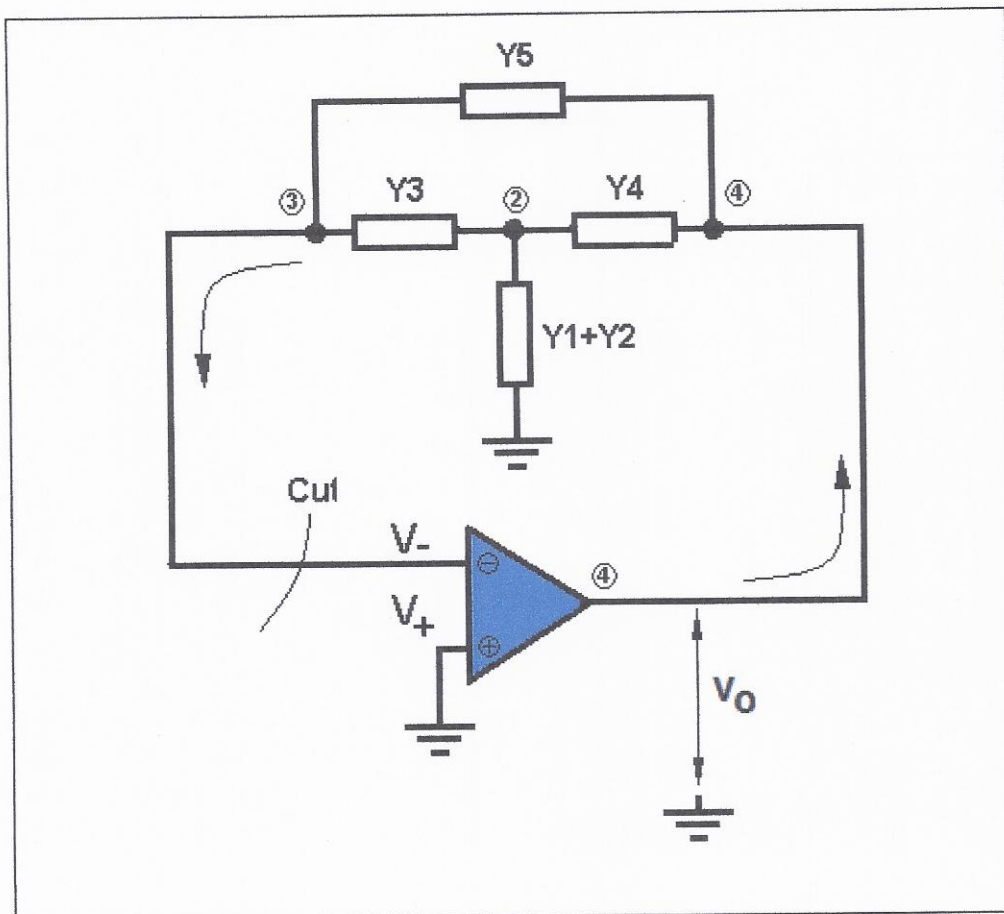


fig.:5.1.8

The voltage generator  $V_T$  has been placed at the input of the op amp, here represented with its equivalent circuit:

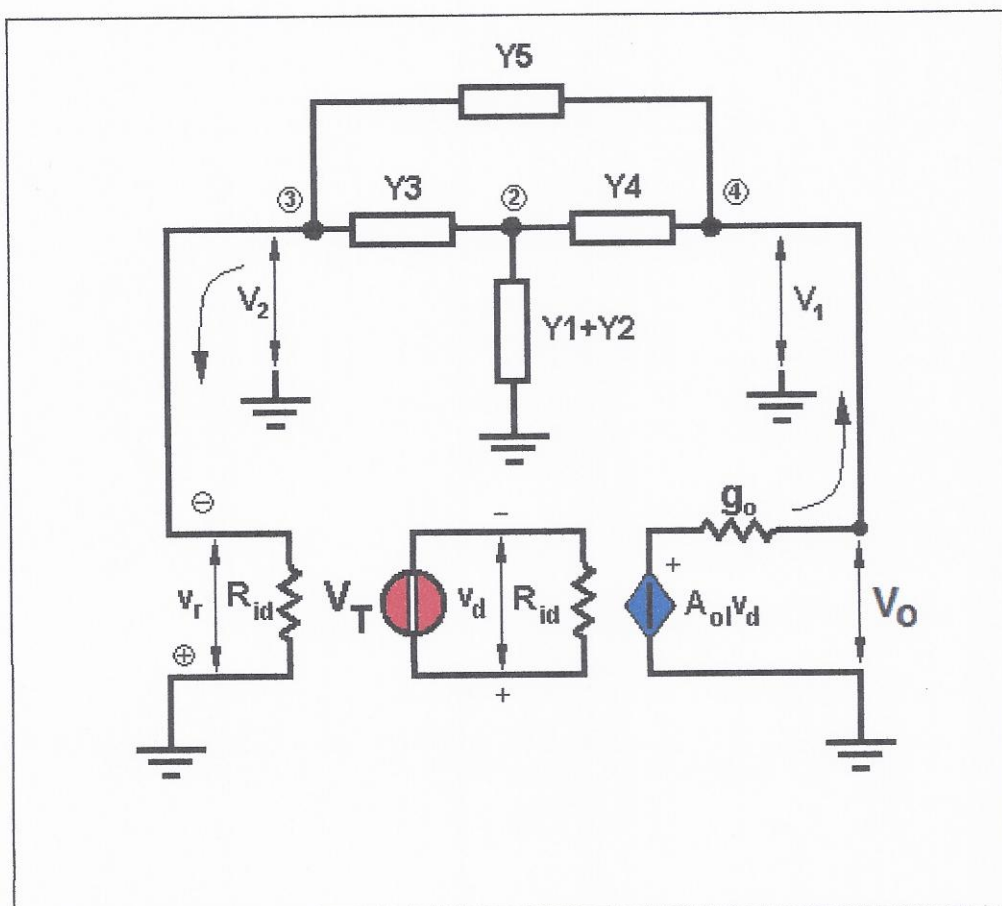


fig.:5.1.9

To simplify the circuit analysis, it has been placed:  $R_{id}=\infty\Omega$ ,  $r_o=0\Omega$ . The equivalent circuit now is:

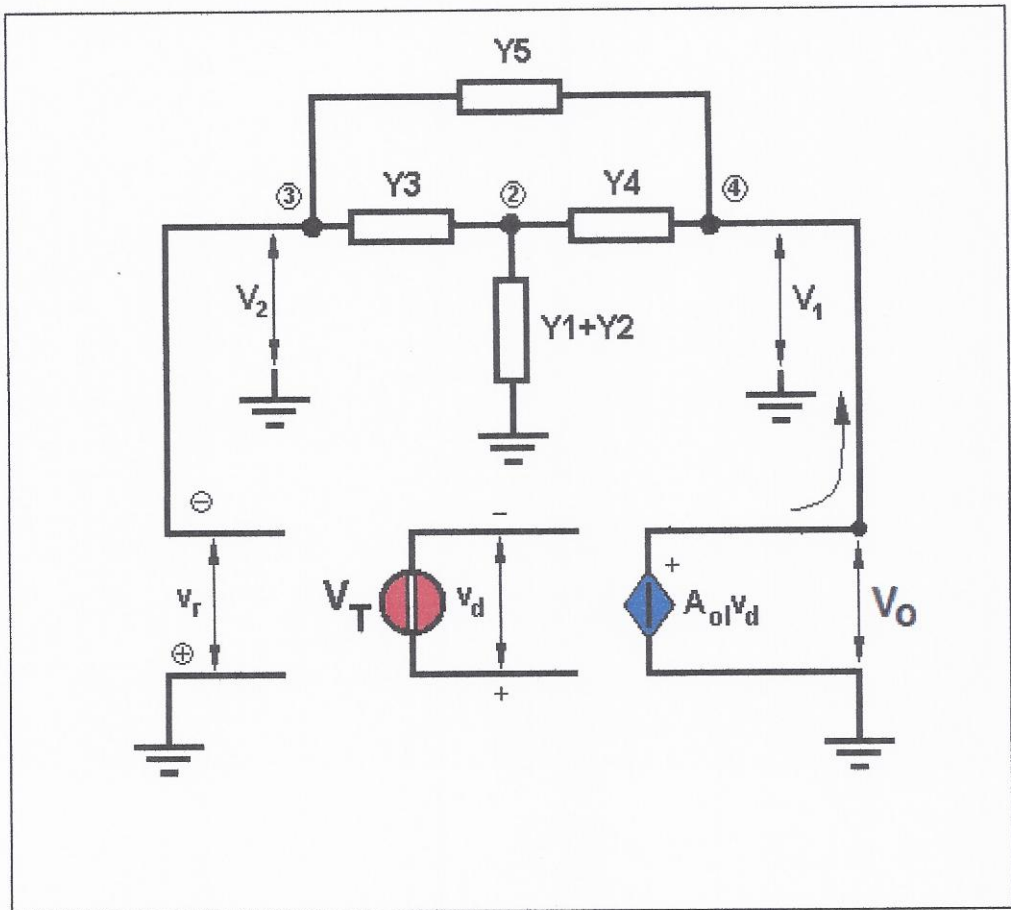


fig.:5.1.10

☑ Computation of V2 by inspection

$$v_r = V_2 = V_3 + V_{12}$$

$$Y_T = (Y_1 + Y_2) \parallel [Y_4 + (Y_5 \parallel Y_3)]$$

$$I_T = \frac{V_1}{Z_T} = V_1 \cdot Y_T = V_1 \cdot [(Y_1 + Y_2) \parallel [Y_4 + (Y_5 \parallel Y_3)]]$$

$$I_T = I_4 + I_5$$

$$I_4 = \frac{Y_4}{Y_4 + Y_5 \parallel Y_3} \cdot I_T$$

$$I_5 = \frac{Y_5 \parallel Y_3}{Y_5 \parallel Y_3 + Y_4} \cdot I_T$$

$$V_3 = Z_3 \cdot I_5 = \frac{I_5}{Y_3} = \frac{Y_5 \parallel Y_3}{Y_5 \parallel Y_3 + Y_4} \cdot \frac{I_T}{Y_3}$$

$$V_{12} = \frac{I_T}{Y_1 + Y_2} = \frac{V_1 \cdot [(Y_1 + Y_2) \parallel [Y_4 + (Y_5 \parallel Y_3)]]}{Y_1 + Y_2}$$

$$v_r = V_3 + V_{12} = \frac{Y_5 \parallel Y_3}{Y_5 \parallel Y_3 + Y_4} \cdot \frac{I_T}{Y_3} + \frac{I_T}{Y_1 + Y_2}$$

$$v_r = \left( \frac{Y_5 \parallel Y_3}{Y_5 \parallel Y_3 + Y_4} \cdot \frac{1}{Y_3} + \frac{1}{Y_1 + Y_2} \right) \cdot [V_1 \cdot [(Y_1 + Y_2) \parallel [Y_4 + (Y_5 \parallel Y_3)]]]$$



$$v_r = \frac{V_1 \cdot [(Y_1 + Y_2 + Y_3 + Y_4) \cdot Y_5 + Y_3 \cdot Y_4]}{(Y_1 + Y_2 + Y_3 + Y_4) \cdot Y_5 + Y_3 \cdot (Y_1 + Y_2 + Y_4)}$$

$v_i$

▣ Computation of  $V_2$  by inspection

The open loop gain results to be:

$$\frac{v_r}{V_T} = \frac{[(Y_1 + Y_2 + Y_3 + Y_4) \cdot Y_5 + Y_3 \cdot Y_4] \cdot A_{ol}}{(Y_1 + Y_2 + Y_3 + Y_4) \cdot Y_5 + Y_3 \cdot (Y_1 + Y_2 + Y_4)} \quad (5.1.12)$$

▣

Explicit calculation of the open loop voltage gain. The admittances are:

$$Y_1(s) = \frac{1}{R_1},$$

$$Y_2(s) = s \cdot C_1,$$

$$Y_3(s) = \frac{1}{R_2},$$

$$Y_4(s) = \frac{1}{R_3},$$

$$Y_5(s) = s \cdot C_2.$$

Substituting into the general form (5.1.12), gets:

$$\frac{[(Y_1 + Y_2 + Y_3 + Y_4) \cdot Y_5 + Y_3 \cdot Y_4] \cdot A_{ol}}{(Y_1 + Y_2 + Y_3 + Y_4) \cdot Y_5 + Y_3 \cdot (Y_1 + Y_2 + Y_4)} \left\{ \begin{array}{l} \text{substitute, } Y_1 = \frac{1}{R_1}, Y_2 = s \cdot C_1, Y_3 = \frac{1}{R_2} \\ \text{substitute, } Y_4 = \frac{1}{R_3}, Y_5 = s \cdot C_2 \\ \text{collect, } s, A_{ol} \end{array} \right. \rightarrow$$

$$GH_5(s) = \left[ \frac{C_1 \cdot C_2 \cdot R_1 \cdot R_2 \cdot R_3 \cdot s^2 + (C_2 \cdot R_1 \cdot R_2 + C_2 \cdot R_1 \cdot R_3 + C_2 \cdot R_2 \cdot R_3) \cdot s + R_1}{C_1 \cdot C_2 \cdot R_1 \cdot R_2 \cdot R_3 \cdot s^2 + (C_1 \cdot R_1 \cdot R_3 + C_2 \cdot R_1 \cdot R_2 + C_2 \cdot R_1 \cdot R_3 + C_2 \cdot R_2 \cdot R_3) \cdot s + R_1 + R_3} \cdot A_{ol} \right]$$

$$GH_5(s) = \left[ \frac{s^2 + \frac{[R_1 \cdot (R_2 + R_3) + R_2 \cdot R_3]}{C_1 \cdot R_1 \cdot R_2 \cdot R_3} \cdot s + \frac{1}{(C_1 \cdot C_2 \cdot R_2 \cdot R_3)}}{s^2 + \frac{[C_2 \cdot (R_1 \cdot R_2 + R_1 \cdot R_3 + R_2 \cdot R_3) + C_1 \cdot R_1 \cdot R_3]}{(C_1 \cdot C_2 \cdot R_1 \cdot R_2 \cdot R_3)} \cdot s + \frac{R_1 + R_3}{C_1 \cdot C_2 \cdot R_1 \cdot R_2 \cdot R_3}} \cdot A_{ol} \right] \quad (5.1.13)$$

▣ Calculation of the Input and output resistances

### Transfer function calculation

For the proposed filter, here reported:

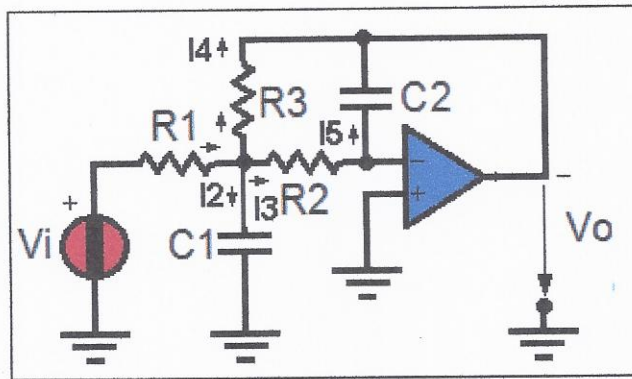


fig.:5.1.18

the admittances are:

$$Y1(s) = \frac{1}{R1},$$

$$Y2(s) = s \cdot C1,$$

$$Y3(s) = \frac{1}{R2},$$

$$Y4(s) = \frac{1}{R3},$$

$$Y5(s) = s \cdot C2.$$

Substituting into the general form (5.1.11), gets:

$$W_{lp}(s) := \frac{-Y1 \cdot Y3}{Y5 \cdot (Y1 + Y2 + Y3 + Y4) + Y3 \cdot Y4}$$

$\left\{ \begin{array}{l} \text{substitute, } Y1 = \frac{1}{R1}, Y2 = s \cdot C1, Y3 = \frac{1}{R2}, Y4 = \frac{1}{R3} \\ \text{substitute, } Y5 = s \cdot C2 \\ \text{collect, } s \end{array} \right. \rightarrow$

The resulting transfer function, then is:

$$W_{lp}(s) = \frac{R3}{C1 \cdot C2 \cdot R1 \cdot R2 \cdot R3 \cdot s^2 + (C2 \cdot R1 \cdot R2 + C2 \cdot R1 \cdot R3 + C2 \cdot R2 \cdot R3) \cdot s + R1}$$

Collecting the term:  $C1 \cdot C2 \cdot R1 \cdot R2 \cdot R3$ , the t. f. becomes:

$$W_{lp}(s) = \frac{\frac{1}{R1 \cdot R2 \cdot C1 \cdot C2}}{s^2 + s \cdot \frac{1}{C1} \cdot \left( \frac{1}{R1} + \frac{1}{R2} + \frac{1}{R3} \right) + \frac{1}{R2 \cdot R3 \cdot C1 \cdot C2}} \quad (5.1.14)$$

In order for this transfer function takes the form of the standard low-pass active filter, below rewritten:

$$W_{lp}(s) = \frac{A5 \cdot \omega_5^2}{s^2 + 2 \cdot \zeta \cdot s + \omega_5^2} \quad (5.1.15)$$

one must place into (5.1.29):

a)  $A5 \cdot \omega_5^2 = \frac{-1}{R1 \cdot R2 \cdot C1 \cdot C2}$

$$b) \quad \omega_5^2 = \frac{1}{R_2 \cdot R_3 \cdot C_1 \cdot C_2} \quad \omega_5 = \frac{1}{\sqrt{R_2 \cdot R_3 \cdot C_1 \cdot C_2}}$$

$$c) \quad 2 \cdot \zeta = \frac{\omega_5}{Q_5} = \frac{1}{C_1} \cdot \left( \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \right)$$

▶ Calculations A5 and Q5

In summary, it can be written: d)  $\omega_5 = \frac{1}{\sqrt{R_2 \cdot R_3 \cdot C_1 \cdot C_2}}$ , (5.1.16)

$$e) \quad A_5 = -\frac{R_3}{R_1},$$

$$f) \quad Q_5 = \frac{1}{\sqrt{\frac{C_2}{C_1} \cdot \left( \frac{\sqrt{R_2 \cdot R_3}}{R_1} + \sqrt{\frac{R_3}{R_2}} + \sqrt{\frac{R_2}{R_3}} \right)}}$$

$$g) \quad \zeta = \frac{1}{2} \cdot \left[ \frac{1}{C_1} \cdot \left( \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \right) \right]$$

A further condition is obtained by observing that the relationships  $R_1 = \frac{R_3}{|A_5|}$  and

$R_2 = \frac{1}{\omega_5^2 \cdot R_3 \cdot C_1 \cdot C_2}$  can be replaced in the expression c)  $\frac{\omega_5}{Q_5} = \frac{1}{C_1} \cdot \left( \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \right)$ , namely:

$$\frac{\omega_5}{Q_5} = \frac{1}{C_1} \cdot \left( \frac{1}{\frac{R_3}{|A_5|}} + \frac{1}{\frac{1}{\omega_5^2 \cdot R_3 \cdot C_1 \cdot C_2}} + \frac{1}{R_3} \right),$$

which yields:

$$\frac{\omega_5}{Q_5} - C_2 \cdot R_3 \cdot \omega_5^2 - \frac{|A_5| + 1}{C_1 \cdot R_3} = 0,$$

it is a quadratic equation in the unknown R3:

$$R_3^2 + \frac{|A_5| + 1}{C_1 \cdot C_2 \cdot \omega_5^2} - \frac{R_3}{C_2 \cdot Q_5 \cdot \omega_5} = 0,$$

whose roots are:

$$A_5 := A_5$$

$$Q_5 := Q_5 \quad \omega_5 := \omega_5$$

$$C_1 := C_1 \quad C_2 := C_2$$

$$R_3^2 + \frac{|A_5| + 1}{C_1 \cdot C_2 \cdot \omega_5^2} - \frac{R_3}{C_2 \cdot Q_5 \cdot \omega_5} \quad \left| \begin{array}{l} \text{solve, R3} \\ \text{simplify, max} \end{array} \right. \rightarrow \left( \begin{array}{c} \frac{\sqrt{\frac{4 \cdot C_2 \cdot Q_5^2 - C_1 + 4 \cdot C_2 \cdot Q_5^2 \cdot |A_5|}{C_1}} + 1}{2 \cdot C_2 \cdot Q_5 \cdot \omega_5} \\ \frac{\sqrt{\frac{4 \cdot C_2 \cdot Q_5^2 - C_1 + 4 \cdot C_2 \cdot Q_5^2 \cdot |A_5|}{C_1}} - 1}{2 \cdot C_2 \cdot Q_5 \cdot \omega_5} \end{array} \right)$$

hence:



$$R3 = \frac{1}{2 \cdot C2 \cdot Q_5 \cdot \omega_5} \cdot \left[ \begin{array}{l} 1 + \sqrt{1 - \frac{4 \cdot C2 \cdot Q_5^2 \cdot (|A_5| + 1)}{C1}} \\ 1 - \sqrt{1 - \frac{4 \cdot C2 \cdot Q_5^2 \cdot (|A_5| + 1)}{C1}} \end{array} \right] \quad (5.1.17)$$

it gives a further condition in order that R3 takes real values, namely:

$$1 - \frac{4 \cdot C2 \cdot Q_5^2 \cdot (|A_5| + 1)}{C1} \geq 0$$

$$\frac{C2}{C1} \leq \frac{1}{4 \cdot Q_5^2 \cdot (|A_5| + 1)}$$

$$\frac{C1}{C2} \geq 4 \cdot Q_5^2 \cdot (|A_5| + 1)$$

$$4 \cdot Q_5^2 \cdot (|A_5| + 1) = 1.283 \times 10^3$$

results:  $C1 \geq 4 \cdot Q_5^2 \cdot (|A_5| + 1) \cdot C2$

(5.1.18)

$$Q_5 = 5.4 \quad A_5 = -10$$

**Poles of the transfer function ( $j = \sqrt{-1}$ ):**

$$\text{WDen} := \text{denom}(W_{lp}(s)) \text{ coeffs, } s \rightarrow \blacksquare$$

$$2 \cdot \zeta = 0.035 \cdot \frac{\text{Grads}}{\text{sec}}$$

$$\text{Search of the poles of } W_{lp}(s): \text{ poles} := \text{polyroots}(\text{WDen}) \blacksquare,$$

$$\text{WDen} = \blacksquare \blacksquare$$

$$A_5 := A_5 \quad s := s \quad \omega_5 := \omega_5 \quad \zeta := \zeta$$

$$\text{denominator polynomial: } p(s) := s^2 + 2 \cdot \zeta \cdot s + \omega_5^2$$

$$\omega_5 = 0.192 \cdot \frac{\text{Grads}}{\text{sec}} \quad \text{poles} := p(s) \left| \begin{array}{l} \text{solve, } s \\ \text{simplify} \end{array} \right. \rightarrow \begin{pmatrix} \sqrt{\zeta^2 - \omega_5^2} - \zeta \\ -\zeta - \sqrt{\zeta^2 - \omega_5^2} \end{pmatrix} \quad (5.1.19)$$

$$\zeta = 0.018 \cdot \frac{\text{Grads}}{\text{sec}} \quad \text{poles} = \begin{pmatrix} -0.018 + 0.191j \\ -0.018 - 0.191j \end{pmatrix} \cdot \frac{\text{Grads}}{\text{sec}}$$

The pulsation  $\omega_5$  is the geometrical mean of the magnitude of the poles:

$$\omega_5 = \sqrt{|\text{poles}_0| \cdot |\text{poles}_1|}$$

$$\omega_5 = 0.192 \cdot \frac{\text{Grads}}{\text{sec}} \quad \sqrt{|\text{poles}_0| \cdot |\text{poles}_1|} = 0.192 \cdot \frac{\text{Grads}}{\text{sec}}$$

For  $\omega = \omega_5$  results:

$$\lim_{\omega \rightarrow \omega_5} (20 \cdot \log(|W(j \cdot \omega)|)) = 20 \cdot \log\left(\frac{|A_5| \cdot \omega_5}{2 \cdot \zeta}\right) = 20 \cdot \log(|A_5| \cdot Q_5)$$

$$20 \cdot \log\left(\frac{|A_5| \cdot \omega_5}{2 \cdot \zeta}\right) = 34.648$$

$$20 \cdot \log(|A_5| \cdot Q_5) = 34.648$$

$$\text{Hence for } \omega = \omega_5 \text{ results: } |A_5| \cdot Q_5 = 1 \quad 20 \cdot \log(|A_5|) = -20 \cdot \log(Q_5)$$

## About Sensitivity

Sensitivity definition given the performance  $\mathcal{P}$  and the parameter  $x_i$ :

$$S_{x,i} = \frac{x_i}{\mathcal{P}} \cdot \frac{\partial \mathcal{P}}{\partial x_i} = \frac{\partial}{\partial \ln x_i} \ln(\mathcal{P})$$

### 5.1.1) Calculation of the circuit performance $A_5$

The filter project could be developed even considering the desired sensitivity of the various parameters.

In this particular case, the performance is the voltage gain:  $\mathcal{P} = A_5 = \frac{-R3}{R1}$

► Sensitivity Calculation

$$\left| \frac{\Delta A_5}{A_5} \right| = |S_{R1}| \cdot \frac{|\Delta R1|}{R1} + |S_{R3}| \cdot \frac{|\Delta R3|}{R3} = \frac{|\Delta R1|}{R1} + \frac{|\Delta R3|}{R3}$$

$$S_{A5} = \frac{\Delta A_5}{A_5} = \frac{|\Delta R1|}{R1} + \frac{|\Delta R3|}{R3} \quad (5.1.1.1)$$



**5.1.2) Calculation of the sensitivity of the circuit performance  $\omega_5$**

$$S_{x,i} = \frac{x_i}{\mathcal{P}} \cdot \frac{\partial \mathcal{P}}{\partial x_i} = \frac{\partial}{\partial \ln x_i} \ln(\mathcal{P})$$

$$\mathcal{P} = \omega_5 = \frac{1}{\sqrt{R_2 \cdot R_3 \cdot C_1 \cdot C_2}}$$

► Sensitivity Calculation

$$S_{\omega_5} = \frac{1}{2} \cdot \left( \frac{|\Delta C_1|}{C_1} + \frac{|\Delta C_2|}{C_2} + \frac{|\Delta R_1|}{R_1} + \frac{|\Delta R_2|}{R_2} \right) \tag{5.1.2.1}$$

$$\frac{|\Delta C_1|}{C_1} + \frac{|\Delta C_2|}{C_2} = 2 \cdot S_{\omega_5} - \left( \frac{|\Delta R_1|}{R_1} + \frac{|\Delta R_2|}{R_2} \right)$$

### 5.1.3) Calculation of the sensitivity of the circuit performance Q

$$S_{x,i} = \frac{x_i}{\mathcal{P}} \cdot \frac{\partial \mathcal{P}}{\partial x_i} = \frac{\partial}{\partial \ln x_i} \ln(\mathcal{P})$$

$$Q_5 = \frac{1}{\sqrt{\frac{C2}{C1} \cdot \left( \frac{\sqrt{R2 \cdot R3}}{R1} + \sqrt{\frac{R3}{R2}} + \sqrt{\frac{R2}{R3}} \right)}}$$

► Sensitivity Calculation

$$S_{Q,5} = \frac{1}{2} \cdot \left( \frac{|\Delta C1|}{C1} + \frac{|\Delta C2|}{C2} \right) + \frac{1}{1 + R1 \cdot \left( \frac{1}{R3} + \frac{1}{R2} \right)} \cdot \frac{|\Delta R1|}{R1} + \frac{1}{2} \cdot \frac{R1 \cdot (R2 - R3) + R2 \cdot R3}{R1 \cdot (R2 + R3) + R2 \cdot R3} \cdot \frac{|\Delta R2|}{R2} \dots$$

$$+ \left| \frac{1}{R3 \cdot \left( \frac{1}{R1} + \frac{1}{R2} \right) + 1} - \frac{1}{2} \right| \cdot \frac{|\Delta R3|}{R3}$$

(5.1.3.1)

$$\text{if: } \frac{|\Delta C1|}{C1} = \frac{|\Delta C2|}{C2} = \frac{|\Delta R1|}{R1} = \frac{|\Delta R2|}{R2} = \frac{|\Delta R3|}{R3} = \text{rtol}$$

$$S_{Q,5} = \left[ \left| \frac{1}{R3 \cdot \left( \frac{1}{R1} + \frac{1}{R2} \right) + 1} - \frac{1}{2} \right| + \frac{1}{\left| R1 \cdot \left( \frac{1}{R2} + \frac{1}{R3} \right) + 1 \right|} + \frac{\left| R2 \cdot \left( \frac{1}{R1} + \frac{1}{R3} \right) - 1 \right|}{2 \cdot \left| R2 \cdot \left( \frac{1}{R1} + \frac{1}{R3} \right) + 1 \right|} + 1 \right] \cdot \text{rtol}$$

### Summary

$$S_{A,5} = \frac{\Delta A_5}{A_5} = \frac{|\Delta R1|}{R1} + \frac{|\Delta R3|}{R3} = 2 \cdot \text{rtol} \quad (5.1.3.2)$$

$$S_{\omega,5} = \frac{1}{2} \cdot \left( \frac{|\Delta C1|}{C1} + \frac{|\Delta C2|}{C2} + \frac{|\Delta R1|}{R1} + \frac{|\Delta R2|}{R2} \right) = 2 \cdot \text{rtol} \quad (5.1.3.3)$$

$$S_{Q,5} = \left[ \left| \frac{1}{R3 \cdot \left( \frac{1}{R1} + \frac{1}{R2} \right) + 1} - \frac{1}{2} \right| + \frac{1}{\left| R1 \cdot \left( \frac{1}{R2} + \frac{1}{R3} \right) + 1 \right|} + \frac{\left| R2 \cdot \left( \frac{1}{R1} + \frac{1}{R3} \right) - 1 \right|}{2 \cdot \left| R2 \cdot \left( \frac{1}{R1} + \frac{1}{R3} \right) + 1 \right|} + 1 \right] \cdot \text{rtol} \quad (5.1.3.4)$$

**Example (5.1.3.1-1)**

are given:  $r_{tol} := 5.0\%$ ,  
 $R1 := 0.047 \cdot k\Omega$ , or  $R1 := 120 \cdot \Omega$   
 $C1 := 1.70 \cdot nF$ ,  
 from the definition of  $A_5$  is:  $R3 := -A_5 \cdot R1$ ,

from the definition of damping factor, is derived:  $R2 := \frac{1}{2 \cdot C1 \cdot \zeta - \left(\frac{1}{R1} + \frac{1}{R3}\right)}$ ,

therefore, in order that  $R2 > 0$ , must be:  $C1 > \frac{1}{2 \cdot \zeta} \cdot \left(\frac{1}{R1} + \frac{1}{R3}\right)$ ,  $\frac{1}{2 \cdot \zeta} \cdot \left(\frac{1}{R1} + \frac{1}{R3}\right) = 0.258 \cdot nF$

namely:  $R2 = 19.546 \cdot \Omega$ .

From the definition of  $\omega_5$ , is derived:  $C2 := \frac{1}{C1 \cdot R2 \cdot R3 \cdot \omega_5^2}$ ,  
 $C2 = 0.683 \cdot pF$ ,

$$\frac{\omega_5}{2 \cdot \zeta} = 5.4$$

$$\frac{\omega_5}{2 \cdot Q_5} = 17.743 \cdot \frac{Mrads}{sec}$$

$$R1 = 0.12 \cdot k\Omega$$

$$4 \cdot Q_5^2 \cdot (|A_5| + 1) \cdot C2 = 0.876 \cdot nF$$

$$R2 = 19.546 \Omega$$

$$C1 = 1.7 \cdot nF$$

$$R3 = 1.2 \cdot k\Omega$$

Voltage gain:  $-\frac{R3}{R1} = -10$

Pole Q factor:  $\frac{1}{\sqrt{\frac{C2}{C1} \cdot \left(\frac{\sqrt{R2 \cdot R3}}{R1} + \sqrt{\frac{R3}{R2}} + \sqrt{\frac{R2}{R3}}\right)}} = 5.4$

Geometric mean of the poles and the resulting frequency:

$$\frac{1}{\sqrt{R2 \cdot R3 \cdot C1 \cdot C2}} = 0.192 \cdot \frac{Grads}{sec}$$

$$\omega_5 = 0.192 \cdot \frac{Grads}{sec}$$

The previous findings ( $R1, R2, R3$ ), after a slight modification of one or more of them, may be used as a guess for a new solution (not always found, also after many attempts) of the system, as follows:

Given

(5.1.3.5)

$$\omega_5 = \frac{1}{\sqrt{R2 \cdot R3 \cdot C1 \cdot C2}}$$

$$A_5 = -\frac{R3}{R1}$$

$$Q_5 = \frac{1}{\sqrt{\frac{C2}{C1} \cdot \left(\frac{\sqrt{R2 \cdot R3}}{R1} + \sqrt{\frac{R3}{R2}} + \sqrt{\frac{R2}{R3}}\right)}}$$



$$\zeta = \frac{1}{2} \cdot \left[ \frac{1}{C1} \cdot \left( \frac{1}{R1} + \frac{1}{R2} + \frac{1}{R3} \right) \right]$$

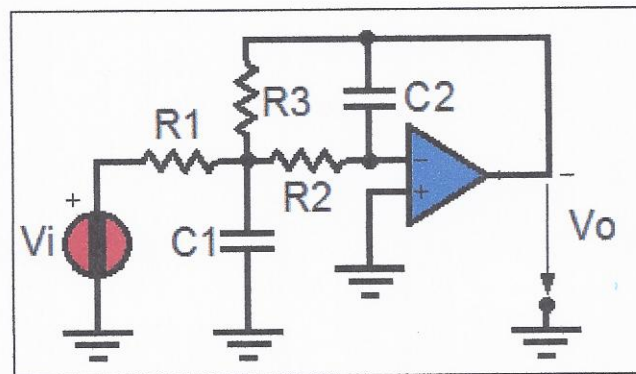
$$R1 > 0.0 \cdot \Omega$$

$$R2 > 0.0 \cdot \Omega$$

$$R3 > 0.0 \cdot \Omega$$

$$\boxed{Rx := \text{Find}(R1, R2, R3)}$$

$$Rx = \begin{pmatrix} 120 \\ 19.546 \\ 1.2 \times 10^3 \end{pmatrix} \cdot \Omega \quad C1 = 1.7 \cdot \text{nF} \\ C2 = 6.83 \times 10^{-4} \cdot \text{nF}$$



$$A_5 = -10$$

$$\omega_5 = 0.192 \cdot \frac{\text{Grads}}{\text{sec}}$$

$$2 \cdot \zeta = 0.035 \cdot \frac{\text{Grads}}{\text{sec}}$$

fig.:5.1.19

### Sensitivity for this example

$$S_{A.5} = \left| \frac{\Delta A_5}{A_5} \right| \quad S_{A.5} := (\text{rtol} + \text{rtol}) \quad \text{rtol} = 5\% \quad (5.1.3.6)$$

$$S_{\omega.5} = \frac{|\Delta \omega_5|}{|\omega_5|} \quad S_{\omega.5} := \frac{1}{2} \cdot (\text{rtol} + \text{rtol} + \text{rtol} + \text{rtol}) \quad (5.1.3.7)$$

$$S_{Q.5} := \left[ \left| \frac{1}{R3 \cdot \left( \frac{1}{R1} + \frac{1}{R2} \right) + 1} - \frac{1}{2} \right| + \left| \frac{1}{R1 \cdot \left( \frac{1}{R2} + \frac{1}{R3} \right) + 1} \right| + \frac{\left| R2 \cdot \left( \frac{1}{R1} + \frac{1}{R3} \right) - 1 \right|}{2 \cdot \left| R2 \cdot \left( \frac{1}{R1} + \frac{1}{R3} \right) + 1 \right|} + 1 \right] \cdot \text{rtol}$$

$$\text{results: } S_{A.5} = 10\% \quad S_{\omega.5} = 10\% \quad S_{Q.5} = 9.862\% \quad (5.1.3.8)$$

## 5.1.4 NYQUIST DIAGRAM

Op Amp open loop voltage gain:  $A_{ol} := 10^5$

After the following substitution:

$$GH_5(s) := \frac{[(Y1 + Y2 + Y3 + Y4) \cdot Y5 + Y3 \cdot Y4] \cdot A_{ol}}{(Y1 + Y2 + Y3 + Y4) \cdot Y5 + Y3 \cdot (Y1 + Y2 + Y4)}$$

substitute,  $Y1 = \frac{1}{R1}, Y2 = s \cdot C1, Y3 = \frac{1}{R2}$

substitute,  $Y4 = \frac{1}{R3}, Y5 = s \cdot C2$

→

results the already found relation (5.1.13), her rewritten:

$$GH_5(s) := \frac{s^2 + \frac{[R1 \cdot (R2 + R3) + R2 \cdot R3]}{C1 \cdot R1 \cdot R2 \cdot R3} \cdot s + \frac{1}{(C1 \cdot C2 \cdot R2 \cdot R3)}}{s^2 + \frac{[C2 \cdot [R1 \cdot (R2 + R3) + R2 \cdot R3] + C1 \cdot R1 \cdot R3]}{(C1 \cdot C2 \cdot R1 \cdot R2 \cdot R3)} \cdot s + \frac{R1 + R3}{C1 \cdot C2 \cdot R1 \cdot R2 \cdot R3}} \cdot A_{ol} \quad (5.1.13')$$

Place:

$$\omega_5 = \frac{1}{\sqrt{R2 \cdot R3 \cdot C1 \cdot C2}}$$

$$\omega_1 := \frac{[R1 \cdot (R2 + R3) + R2 \cdot R3]}{C1 \cdot R1 \cdot R2 \cdot R3}$$

$$\omega_2 = \frac{R1 \cdot (R2 + R3) + R2 \cdot R3}{C1 \cdot R1 \cdot R2 \cdot R3} + \frac{1}{C2 \cdot R2} = \omega_1 + \frac{1}{R2 \cdot C2}$$

$$\omega_3 := \sqrt{\frac{R1 + R3}{R1} \cdot \omega_5^2}$$

$$\omega_2 := \omega_1 + \frac{1}{R2 \cdot C2}$$

$$\omega_5 = 0.192 \cdot \frac{\text{Grads}}{\text{sec}}$$

$$\omega_1 = 0.035 \cdot \frac{\text{Grads}}{\text{sec}}$$

$$\omega_2 = 74.946 \cdot \frac{\text{Grads}}{\text{sec}}$$

$$\omega_3 = 0.636 \cdot \frac{\text{Grads}}{\text{sec}}$$

$$\omega_3^2 = 0.404 \cdot \left(\frac{\text{Grads}}{\text{sec}}\right)^2$$

$s := s$

so that one can write:

$$GH_5(s) := \frac{s^2 + \omega_1 \cdot s + \omega_5^2}{s^2 + \omega_2 \cdot s + \omega_3^2} \cdot A_{ol} \quad (5.1.4.2)$$

$$\omega := \frac{\omega_1}{10^1}, \frac{\omega_1}{10^1} + \frac{\left(2 \cdot 10^2 \cdot \omega_2 - \frac{\omega_1}{10^1}\right)}{10^4} \cdot 2 \cdot 10^2 \cdot \omega_2 \quad \frac{\omega_1}{10^1} = 3.549 \times 10^{-3} \cdot \frac{\text{Grads}}{\text{sec}}$$

### Magnitude of the Open Loop Gain GH5

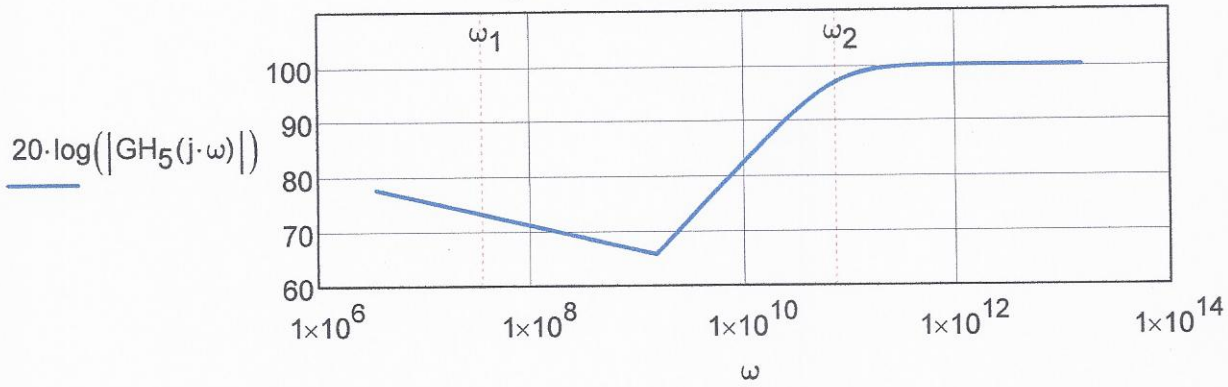


fig.:5.1.4.1

### Phase of the Open Loop Gain GH5

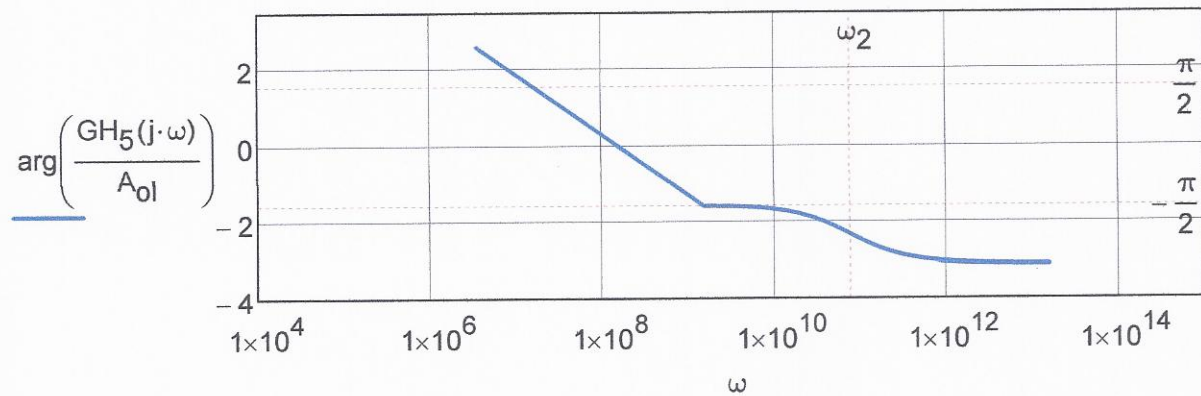


fig.:5.1.4.2

$$\omega_x := -2 \cdot 10^2 \cdot \omega_2, -2 \cdot 10^2 \cdot \omega_2 + \frac{2 \cdot 10^2 \cdot \omega_2 + 2 \cdot 10^2 \cdot \omega_2}{10^5} .. 2 \cdot 10^2 \cdot \omega_2$$

To let see the inner loop of the Nyquist diagram it has been defined the following new interval:

$$\omega_x := -4 \cdot 10^{-4} \cdot \omega_2, -4 \cdot 10^{-4} \cdot \omega_2 + \frac{4 \cdot 10^{-4} \cdot \omega_2 + 4 \cdot 10^{-4} \cdot \omega_2}{10^5} .. 4 \cdot 10^{-4} \cdot \omega_2$$

The Nyquist diagram of the normalized open loop gain is composed by two circles, both tangent to the Origin:

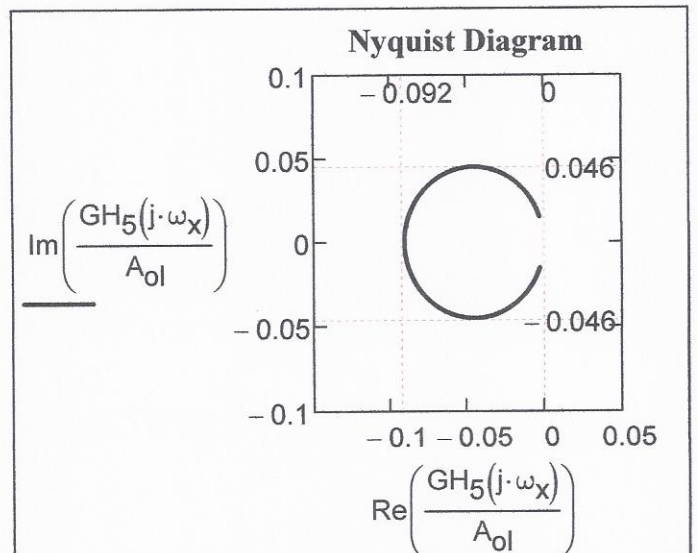
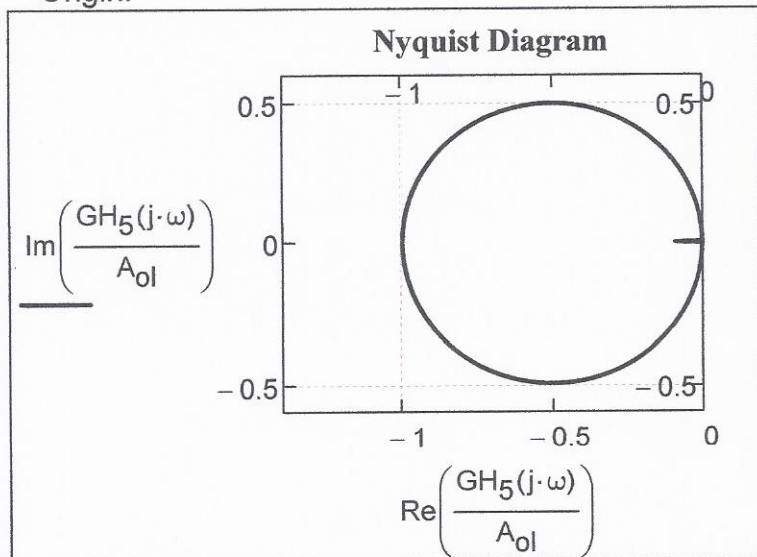




fig.:5.1.4.3

Desensitization:

$$D(s) = 1 + GH_5(s) = \frac{[[Y_5 \cdot (Y_1 + Y_2 + Y_3 + Y_4) + Y_3 \cdot Y_4] \cdot (1 - A_{ol}) + Y_3 \cdot (Y_1 + Y_2)]}{[(Y_1 + Y_2 + Y_3 + Y_4) \cdot Y_5 + Y_3 \cdot (Y_1 + Y_2 + Y_4)]} \quad (5.1.4.3)$$

$$\frac{1}{D(s)} = \frac{[(Y_1 + Y_2 + Y_3 + Y_4) \cdot Y_5 + Y_3 \cdot (Y_1 + Y_2 + Y_4)]}{[[Y_5 \cdot (Y_1 + Y_2 + Y_3 + Y_4) + Y_3 \cdot Y_4] \cdot (1 - A_{ol}) + Y_3 \cdot (Y_1 + Y_2)]} \quad (5.1.4.4)$$

$$W(s) = \lim_{A_{ol} \rightarrow \infty} \frac{(Y_1 + Y_2 + Y_3 + Y_4) \cdot Y_5 + Y_3 \cdot (Y_1 + Y_2 + Y_4)}{[Y_5 \cdot (Y_1 + Y_2 + Y_3 + Y_4) + Y_3 \cdot Y_4] \cdot \left(\frac{1 - A_{ol}}{A_{ol}}\right) + \frac{Y_3 \cdot (Y_1 + Y_2)}{A_{ol}}}$$

$$W(s) = \frac{(Y_1 + Y_2 + Y_3 + Y_4) \cdot Y_5 + Y_3 \cdot (Y_1 + Y_2 + Y_4)}{Y_5 \cdot (Y_1 + Y_2 + Y_3 + Y_4) + Y_3 \cdot Y_4}$$

## 5.1.5 Nichols chart

From the analysis of a generic L.T.I. system with negative feedback, can be drawn some conclusion summarized here below:

### NEGATIVE FEEDBACK SYSTEM

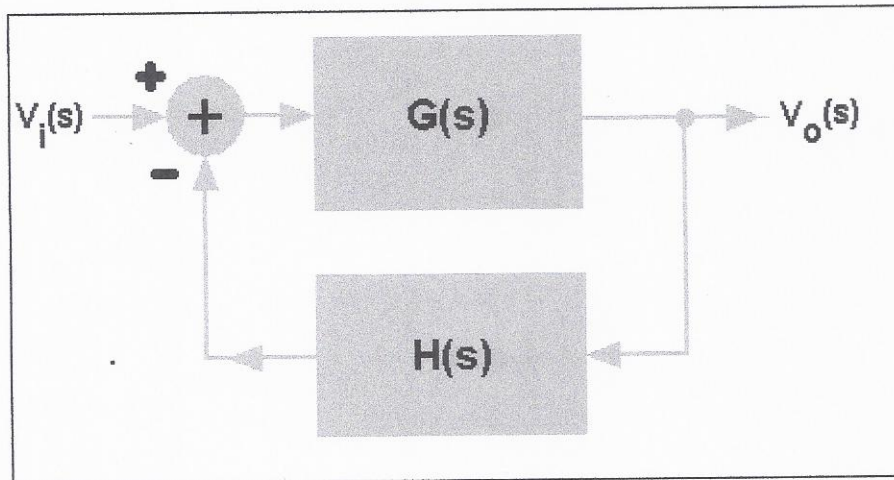


fig.:5.1.5.1

System transfer function with negative feedback:

$$G_f(s) = \frac{G(s)}{1 + G(s) \cdot H(s)} \quad 5.1.5.1$$

System's parameters variation: desensitization.

Ratio of the relative gain variations with and without feedback:

$$\left| \frac{\frac{\Delta G_f(s)}{G_f(s)}}{\frac{\Delta G(s)}{G(s)}} \right| = \left| \frac{1}{D(s)} \right| \quad 5.1.5.2$$

Low frequency voltage gain in dB:  $W_{lpp} := 20 \cdot \log(|A_5|)$   $W_{lpp} = 20$

Open loop gain( $GH_5(s) = G(s) \cdot H(s) = \frac{s^2 + \omega_1 \cdot s + \omega_5^2}{s^2 + \omega_2 \cdot s + \omega_3^2} \cdot A_{ol}$ ):

$$GH_5(s) := \frac{s^2 + \omega_1 \cdot s + \omega_5^2}{s^2 + \omega_2 \cdot s + \omega_3^2} \cdot A_{ol} \quad 5.1.5.3$$

In the following Nichols chart, the magnitude in dB of the open loop gain is in **purple red** depicted. The **red scarlet** line refers to the Magnitude in dB of the **open loop gain = 0dB**. The yellow arc is

related to  $\phi = \pm \frac{\pi}{2}$ .

$$\phi(\omega) = \arg(G(j \cdot \omega) \cdot H(j \cdot \omega)) \quad \alpha_{nch5}(\omega) = \arg(G(j \cdot \omega) \cdot H(j \cdot \omega)) \quad 5.1.5.4$$

**Scroll The Slider to Zoom In or Out The Nichols Chart**

$mnp_{nch} :=$

Nichol's chart lower left corner:  $m_{nch5} := 1.5$

Nichol's chart lower right corner:  $n_{nch} := 0.0$

Nichol's chart upper left corner:  $p_{nch} := mnp_{nch} \cdot 10$       $p_{nch} = 100$

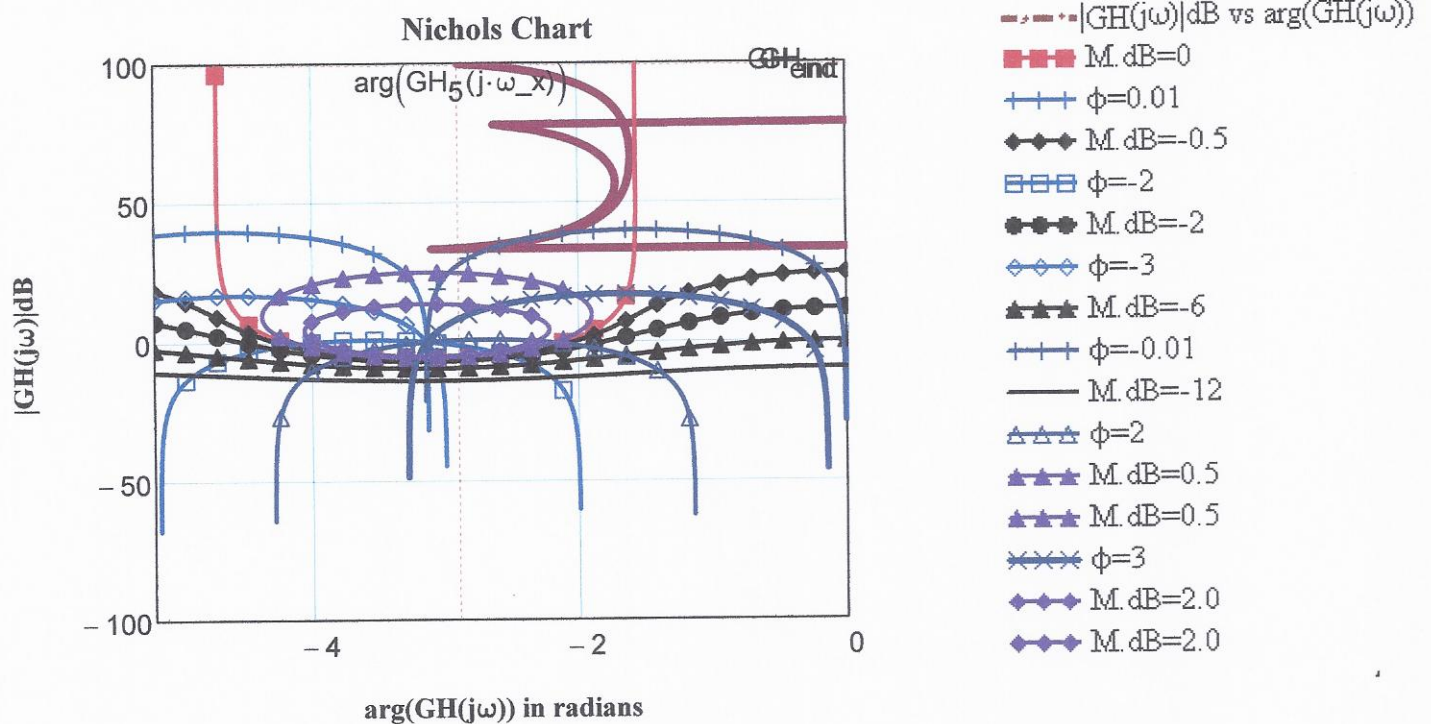
$$\alpha_{nch5} := -\pi \cdot m_{nch5} \cdot 1.1, -\pi \cdot m_{nch5} \cdot 1.1 + \frac{2 \cdot \pi \cdot m_{nch5}}{10^4} .. m_{nch5} \cdot \pi$$

The values of the *gain margin* and the *phase margin* can be deduced observing the graph. Therefore have a measure of the degree of stability of the system with feedback.

$$\omega_x := 4 \cdot 10^0 \cdot \omega_2 \quad GH_{init} := 20 \log[|GH_5[j \cdot (-\omega_x)]|] \quad GH_{end} := 20 \log[|GH_5[j \cdot (\omega_x)]|]$$

$$\omega_1 = 0.035 \cdot \frac{\text{Grads}}{\text{sec}} \quad \omega := -\omega_x, -\omega_x + \frac{\omega_x + \omega_x}{10^5 \cdot 2} .. \omega_x$$

$$\omega_{test} = 0.383 \cdot \frac{\text{Grads}}{\text{sec}} \quad \omega_2 = 74.946 \cdot \frac{\text{Grads}}{\text{sec}} \quad Q_5 = 5.4$$



The *gain margin* is defined as  $g_{mg} = \frac{1}{|GH_5(j \cdot \omega_x)|}$  and is calculated at the frequency  $\omega_x$  where

$$arg(GH(j \cdot \omega_x)) = -\pi \text{ or } \pi. \text{ (For amplifiers it should be } g_{mg} \geq 4)$$

The *phase margin*, on the other hand, is defined as  $ph_{mg} = \pi - |arg(GH_5(j \cdot \omega_y))|$  calculated at the

frequency  $\omega_y$  where  $|GH(j \cdot \omega_y)| = 1$ . (For amplifiers it should be  $ph_{mg} \geq \frac{\pi}{3}$ )



### 5.1.6 Four particular cases of the transfer function .

1a)  $R_1 = R_2 = R_3 = R$ , and  $C_1 = C_2 = C$ ,

2a)  $R_2 = R_3 = R$ , and  $C_1 = C_2 = C$ ,

3a)  $R_1 = R_2 = R$ , and  $C_1 = C_2 = C$ ,

4a)  $R_1 = R_2 = R_3 = R$ ,

#### 1a) $R_1=R_2=R_3=R$ , and $C_1=C_2=C$

Substituting in the transfer function,  $W_{lp}(s) = \frac{\frac{1}{R_1 \cdot R_2 \cdot C_1 \cdot C_2}}{s^2 + s \cdot \frac{1}{C_1} \cdot \left( \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \right) + \frac{1}{R_2 \cdot R_3 \cdot C_1 \cdot C_2}}$ ,

it becomes:  $W_{lp}(s) = \frac{\frac{1}{R^2 \cdot C^2}}{s^2 + s \cdot \frac{3}{C \cdot R} + \frac{1}{R^2 \cdot C^2}}$  (5.1.6.1)

$$A_5 \cdot \omega_5^2 = -\frac{1}{R^2 \cdot C^2} \quad R_3 = R = \frac{1}{2 \cdot C \cdot Q_5 \cdot \omega_5} \cdot \left[ \frac{1 + \sqrt{1 - 4 \cdot Q_5^2 \cdot (|A_5| + 1)}}{1 - \sqrt{1 - 4 \cdot Q_5^2 \cdot (|A_5| + 1)}} \right] \quad (5.1.6.2)$$

condition in order that R takes real values  $|A_5| = 1$

$$\frac{1}{2 \cdot \sqrt{2}} = 0.354 \quad Q_5 \leq \frac{1}{2 \cdot \sqrt{|A_5| + 1}} = \frac{1}{2 \cdot \sqrt{2}}$$

In summary, it can be written: a)  $\omega_5 = \frac{1}{\sqrt{R_2 \cdot R_3 \cdot C_1 \cdot C_2}} = \frac{1}{\sqrt{R \cdot R \cdot C \cdot C}} = \frac{1}{R \cdot C}$ , (5.1.6.3)

b)  $A_5 = \frac{R_3}{R_1} = -1$ , (5.1.6.4)

c)  $Q_5 = \frac{1}{\sqrt{\frac{C_2}{C_1} \cdot \left( \frac{\sqrt{R_2 \cdot R_3}}{R_1} + \sqrt{\frac{R_3}{R_2}} + \sqrt{\frac{R_2}{R_3}} \right)}} = \frac{1}{3}$ , (5.1.6.5)

d)  $\zeta = \frac{1}{2} \cdot \left[ \frac{1}{C_1} \cdot \left( \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \right) \right] = \frac{3}{2 \cdot R \cdot C}$ , (5.1.6.6)

e)  $\frac{\omega_5}{2 \cdot \zeta} = \frac{1}{3}$ . (5.1.6.7)

2a)  $R_2=R_3=R$ , and  $C_1=C_2=C$

$$A_5 \cdot \omega_5^2 = \frac{1}{R_1 \cdot R \cdot C \cdot C}$$

$$W_{lp}(s) = \frac{\frac{1}{R_1 \cdot R \cdot C^2}}{s^2 + s \cdot \frac{1}{C} \cdot \left( \frac{1}{R_1} + \frac{2}{R} \right) + \frac{1}{R^2 \cdot C^2}} \quad (5.1.6.8)$$

$$R = \frac{1}{2 \cdot C \cdot Q_5 \cdot \omega_5} \cdot \left[ \frac{1 + \sqrt{1 - 4 \cdot Q_5^2 \cdot (|A_5| + 1)}}{1 - \sqrt{1 - 4 \cdot Q_5^2 \cdot (|A_5| + 1)}} \right] \quad (5.1.6.9)$$

Condition in order that  $R_3$  takes real values:

$$1 \geq 4 \cdot Q_5^2 \cdot (|A_5| + 1) \quad (5.1.6.10)$$

$$Q_5 \leq \frac{1}{2 \cdot \sqrt{|A_5| + 1}} \quad \frac{1}{2 \cdot \sqrt{|A_5| + 1}} = 0.151$$

In summary, it can be written: a)  $\omega_5 = \frac{1}{\sqrt{R_2 \cdot R_3 \cdot C_1 \cdot C_2}} = \frac{1}{\sqrt{R \cdot R \cdot C \cdot C}} = \frac{1}{R \cdot C}$ , (5.1.6.11)

b)  $A_5 = \frac{R_3}{R_1} = \frac{R}{R_1}$ , (5.1.6.12)

c)  $Q_5 = \frac{1}{\sqrt{\frac{C_2}{C_1} \cdot \left( \frac{\sqrt{R_2 \cdot R_3}}{R_1} + \sqrt{\frac{R_3}{R_2}} + \sqrt{\frac{R_2}{R_3}} \right)}} = \frac{1}{\left( \frac{R}{R_1} + 2 \right)} = \frac{1}{2 - A_5}$ ,  
 that holds for  $Q_5 < \frac{1}{2}$  or  $A_5 < 2$ . (5.1.6.13)

d)  $\zeta = \frac{1}{2} \cdot \left[ \frac{1}{C_1} \cdot \left( \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \right) \right] = \frac{3}{2 \cdot R \cdot C}$ , (5.1.6.14)

e)  $\frac{\omega_5}{2 \cdot \zeta} = \frac{1}{3} = \frac{1}{2 - A_5}$ . (5.1.6.15)

3a)  $R1=R2=R$ , and  $C1=C2=C$

$$W_{lp}(s) = \frac{\frac{1}{R^2 \cdot C^2}}{s^2 + s \cdot \frac{1}{C} \cdot \left( \frac{2}{R} + \frac{1}{R3} \right) + \frac{1}{R \cdot R3 \cdot C^2}} \quad (5.1.6.16)$$

$$R3 = \frac{1}{2 \cdot C \cdot Q_5 \cdot \omega_5} \cdot \left[ \frac{1 + \sqrt{1 - 4 \cdot Q_5^2 \cdot (|A_5| + 1)}}{1 - \sqrt{1 - 4 \cdot Q_5^2 \cdot (|A_5| + 1)}} \right] \quad (5.1.6.17)$$

condition in order that R3 takes real values

$$A_5 = -10 \quad Q_5 \leq \frac{1}{2 \cdot \sqrt{|A_5| + 1}} \quad \frac{1}{2 \cdot \sqrt{|A_5| + 1}} = 0.151$$

In summary, it can be written:

$$a) \quad \omega_5 = \frac{1}{\sqrt{R \cdot R3 \cdot C \cdot C}} = \frac{1}{C \cdot \sqrt{R \cdot R3}}, \quad (5.1.6.18)$$

$$A_5 \cdot \omega_5^2 = \frac{A_5}{(R \cdot R3 \cdot C \cdot C)} = \frac{1}{R^2 \cdot C^2}, \quad (5.1.6.19)$$

$$b) \quad A_5 = -\frac{R3}{R}, \quad (5.1.6.20)$$

$$c) \quad Q_5 = \frac{1}{\sqrt{\frac{C}{C} \cdot \left( \frac{\sqrt{R \cdot R3}}{R} + \sqrt{\frac{R3}{R}} + \sqrt{\frac{R}{R3}} \right)}} = \frac{1}{\left( 2 \cdot \sqrt{\frac{R3}{R}} + \sqrt{\frac{R}{R3}} \right)} = \frac{1}{\left( 2 \cdot \sqrt{|A_5|} + \sqrt{\frac{1}{|A_5|}} \right)}, \quad (5.1.6.2)$$

$$d) \quad \zeta = \frac{1}{2} \cdot \frac{1}{C} \cdot \left( \frac{1}{R} + \frac{1}{R} + \frac{1}{R3} \right) = \frac{1}{2 \cdot C} \cdot \left( \frac{2}{R} + \frac{1}{R3} \right), \quad (5.1.6.22)$$

$$e) \quad \frac{\omega_5}{2 \cdot \zeta} = \frac{1}{\sqrt{\frac{R}{R3} + 2 \cdot \sqrt{\frac{R3}{R}}}} \quad (5.1.6.23)$$

from b) it follows:  $\frac{R3}{R} = -A_5 \quad A_5 < 0 \quad \frac{R3}{R} = |A_5|$

$$Q_5 = \frac{1}{2 \cdot \sqrt{|A_5|} + \frac{1}{\sqrt{|A_5|}}} \quad (5.1.6.24)$$

$$\sqrt{|A_5|} = \left( \frac{\sqrt{1 - 8 \cdot Q_5^2 + 1}}{4 \cdot Q_5} \right) \quad \sqrt{|A_5|} = \frac{\sqrt{1 - 8 \cdot Q_5^2 + 1}}{4 \cdot Q_5}$$

$$\left( \frac{\sqrt{1 - 8 \cdot Q_5^2 - 1}}{4 \cdot Q_5} \right)$$

$$1 - 8 \cdot Q_5^2 > 0$$

condition in order that R3 takes real values

$$Q_5 \leq \frac{1}{2 \cdot \sqrt{|A_5| + 1}} \quad \boxed{0 < Q_5 < \frac{1}{2 \cdot \sqrt{2}}} \quad (5.1.6.25)$$

$$\frac{1}{2 \cdot \sqrt{2}} = 0.3536 \quad \frac{1}{2 \cdot \sqrt{|A_5| + 1}} = 0.151$$

4a)  $R_1=R_2=R_3=R$

$$W_{lp}(s) = \frac{\frac{1}{R^2 \cdot C_1 \cdot C_2}}{s^2 + s \cdot \frac{1}{C_1} \cdot \frac{3}{R} + \frac{1}{R^2 \cdot C_1 \cdot C_2}} \quad (5.1.6.26)$$

$$R_3 = R = \frac{1}{2 \cdot C_2 \cdot Q_5 \cdot \omega_5} \cdot \left[ \begin{array}{l} 1 + \sqrt{1 - \frac{4 \cdot C_2 \cdot Q_5^2 \cdot (|A_5| + 1)}{C_1}} \\ 1 - \sqrt{1 - \frac{4 \cdot C_2 \cdot Q_5^2 \cdot (|A_5| + 1)}{C_1}} \end{array} \right] \quad (5.1.6.27)$$

condition in order that R3 takes real values:

$$\boxed{\frac{C_1}{C_2} \geq 4 \cdot Q_5^2 \cdot (|A_5| + 1)} \quad Q_5 \leq \frac{1}{2} \cdot \sqrt{\frac{C_1}{C_2 \cdot (|A_5| + 1)}} \quad (5.1.6.28)$$

In summary, it can be written: a)  $\omega_5 = \frac{1}{\sqrt{R_2 \cdot R_3 \cdot C_1 \cdot C_2}} = \frac{1}{R \cdot \sqrt{C_1 \cdot C_2}}$ , (5.1.6.29)

b)  $A_5 = -\frac{R_3}{R_1} = -1$ , (5.1.6.30)

c)  $Q_5 = \frac{1}{\sqrt{\frac{C_2}{C_1} \cdot \left( \frac{\sqrt{R_2 \cdot R_3}}{R_1} + \sqrt{\frac{R_3}{R_2}} + \sqrt{\frac{R_2}{R_3}} \right)}} = \frac{1}{3} \cdot \sqrt{\frac{C_1}{C_2}}$  (5.1.6.31)

$$\zeta = \frac{\omega_5}{2 \cdot Q_5} = \frac{\frac{1}{R \cdot \sqrt{C_1 \cdot C_2}}}{2 \cdot \frac{1}{\sqrt{\frac{C_2}{C_1}} \cdot 3}} = \frac{3 \cdot \sqrt{\frac{C_2}{C_1}}}{2 \cdot R \cdot \sqrt{C_1 \cdot C_2}} = \frac{3}{2 \cdot R \cdot C_1} \quad (5.1.6.32)$$

condition in order that R3 takes real values  $Q_5 \leq \frac{1}{2} \cdot \sqrt{\frac{C_1}{C_2 \cdot (|A_5| + 1)}}$  (5.1.6.33)

$$\frac{1}{2} \cdot \sqrt{\frac{C_1}{C_2 \cdot (|A_5| + 1)}} = 7.521 \quad \frac{1}{3} \cdot \sqrt{\frac{C_1}{C_2}} = 16.631$$



## 5.1.7 Pulse response.

General case:  $R1 \neq R2 \neq R3, C1 \neq C2$

Known values:

Voltage gain:  $A_5 = -10$

Pole Q factor:  $Q_5 = 5.4$  ,

Pole pulsation:  $\omega_5 = 0.192 \cdot \frac{\text{Grads}}{\text{sec}}$

damping factor:  $\zeta = 0.018 \cdot \frac{\text{Grads}}{\text{sec}}$

Graph of the pulse response:

Transfer function:  $W_{lp}(s) := \begin{cases} \frac{A_5 \cdot \omega_5^2}{s^2 + 2 \cdot \zeta \cdot s + \omega_5^2} & \text{if } \zeta \neq \omega_5 \\ A_5 \cdot \frac{\omega_5^2}{(s + \omega_5)^2} & \text{otherwise} \end{cases}$  (5.1.7.1)

$A_5 := A_5 \quad s := s \quad a := a \quad \omega_5 := \omega_5 \quad \zeta := \zeta$

Calculation of the pulse response as the inverse Laplace transform of the t. f.:

$w(t) := \frac{A_5 \cdot \omega_5^2}{s^2 + 2 \cdot \zeta \cdot s + \omega_5^2} \begin{cases} \text{invlaplace, s, t} \\ \text{rewrite, exp} \rightarrow \\ \text{simplify, max} \end{cases}$

Dirac pulse response chosen:

Pulse response:  $w(t) := A_5 \cdot \omega_5^2 \cdot t \cdot \text{sinc}\left(t \cdot \sqrt{\omega_5^2 - \zeta^2}\right) \cdot e^{-\zeta \cdot t} \cdot \Phi(t)$  (5.1.7.2)

Search of the minimum:

$\frac{\partial}{\partial t} w(t) = 0$  for  $t = tx2 := \begin{cases} \frac{2 \cdot \text{atan}\left(\frac{\zeta - \omega_5}{\sqrt{\omega_5^2 - \zeta^2}}\right)}{\sqrt{\omega_5^2 - \zeta^2}} & \text{if } \zeta \neq \omega_5 \\ \frac{1}{\zeta} & \text{otherwise} \end{cases}$  , (5.1.7.3)

$tx2 = 7.747 \cdot \text{ns}$

Minimum:  $w(tx2) = -1.67 \cdot \frac{\text{Grads}}{\text{sec}}$

Initial value theorem:  $\lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} (s \cdot F(s))$  ,

Final value theorem:  $\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} (s \cdot F(s))$

$A_5 := A_5 \quad s := s \quad a := a \quad \omega_5 := \omega_5$

$\zeta := \zeta$

$\lim_{s \rightarrow \infty} \left( s \cdot \frac{A_5 \cdot \omega_5^2}{s^2 + 2 \cdot \zeta \cdot s + \omega_5^2} \right) \rightarrow 0 \quad \omega_5^2 - \zeta^2 = 0.036 \cdot \left( \frac{\text{Grads}}{\text{sec}} \right)^2$

$A_5 := A_5 \quad s := s \quad a := a \quad \omega_5 := \omega_5$

$\zeta := \zeta$

$\lim_{s \rightarrow 0} \left( s \cdot \frac{A_5 \cdot \omega_5^2}{s^2 + 2 \cdot \zeta \cdot s + \omega_5^2} \right) \rightarrow 0$

$tx2 = 7.747 \cdot ns$

$t_w := -1 \cdot T_{test}, -1 \cdot T_{test} + \frac{20 \cdot T_5 + 1 \cdot T_{test}}{10000} .. 20 \cdot T_5$

▶ Right Lower Corner Graph Control

Graph of the impulse response.

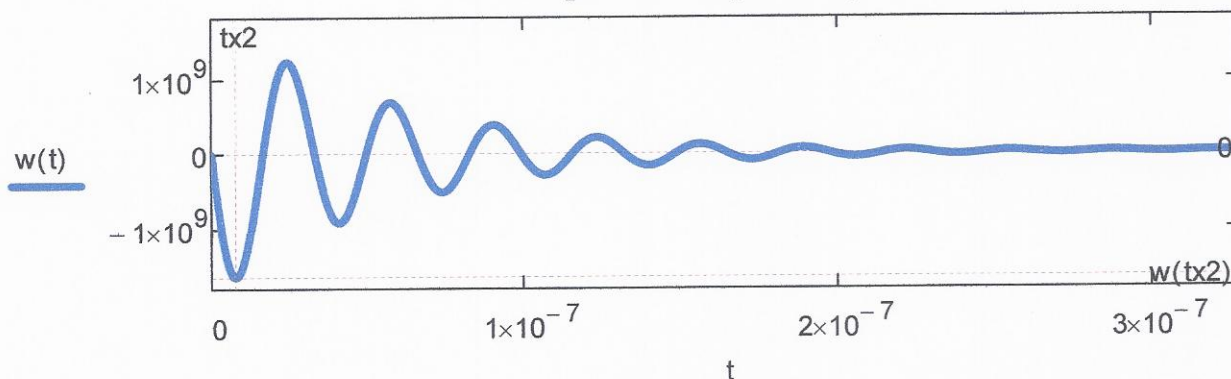


fig.:5.1.7.1

**5.1.8 BODE PLOTS (Low Pass II° order):**

For time harmonic signal place:  $s=j\omega$  and call the magnitude in dB of the frequency response as follows:

$WlpdB(\omega) := 20 \cdot \log(|Wlp(j \cdot \omega)|) \quad WlpdB(\omega_5) = 34.648$

now proceed to its computing:

$A_5 := A_5 \quad s := s \quad a := a \quad \omega_5 := \omega_5 \quad \zeta := \zeta \quad \omega := \omega$

$Wlp_-(\omega) := \frac{A_5 \cdot \omega_5^2}{s^2 + 2 \cdot \zeta \cdot s + \omega_5^2} \quad \left| \begin{array}{l} \text{substitute, } s = j \cdot \omega \\ \text{simplify, max} \\ \text{collect, } \omega \end{array} \right. \rightarrow$

$$W_{lp\_}(\omega) := \frac{\omega^2 - \omega_5^2 + 2j\zeta\omega}{4\zeta^2\omega^2 + \omega^4 - 2\omega^2\omega_5^2 + \omega_5^4} \cdot \omega_5^2 \cdot A_5 \quad (5.1.8.1)$$

Considering the poles, the transfer function can be rewritten as:

$$W_{lp\_}(\omega) := \frac{A_5 \cdot \omega_5^2}{[j\omega - (\sqrt{\zeta^2 - \omega_5^2} - \zeta)] \cdot (j\omega + \zeta + \sqrt{\zeta^2 - \omega_5^2})} \quad (5.1.8.2)$$

Hence the magnitude of the frequency response in dB is:

$$A_5 = -|A_5| \quad W_{lpdB}(\omega) := 20 \cdot \log \left[ |A_5| \cdot \omega_5^2 \cdot \frac{\sqrt{(-\omega^2 + \omega_5^2)^2 + (-2\zeta\omega)^2}}{\omega^4 + \omega_5^4 + \omega^2 \cdot (4\zeta^2 - 2\omega_5^2)} \right] \quad (5.1.8.3)$$

The phase response is:

$$\varphi_S(\omega) := \pi - \operatorname{atan} \left[ \frac{\omega}{-(\sqrt{\zeta^2 - \omega_5^2} - \zeta)} \right] - \operatorname{atan} \left[ \frac{\omega}{(\zeta + \sqrt{\zeta^2 - \omega_5^2})} \right] \quad (5.1.8.4)$$

If  $Q_5 > 0.5$  the frequency response presents a overshoot at:

$$\omega_{pick} := \begin{cases} \sqrt{\omega_5^2 - 2\zeta^2} & \text{if } \zeta \neq \omega_5 \wedge \omega_5 > \sqrt{2}\zeta \\ \omega_5 & \text{otherwise} \end{cases}$$

$$\omega_{pick} = 0.19 \cdot \frac{\text{Grads}}{\text{sec}}$$

and the pick amplitude is

$$W_{lpdB}_{pick} := \begin{cases} 20 \cdot \log \left[ \frac{\omega_5^2 \cdot |A_5|}{2 \cdot \sqrt{\zeta^2 \cdot (\omega_5^2 - \zeta^2)}} \right] & \text{if } \zeta \neq \omega_5 \wedge \omega_5 > \sqrt{2}\zeta \\ 20 \cdot \log(|A_5|) & \text{otherwise} \end{cases} \quad (5.1.8.5)$$

$$\text{pick amplitude } W_{lpdB}_{pick} = 34.685$$

Bandwidth calculation.

The bandwidth is given by the frequency at which the magnitude of the transfer function is  $\frac{|A_5|}{\sqrt{2}}$ , ( $s=j\omega$ ), namely:

$$|A_5| \cdot \omega_5^2 \cdot \frac{\sqrt{(-\omega^2 + \omega_5^2)^2 + (-2\zeta\omega)^2}}{\omega^4 + \omega_5^4 + \omega^2 \cdot (4\zeta^2 - 2\omega_5^2)} = \frac{|A_5|}{\sqrt{2}}$$



Bandwidth:

$$\boxed{Bw := \frac{\sqrt{\omega_5^2 - 2 \cdot \zeta^2 + \sqrt{2} \cdot \sqrt{2 \cdot \zeta^2 \cdot (\zeta^2 - \omega_5^2) + \omega_5^4}}}{2 \cdot \pi}} \quad (5.1.8.6)$$

Knowing the bandwidth one can determine  $\omega_5$  vs.  $Q_5$  and Bw:

$$Bw = 47.1 \cdot \text{MHz}$$

$$\omega_5 = 0.192 \cdot \frac{\text{Grads}}{\text{sec}} \quad \omega_5 := \sqrt{\frac{(Bw \cdot 2 \cdot \pi)^2 \cdot \left[ \sqrt{4 \cdot Q_5^2 \cdot (2 \cdot Q_5^2 - 1) + 1} - 2 \cdot Q_5^2 + 1 \right]}{2 \cdot Q_5^2}} \quad (5.1.8.7)$$

if  $Q_5=0.5$ , it results.

$$\omega_5 = 2 \cdot \pi \cdot Bw \cdot \sqrt{\sqrt{2} + 1}$$

Knowing  $Q_5$  and  $\omega_5$ , one can obtain Bw:

$$Bw = \frac{\sqrt{2} \cdot \omega_5 \cdot Q_5}{2 \cdot \pi \cdot \sqrt{\sqrt{4 \cdot Q_5^2 \cdot (2 \cdot Q_5^2 - 1) + 1} - 2 \cdot Q_5^2 + 1}} \quad (5.1.8.8)$$

Numerical result:

$$f_5 = 30.498 \cdot \text{MHz} \quad \frac{\sqrt{2} \cdot \omega_5 \cdot Q_5}{2 \cdot \pi \cdot \sqrt{\sqrt{4 \cdot Q_5^2 \cdot (2 \cdot Q_5^2 - 1) + 1} - 2 \cdot Q_5^2 + 1}} = 47.1 \cdot \text{MHz}$$

$$Bw = 47.1 \cdot \text{MHz}$$

If  $\zeta = \omega_5$  results:

$$Bw = \frac{\sqrt{\omega_5^2 - 2 \cdot \omega_5^2 + \sqrt{2} \cdot \sqrt{2 \cdot \omega_5^2 \cdot (\omega_5^2 - \omega_5^2) + \omega_5^4}}}{2 \cdot \pi} = \frac{\omega_5}{2 \cdot \pi} \cdot \sqrt{\sqrt{2} - 1}$$

$$\sqrt{\sqrt{2} - 1} = 0.644$$

For  $\omega = Bw$ , the voltage gain in dB takes the value  $Wl_{\text{pdB}}(2 \cdot \pi \cdot Bw) = 16.99 \cdot \text{dB}$   $Wl_{\text{pp}} - \text{dB}_{\text{gd}} = 16.99$

Low frequency voltage gain:  $Wl_{\text{pp}} = 20 \cdot \text{dB}$   $\zeta = 0.018 \cdot \frac{\text{Grads}}{\text{sec}}$   $\omega_5 = 0.192 \cdot \frac{\text{Grads}}{\text{sec}}$   $Q_5 = 5.4$

The angular frequency for which  $Wl_{\text{pdB}}(\omega) = 0 \text{dB}$  is:

$$\boxed{\omega_{5\text{dB0}} := \sqrt{\sqrt{4 \cdot \zeta^2 \cdot (\zeta^2 - \omega_5^2) + \omega_5^4 \cdot (|A_5|)^2} - 2 \cdot \zeta^2 + \omega_5^2}}$$

$$\omega_{5\text{dB0}} = 0.635 \cdot \frac{\text{Grads}}{\text{sec}} \quad Wl_{\text{pdB}}(\omega_{5\text{dB0}}) = 0$$

If  $\zeta = \omega_5$ , the corresponding angular frequency for which  $Wl_{\text{pdB}}(\omega) = 0 \text{dB}$  is:  $\omega_{5\text{dB0}} := \omega_5 \cdot \sqrt{|A_5| - 1}$

$$\zeta = 0.018 \cdot \frac{\text{Grads}}{\text{sec}} \quad \omega_5 = 0.192 \cdot \frac{\text{Grads}}{\text{sec}} \quad \omega_{5\text{dB0}} = 0.635 \cdot \frac{\text{Grads}}{\text{sec}} \quad Wl_{\text{pdB}}(\omega_{5\text{dB0}}) = 1.929 \times 10^{-15}$$



## System modes.

Knowing the transfer function:  $W_{lp}(s) := \frac{A_5 \cdot \omega_5^2}{s^2 + 2 \cdot \zeta \cdot s + \omega_5^2}$  one can know which the system modes.

► Modes calculation

$$\text{modes} = \text{"Pseudoperiodics"} \quad Q_5 = 5.4 \quad (5.1.8.9)$$

## Bode Plots

$$W_{lp}(s) := \begin{cases} \frac{A_5 \cdot \omega_5^2}{s^2 + 2 \cdot \zeta \cdot s + \omega_5^2} & \text{if } \zeta \neq \omega_5 \\ \frac{A_5 \cdot \omega_5^2}{(s + \omega_5)^2} & \text{otherwise} \end{cases} \quad (5.1.8.10)$$

$$\text{poles}^T = (-0.018 + 0.191j \quad -0.018 - 0.191j) \cdot \frac{\text{Grads}}{\text{sec}}$$

$$|\text{poles}_0| = 0.192 \cdot \frac{\text{Grads}}{\text{sec}}$$

$$\sqrt{|\text{poles}_0| \cdot |\text{poles}_1|} = 0.192 \cdot \frac{\text{Grads}}{\text{sec}} \quad \omega_5 = 0.192 \cdot \frac{\text{Grads}}{\text{sec}}$$

$$\omega_{\text{res}} := \frac{\omega_5}{1000}, \frac{\omega_5}{1000} + \frac{40 \cdot \omega_5 - \frac{\omega_5}{1000}}{1000} \dots 40 \cdot \omega_5$$

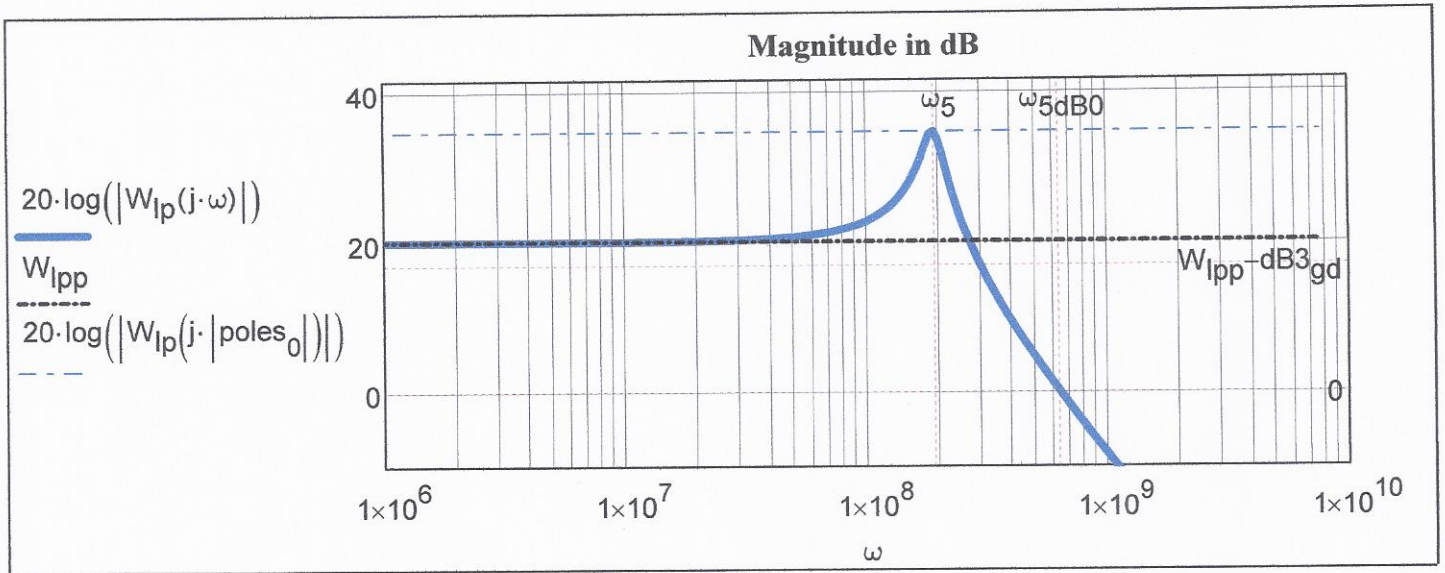
$$W_{lpdB}(\omega_5) = 34.648$$

$$W_{lpp} = 20 \cdot \text{dB}$$

Pick amplitude of the frequency response if  $Q_5 > 0.5$  rpeak :=  $20 \cdot \log\left(\left|\frac{A_5 \cdot \omega_5}{2 \cdot \zeta}\right|\right)$

► Plots Control

$$Q_5 = 5.4$$



Bw = 47.1·MHz

fig.:5.1.8.1

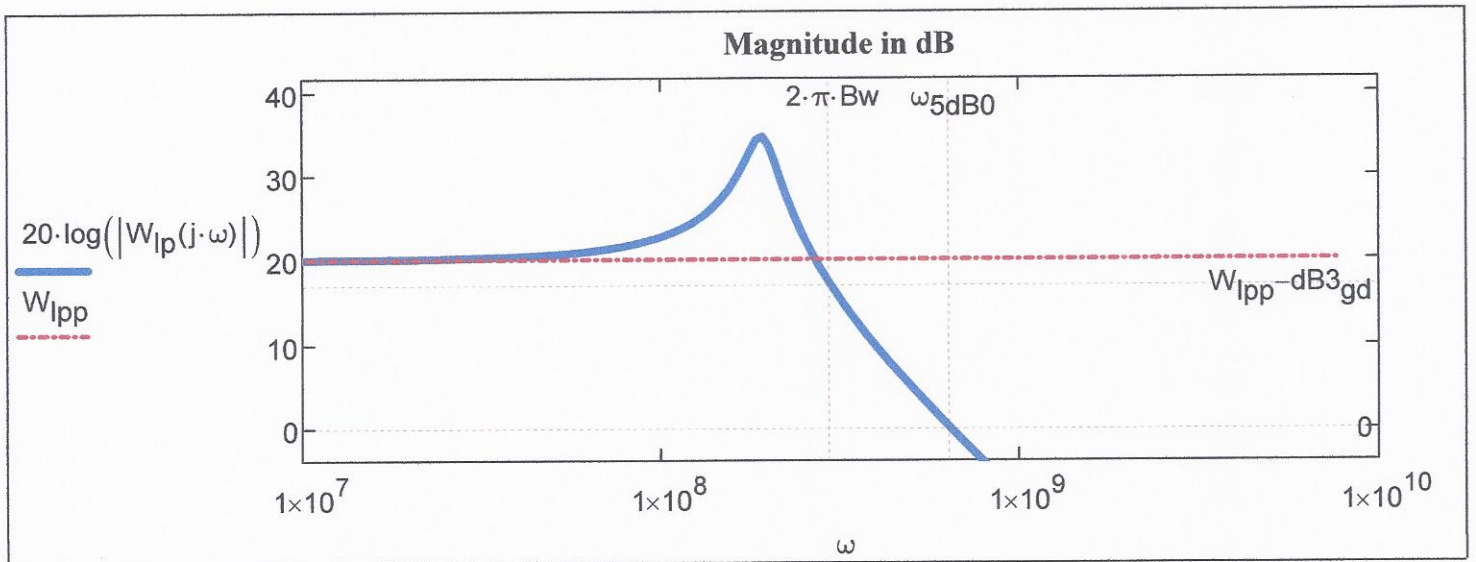


fig.:5.1.8.1'

pick amplitude  $WlpdB_{pick} = 34.685$

$W_{lpp} = 20$

$\omega_{pick} = 0.19 \cdot \frac{\text{Grads}}{\text{sec}}$

$\omega_5 = 0.192 \cdot \frac{\text{Grads}}{\text{sec}}$

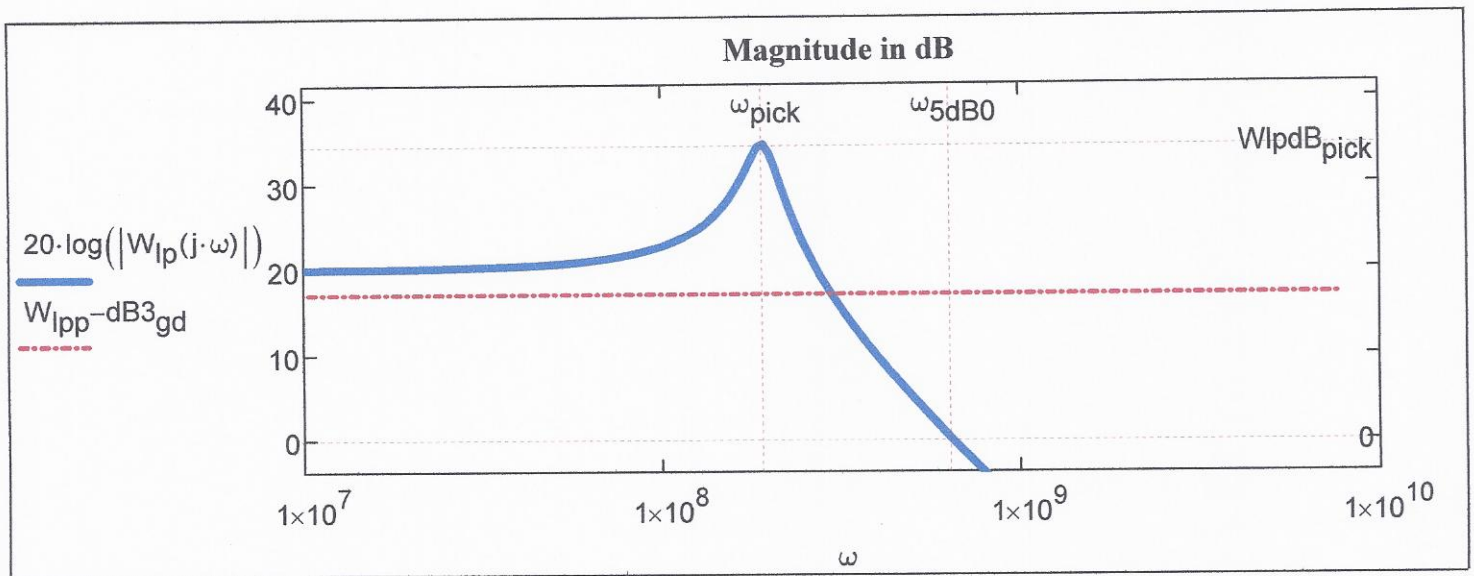


fig.:5.1.8.1"

$$\omega_{pick} = 0.19 \cdot \frac{\text{Grads}}{\text{sec}}$$

$$\omega_5 = 0.192 \cdot \frac{\text{Grads}}{\text{sec}}$$

$$\omega_5 - \omega_{pick} = 1.65 \cdot \frac{\text{Mrads}}{\text{sec}}$$

$$W_{lpp} - 2 \cdot \text{dB3}_{gd} = r_{peak} = 20 \cdot \log \left( \left| \frac{A_5 \cdot \omega_5}{2 \cdot \zeta} \right| \right)$$

$$W_{lpp} - 2 \cdot \text{dB3}_{gd} = 13.979$$

$$r_{peak} = 34.648$$

$$20 \cdot \log \left( \left| \frac{A_5 \cdot \omega_5}{2 \cdot \zeta} \right| \right) = 34.648$$

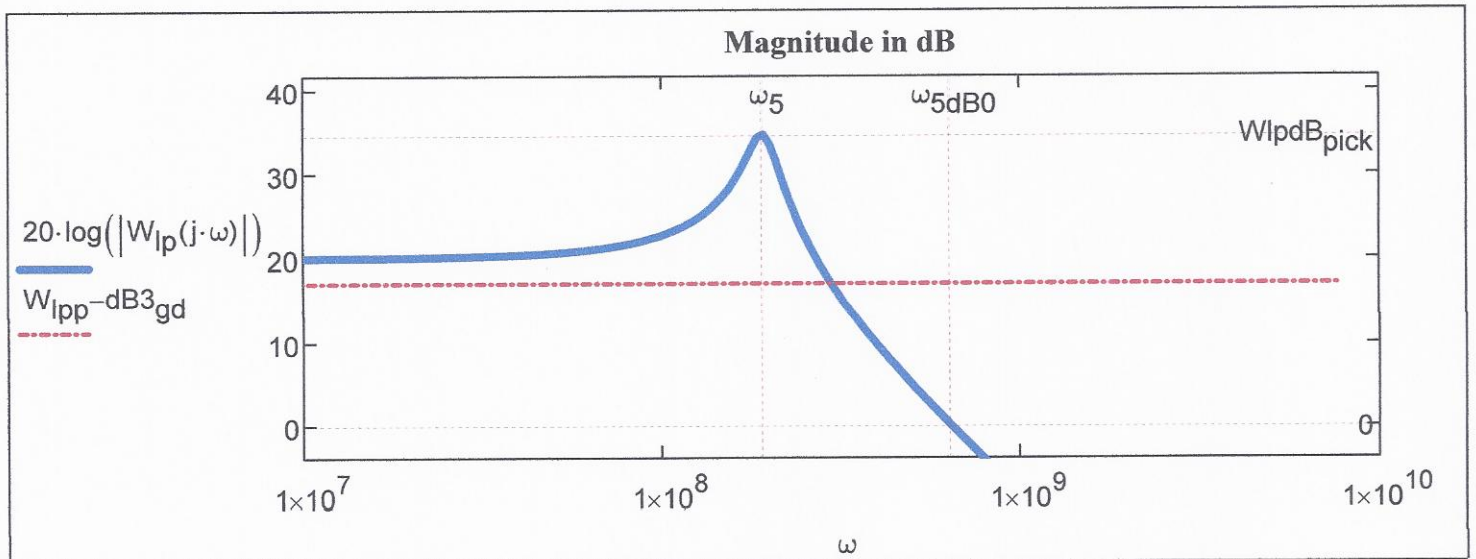


fig.:5.1.8.1'''

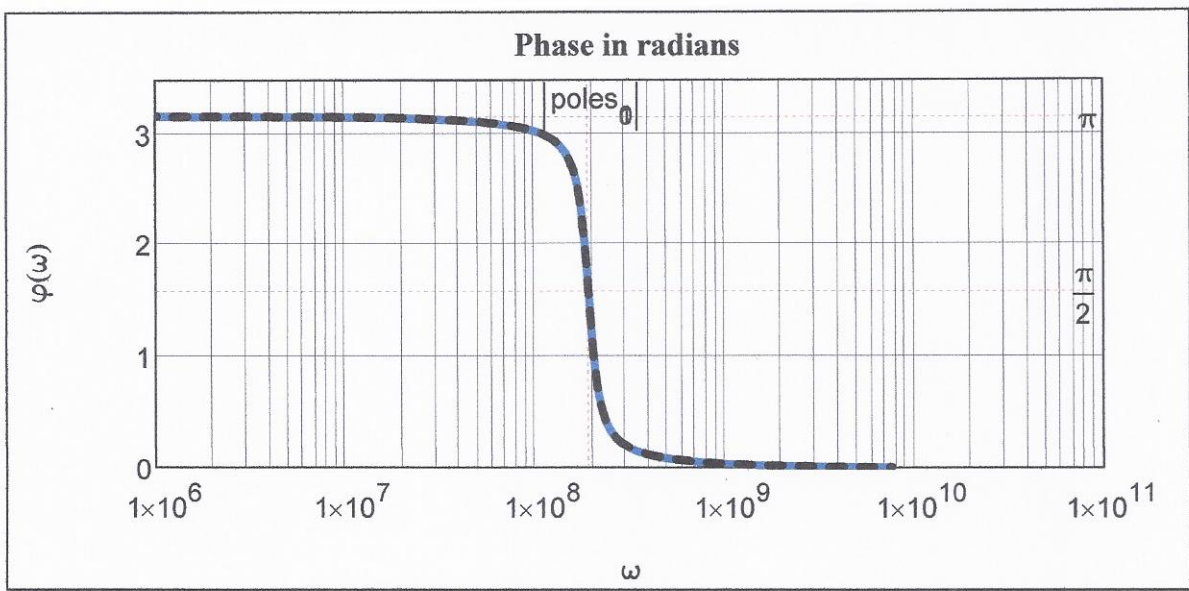


fig.:5.1.8.2

Knowing the poles of the transfer function, it is immediate to see the system stability:

► Stability Type

**stability** = " System Exponentially Stable"



# 5.2

## ANALOG FILTER OUTPUT ANALYSIS

For a signal definition refer to the file "Signal List.xmcd"

Chosen period of the test signal,  $T_{\text{test}} = 0.016 \cdot \mu\text{s}$ . At the corresponding frequency, the voltage gain of the filter is  $20 \cdot \log(|W_{\text{lp}}(j \cdot \omega_{\text{test}})|) = 10.392 \cdot \text{dB}$ . As seen the pulse response waveform is:

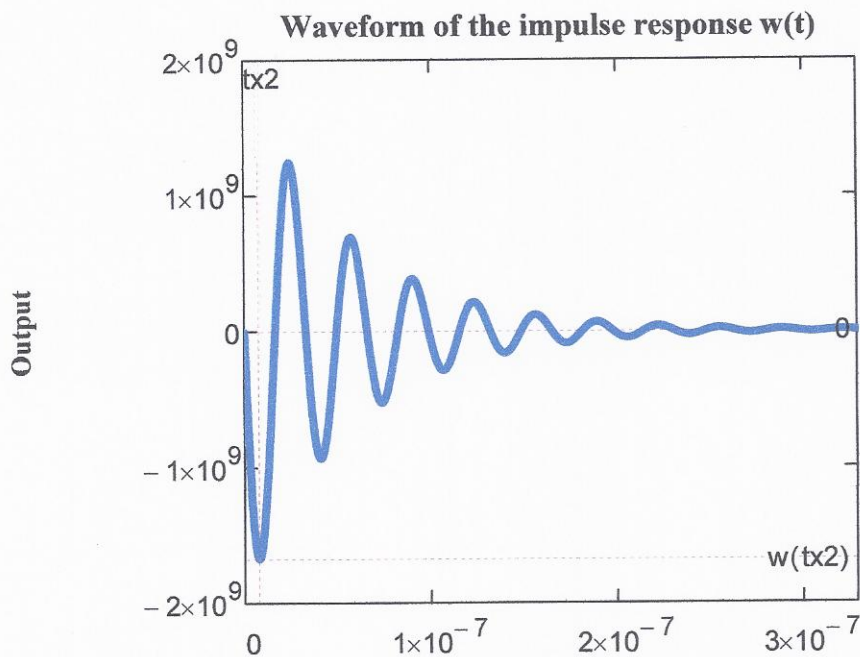


fig.:5.2.1

$$w(\text{tx2}) = -1.67 \cdot \frac{\text{Grads}}{\text{sec}}$$

### 5.2 ANALOG FILTER OUTPUT ANALYSIS

#### 5.2.1 Voltage step response - Analytical solution

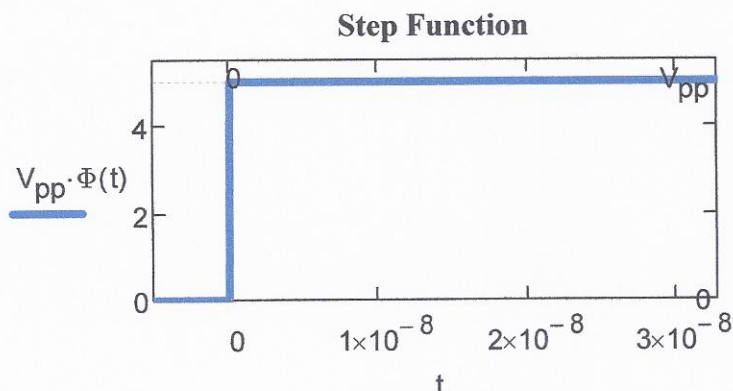


fig.:5.2.1.1

$$A_5 := A_5 \quad \omega_5 := \omega_5$$

$$V_{pp} := V_{pp} \quad V_{pp} = 5V$$

The evaluation is disabled because the result exceeds the page margins.

$$y_{sr}(t) := \frac{A_5 \cdot \omega_5^2}{s^2 + 2 \cdot \zeta \cdot s + \omega_5^2} \cdot \frac{V_{pp}}{s} \quad \left| \begin{array}{l} \text{invlaplace, s} \\ \text{simplify, max} \\ \text{collect, } A_5 \cdot V_{pp}, e^{-\zeta \cdot t} \end{array} \right. \rightarrow$$

Step response:

define the function:

$$g_{sr}(t, A_5, \zeta, \omega_5) := \left[ \left( \cosh\left(t \cdot \sqrt{\zeta^2 - \omega_5^2}\right) + \frac{\zeta \cdot \sinh\left(t \cdot \sqrt{\zeta^2 - \omega_5^2}\right)}{\sqrt{\zeta^2 - \omega_5^2}} \right) \cdot e^{-\zeta \cdot t} - 1 \right],$$

the output waveform is:

$$y_{sr}(t) := A_5 \cdot V_{pp} \cdot \begin{cases} g_{sr}(t, A_5, \zeta, \omega_5) \cdot \Phi(t) & \text{if } \zeta \neq \omega_5 \\ \left[ 1 - e^{-t \cdot \omega_5} \cdot (t \cdot \omega_5 + 1) \right] \cdot \Phi(t) & \text{otherwise} \end{cases} \quad (5.2.1.1)$$

Calculation of the initial and final values of the output:

**Initial value theorem:**  $\lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} (s \cdot F(s))$ ,

**Final value theorem:**  $\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} (s \cdot F(s))$

**Output's Initial value:**

**Input signal:**  $V_i(s) = \frac{V_{pp}}{s}$  (5.2.1.2)

$$\lim_{s \rightarrow \infty} (s \cdot V_O(s)) = \lim_{s \rightarrow \infty} (s \cdot W(s) \cdot V_i(s)) = V_{pp} \cdot \lim_{s \rightarrow \infty} (W(s))$$

$$A_5 := A_5 \quad s := s \quad a := a \quad \omega_5 := \omega_5 \quad \zeta := \zeta$$

$$V_{pp} \cdot \lim_{s \rightarrow \infty} \left( \frac{A_5 \cdot \omega_5^2}{s^2 + 2 \cdot \zeta \cdot s + \omega_5^2} \right) = 0 \cdot \text{volt}$$

**Output's final value:**

**Input signal:**  $V_i(s) = \frac{V_{pp}}{s}$

$$\lim_{s \rightarrow 0} (s \cdot V_O(s)) = \lim_{s \rightarrow 0} (s \cdot W(s) \cdot V_i(s)) = V_{pp} \cdot \lim_{s \rightarrow 0} (W(s))$$

$$A_5 := A_5 \quad s := s \quad a := a \quad \omega_5 := \omega_5 \quad \zeta := \zeta$$

$$V_{pp} \cdot \lim_{s \rightarrow 0} \left( \frac{A_5 \cdot \omega_5^2}{s^2 + 2 \cdot \zeta \cdot s + \omega_5^2} \right) \rightarrow \begin{cases} A_5 \cdot V_{pp} & \text{if } \omega_5 \neq 0 \\ \text{undefined} & \text{otherwise} \end{cases}$$

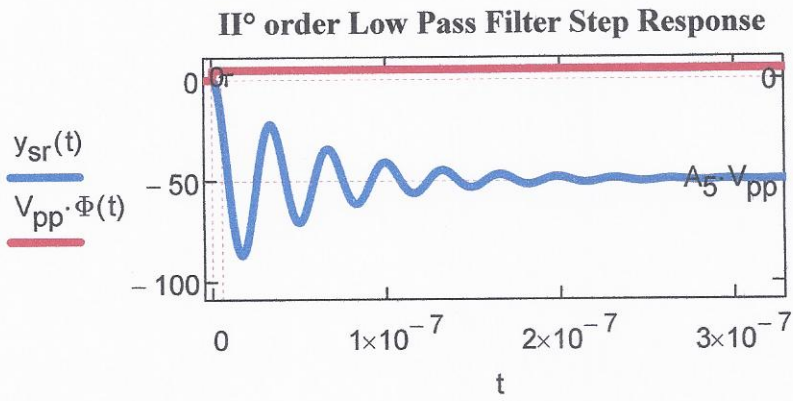


fig.:5.2.1.2

$$\zeta = 17.743 \cdot \frac{\text{Mrads}}{\text{sec}}$$

$$\omega_5 = 191.627 \cdot \frac{\text{Mrads}}{\text{sec}}$$

$$V_{pp} = 5 \times 10^3 \cdot \text{mV}$$

$$f_5 = 30.498 \cdot \text{MHz}$$

$$Q_5 = 5.4$$

## Bode plots

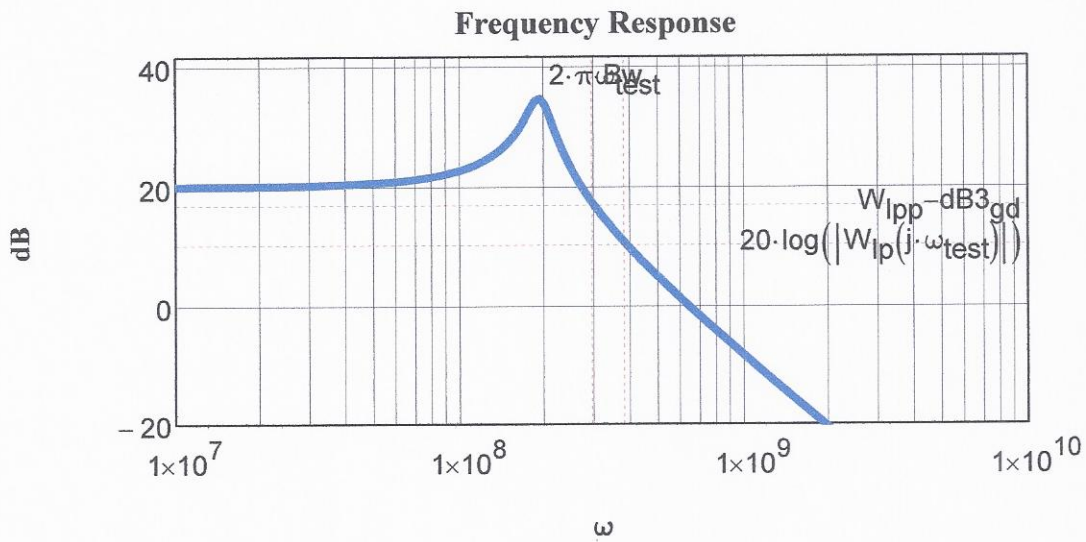


fig.:5.2.1.3

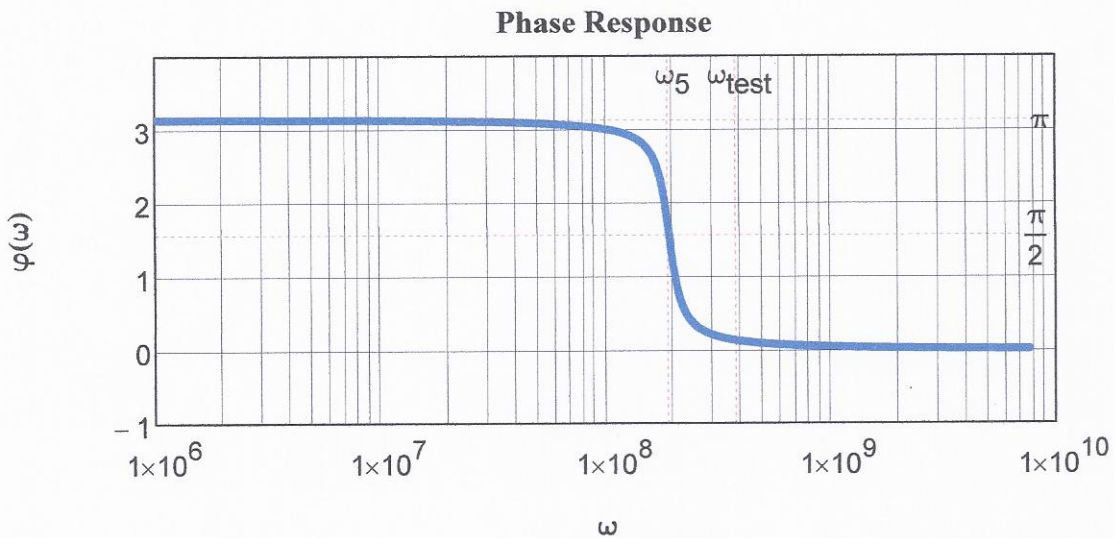


fig.:5.2.1.4



At the chosen test frequency, the voltage gain assumes the following value:

$$20 \cdot \log(|W_{Ip}(j \cdot \omega_{\text{test}})|) = 10.392 \cdot \text{dB} \quad 20 \cdot \log\left(\left|\frac{A_5 \cdot \omega_5}{2 \cdot \zeta}\right|\right) = 34.648 \cdot \text{dB}$$

while for  $\omega = \omega_{5\text{dB0}}$  the voltage gain is 0dB, being:  $\frac{|A_5| \cdot \omega_{5\text{dB0}}}{2 \cdot \zeta} = 178.944$

$$20 \cdot \log(|W_{Ip}(j \cdot \omega_{5\text{dB0}})|) = 0 \cdot \text{dB}$$

Angular frequency for 0 dB Voltage gain:  $\omega_{5\text{dB0}} = 0.635 \cdot \frac{\text{Grads}}{\text{sec}}$

Sampling of the step response

Signal frequency:  $f_{\text{test}} = 60.997 \cdot \text{MHz}$ ,

arbitrary sampling frequency:  $f_{\text{sstp}} := 10 \cdot f_{\text{test}}$ ,  $f_{\text{sstp}} = 609.968 \cdot \text{MHz}$  (5.2.1.3)

sampling angular frequency:  $\omega_{\text{smp}} := 2 \cdot \pi \cdot f_{\text{sstp}}$ ,  $\omega_{\text{smp}} = 3.833 \cdot \frac{\text{Grads}}{\text{sec}}$ ,

sampling period:  $T_{\text{sstp}} := \frac{1}{f_{\text{sstp}}}$ ,  $T_{\text{sstp}} = 1.639 \cdot \text{ns}$ ,

generic pulse delay time:  $\tau_5 := 0.4 \cdot T_{\text{test}}$

sampling time step:  $n\text{stp}_k := \frac{k}{f_{\text{sstp}}}$ ,  $N_0 = 256$  (5.2.1.4)

$$\frac{N_0}{f_{\text{sstp}}} \cdot f_5 = 12.8 \quad (5.2.1.5)$$

sampling time step:

$n\text{stp}^T =$	0	1	2	3	4	5	6	$\cdot \mu\text{s}$
	0	$1.639 \cdot 10^{-3}$	$3.279 \cdot 10^{-3}$	$4.918 \cdot 10^{-3}$	$6.558 \cdot 10^{-3}$	$8.197 \cdot 10^{-3}$	...	

$$N_0 = 256 \quad y_{srk} := \frac{y_{sr}(n\text{stp}_k)}{\text{volt}} \quad (5.2.1.6)$$

$$T_5 = 0.033 \cdot \mu\text{s}$$

$$\tau = 5.218 \times 10^{-3} \cdot \mu\text{s}$$

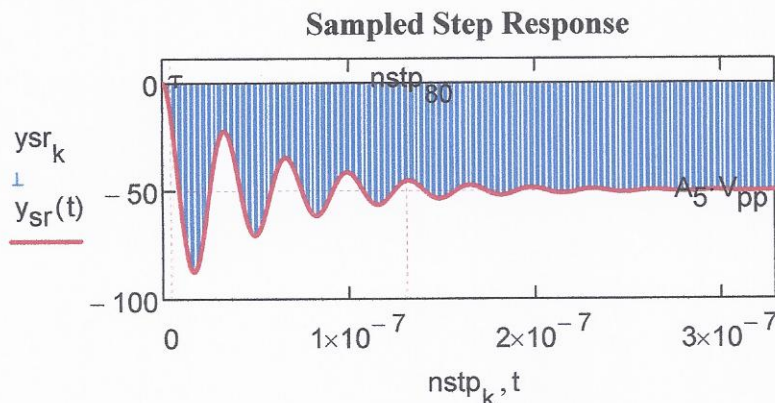


fig.:5.2.1.5



$$\zeta = 17.743 \cdot \frac{\text{Mrads}}{\text{sec}}$$

$$\omega_5 = 0.192 \cdot \frac{\text{Grads}}{\text{sec}}$$

Samples:

$$\text{ysr}^T = \begin{array}{|c|c|c|c|c|c|c|} \hline & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 0 & -2.4 & -9.192 & -19.48 & -32.089 & \dots \\ \hline \end{array}$$

sampling time step:

$$\text{nstp}^T = \begin{array}{|c|c|c|c|c|c|} \hline & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 0 & 1.639 \cdot 10^{-3} & 3.279 \cdot 10^{-3} & 4.918 \cdot 10^{-3} & \dots \\ \hline \end{array} \cdot \mu\text{s}$$

Fourier Transform of the test signal

$$f_{\text{test}} = 0.061 \cdot \text{GHz} \quad \frac{f_{\text{sstp}}}{f_{\text{test}}} = 10 \quad \frac{N_0}{f_{\text{sstp}} \cdot T_{\text{test}}} = 25.6$$

Fourier Transform:  $F_{y_{sr}} := \text{fft}(\text{ysr})$

(5.2.1.7)

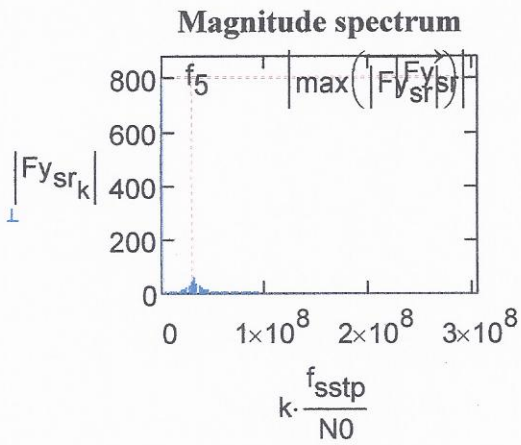
$$F_{y_{sr}}^T = \begin{array}{|c|c|c|c|c|c|} \hline & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & -796.601 & 3.421-0.761j & 3.491-1.549j & 3.615-2.392j & \dots \\ \hline \end{array}$$


fig.:5.2.1.6

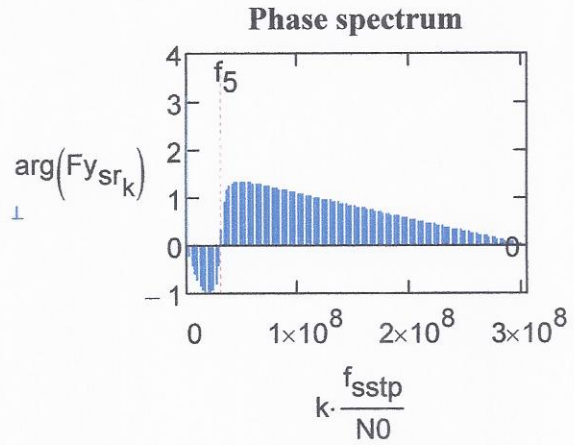


fig.:5.2.1.7

## 5.2 ANALOG FILTER OUTPUT ANALYSIS

### 5.2.2 Short Voltage Pulse response

Description of the waveform's parameters:

$$\text{Definition: } V_4(t, \tau_5, \tau_{pw}, V_{in}) = V_{in} \cdot \text{rect1}(t, \tau_5, \tau_{pw}), \quad V_{in} = \frac{V_{pp}}{V} \quad (5.2.2.1)$$

$$V_4(t, \tau_5, \tau_{pw}, V_{in}) = V_4(\text{time, Rising Edge, Pulse Width, Dimensionless Amplitude}).$$

$$\text{Pulse amplitude: } V_{pp} = 5 \times 10^3 \cdot \text{mV} \quad \text{Pulse width: } \tau_{pw} := T_5 \cdot 20 \quad \tau_{pw} = 655.772 \cdot \text{ns}$$

$$\text{Pulse displacement from the origin: } \xi_{sl} := 0.8, \quad \tau_{pw} = \tau_{pw} \cdot (1 - \xi_{sl}) + \xi_{sl} \cdot \tau_{pw} \quad (5.2.2.2)$$

$$\text{Time delay from the origin: } \tau_{5a} := -\tau_{pw} \cdot (1 - \xi_{sl}), \quad \text{risingedge} = \tau_5, \quad \text{width} = \tau_{pw}$$

Generic pulse definition defined in "Fourier Series.xmcd":

$$\text{Input signal defined in "Test Signal.xmcd": } V_w(t) := V_4(t, \tau_5, \tau_{pw}, V_{pp}) \quad (5.2.2.3)$$

Consider a Short Voltage Pulse delayed  $\tau_5$   $\tau_5 = -0.131 \cdot \mu\text{s}$   $T_{sstp} = 1.639 \cdot \text{ns}$   
seconds:

$$t := -2 \cdot \tau_{pw}, -2 \cdot \tau_{pw} + \frac{4 \cdot \tau_{pw}}{5000} .. 2 \cdot \tau_{pw}$$

$$V_w(\tau_{pw}) = 0$$

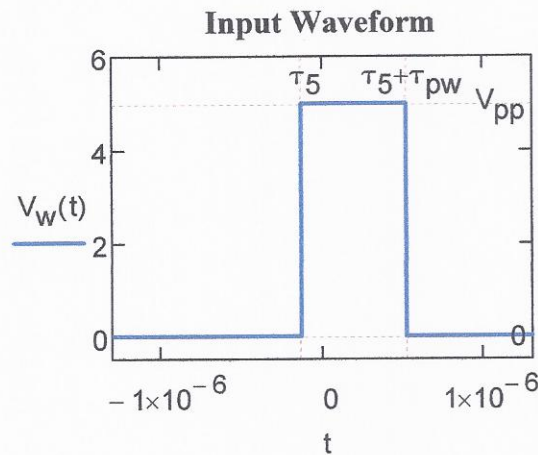


fig.:5.2.2.1

Consider now the same signal repeated periodically, with period  $T_{vp} := 4 \cdot (\tau_{pw} + \tau_5)$ , in such a way that it is possible to calculate the bandwidth using the program BCSA defined in "Fourier Analysis.xmcd":

Description of the program's parameters:

BCSA(*Dimensionless signal name, relative error, polynomial degree, start time, signal period*)  
BCSA stands for "Bandwidth Calculation and Signal Analysis"

$$Sb_{vp0} := \text{BCSA}[V_w, rt_{gd}, 50, 0.0, 2 \cdot (\tau_{pw} + \tau_5)] \quad rt_{gd} = 1\% \quad (5.2.2.4)$$

▶ Bandwidth Calculation

Signal bandwidth:  $B_{vp0} = 0.046 \cdot \text{GHz}$

$f_{\text{test}} = 0.061 \cdot \text{GHz}$

$\text{Parseval}_{vp0} = 24.899 \text{V}^2$

$\text{Average}_{vp0} = 2.5 \text{V}$

$\text{RMS}_{vp0} = 3.536 \text{V}$

Sampling frequency:  $f_{\text{samp}} = \frac{1}{T_{\text{samp}}} \geq 2 \cdot f_1$

Chosen sampling frequency (Nyquist rate):  $f_{\text{svp0}} := 2 \cdot B_{\text{vp0}} \quad f_{\text{svp0}} = 0.091 \cdot \text{GHz} \quad (5.2.2.5)$

$$T_{\text{svp0}} := \frac{1}{f_{\text{svp0}}} \quad (5.2.2.6)$$

$$n_{\text{svp0}_k} := k \cdot T_{\text{svp0}} + \tau_5 \quad \frac{N_0}{f_{\text{svp0}}} \cdot \frac{1}{T_{\text{test}}} = 170.667$$

$$V_{\text{pp}} = 5\text{V}$$

Pulse sampling:  $u_{44k} := V_{\text{w}}(n_{\text{svp0}_k}) \quad (5.2.2.7)$

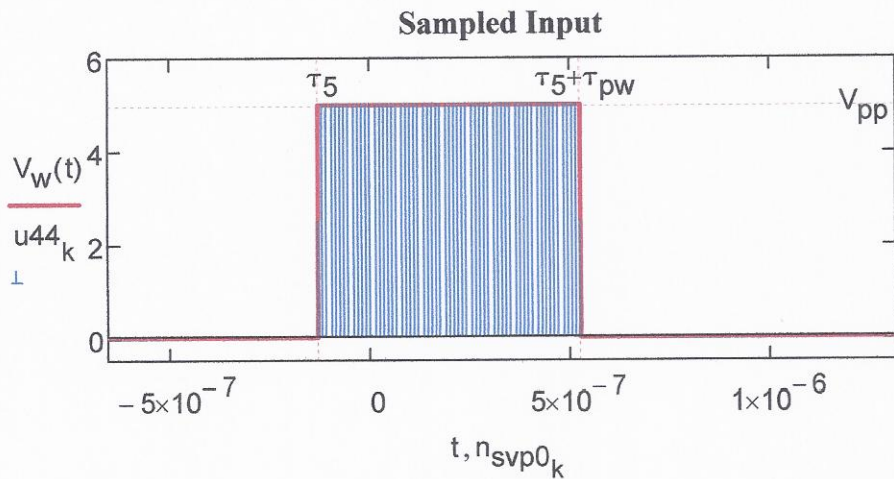


fig.:5.2.2.2

**Filter response:**

Input signal:  $V_{\text{w}}(t) = V_4(t, \tau_5, \tau_{\text{pw}}, V_{\text{pp}}) = V_{\text{pp}} \cdot \text{rect1}(t, \tau_5, \tau_{\text{pw}}) \quad (5.2.2.8)$

or:  $V_4(t) = V_{\text{pp}} \cdot (\Phi(t - \tau_5) - \Phi(t - \tau_{\text{pw}} - \tau_5)) \quad (5.2.2.9)$

Laplace transform of the input signal:  $V_4(s) = V_{\text{pp}} \cdot \left[ \frac{e^{-\tau_5 \cdot s}}{s} - \frac{e^{-(\tau_5 + \tau_{\text{pw}}) \cdot s}}{s} \right] \quad (5.2.2.10)$

$$V_4(s) = \frac{V_{\text{pp}}}{s} \cdot e^{-\tau_5 \cdot s} \cdot (1 - e^{-\tau_{\text{pw}} \cdot s}) \quad (5.2.2.11)$$

Laplace transform of the output signal:  $Y_{\text{vp}}(s) = W(s) \cdot V_4(s)$  where  $W(s)$  is the t. f.:

$$Y_{\text{vp}}(s) = \begin{cases} \left[ \frac{V_{\text{pp}}}{s} \cdot e^{-\tau_5 \cdot s} \cdot (1 - e^{-\tau_{\text{pw}} \cdot s}) \cdot \frac{A_5 \cdot \omega_5^2}{s^2 + 2 \cdot \zeta \cdot s + \omega_5^2} \right] & \text{if } \zeta \neq \omega_5 \\ \left[ \frac{V_{\text{pp}}}{s} \cdot e^{-\tau_5 \cdot s} \cdot (1 - e^{-\tau_{\text{pw}} \cdot s}) \cdot \frac{A_5 \cdot \omega_5^2}{(s + \omega_5)^2} \right] & \text{otherwise} \end{cases} \quad (5.2.2.12)$$

**1) First case:  $\zeta \neq \omega_5$**



Rewrite the Laplace transform of the response in this form:

$$Y_{vp}(s) = V_{pp} \cdot \frac{e^{-\tau_5 \cdot s}}{s} \cdot \frac{A_5 \cdot \omega_5^2}{s^2 + 2 \cdot \zeta \cdot s + \omega_5^2} - V_{pp} \cdot \frac{e^{-(\tau_5 + \tau_{pw}) \cdot s}}{s} \cdot \frac{A_5 \cdot \omega_5^2}{s^2 + 2 \cdot \zeta \cdot s + \omega_5^2}$$

namely:

$$Y_{vp}(s) = V_{pp} \cdot A_5 \cdot \omega_5^2 \cdot \left[ \frac{e^{-\tau_5 \cdot s}}{s \cdot (s^2 + 2 \cdot \zeta \cdot s + \omega_5^2)} - \frac{e^{-(\tau_5 + \tau_{pw}) \cdot s}}{s \cdot (s^2 + 2 \cdot \zeta \cdot s + \omega_5^2)} \right] \quad (5.2.2.13)$$

$$\text{and call } F(s) = \frac{1}{s \cdot (s^2 + 2 \cdot \zeta \cdot s + \omega_5^2)} \quad (5.2.2.14)$$

$$\text{results that: } \frac{e^{-\tau_5 \cdot s}}{s \cdot (s^2 + 2 \cdot \zeta \cdot s + \omega_5^2)} = e^{-\tau_5 \cdot s} \cdot F(s) = \mathcal{L}\{f(t - \tau_5)\}$$

$$\text{and: } \frac{e^{-(\tau_5 + \tau_{pw}) \cdot s}}{s \cdot (s^2 + 2 \cdot \zeta \cdot s + \omega_5^2)} = e^{-(\tau_5 + \tau_{pw}) \cdot s} \cdot F(s) = \mathcal{L}\{f[t - (\tau_5 + \tau_{pw})]\}$$

so that one can write as well:

$$Y_{vp}(s) = V_{pp} \cdot A_5 \cdot \omega_5^2 \cdot \left[ e^{-\tau_5 \cdot s} \cdot F(s) - e^{-(\tau_5 + \tau_{pw}) \cdot s} \cdot F(s) \right] \quad (5.2.2.15)$$

Now calculate the inverse Laplace transform of F(s):

$$\text{call it: } f_1(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left[\frac{1}{s \cdot (s^2 + 2 \cdot \zeta \cdot s + \omega_5^2)}\right]$$

$$\frac{1}{s \cdot (s^2 + 2 \cdot \zeta \cdot s + \omega_5^2)} \left| \begin{array}{l} \text{invlaplace, } s \\ \text{simplify, max } \rightarrow \\ \text{collect, } e^{-\zeta \cdot t} \end{array} \right.$$

$$\text{results: } f_1(t) := \left[ \left( -\cosh\left(t \cdot \sqrt{\zeta^2 - \omega_5^2}\right) - \frac{\zeta \cdot \sinh\left(t \cdot \sqrt{\zeta^2 - \omega_5^2}\right)}{\sqrt{\zeta^2 - \omega_5^2}} \right) \cdot e^{-\zeta \cdot t} + 1 \right] \cdot \frac{1}{\omega_5^2} \quad (5.2.2.16)$$

$$\text{The output is: } Y_{vp}(s) = V_{pp} \cdot A_5 \cdot \omega_5^2 \cdot \left[ e^{-\tau_5 \cdot s} \cdot F(s) - e^{-(\tau_5 + \tau_{pw}) \cdot s} \cdot F(s) \right]$$

whose inverse Laplace transform is:

$$y_{1vp}(t) := V_{pp} \cdot A_5 \cdot \omega_5^2 \cdot \left[ f_1(t - \tau_5) \cdot \Phi(t - \tau_5) - f_1[t - (\tau_5 + \tau_{pw})] \cdot \Phi[t - (\tau_5 + \tau_{pw})] \right] \quad (5.2.2.17)$$

## 2) case: $\zeta = \omega_5$

Rewrite the Laplace transform of the response in this form:



$$Y_{vp}(s) = V_{pp} \cdot \frac{e^{-\tau_5 \cdot s}}{s} \cdot \frac{A_5 \cdot \omega_5^2}{(s + \omega_5)^2} - V_{pp} \cdot \frac{e^{-(\tau_5 + \tau_{pw}) \cdot s}}{s} \cdot \frac{A_5 \cdot \omega_5^2}{(s + \omega_5)^2} \quad (5.2.2.18)$$

namely:

$$Y_{vp}(s) = V_{pp} \cdot A_5 \cdot \omega_5^2 \cdot \left[ \frac{e^{-\tau_5 \cdot s}}{s \cdot (s + \omega_5)^2} - \frac{e^{-(\tau_5 + \tau_{pw}) \cdot s}}{s \cdot (s + \omega_5)^2} \right] \quad (5.2.2.19)$$

$$\text{call: } G(s) = \frac{1}{s \cdot (s + \omega_5)^2} \quad (5.2.2.20)$$

so that the first term of the second member can be also written:

$$\frac{e^{-\tau_5 \cdot s}}{s \cdot (s + \omega_5)^2} = e^{-\tau_5 \cdot s} \cdot G(s) = \mathcal{L}(f(t - \tau_5))$$

While the second term is:

$$\frac{e^{-(\tau_5 + \tau_{pw}) \cdot s}}{s \cdot (s + \omega_5)^2} = e^{-(\tau_5 + \tau_{pw}) \cdot s} \cdot G(s) = \mathcal{L}[f[t - (\tau_5 + \tau_{pw})]]$$

The Laplace transform of the output can be written:

$$Y_{vp}(s) = V_{pp} \cdot A_5 \cdot \omega_5^2 \cdot \left[ e^{-\tau_5 \cdot s} \cdot G(s) - e^{-(\tau_5 + \tau_{pw}) \cdot s} \cdot G(s) \right] \quad (5.2.2.21)$$

To calculate the response, it is sufficient to know the inverse Laplace transform of  $G(s)$ :

place :

$$g_1(t) = \mathcal{L}^{-1}(G(s)) = \mathcal{L}^{-1} \left[ \frac{1}{s \cdot (s + \omega_5)^2} \right]$$

Calculation of the inverse Laplace transform of  $G(s)$ :

$$s := s \quad \zeta := \zeta$$

$$\frac{1}{s \cdot (s + \zeta)^2} \left| \begin{array}{l} \text{invlaplace, s} \\ \text{simplify, max} \\ \text{factor} \rightarrow \\ \text{collect, } e^{-\zeta \cdot t} \\ \text{collect, } \frac{1}{\zeta^2} \end{array} \right.$$

so, the result is:

$$g_1(t) := \frac{1 - e^{-\zeta \cdot t} \cdot (\zeta \cdot t + 1)}{\zeta^2} \quad (5.2.2.22)$$

Finally the inverse Laplace transform of the filter's output:

$$Y_{2vp}(s) = V_{pp} \cdot A_5 \cdot \omega_5^2 \cdot \left[ e^{-\tau_5 \cdot s} \cdot G(s) - e^{-(\tau_5 + \tau_{pw}) \cdot s} \cdot G(s) \right]$$

$$is: y_{2vp}(t) := V_{pp} \cdot A_5 \cdot \omega_5^2 \cdot [g_1(t - \tau_5) \cdot \Phi(t - \tau_5) - g_1[t - (\tau_5 + \tau_{pw})] \cdot \Phi[t - (\tau_5 + \tau_{pw})]]$$

As one can see, the two waveforms, according to which  $\zeta = \omega_{5dB}$  or not, are slightly different.

$$\zeta = 0.018 \cdot \frac{\text{Grads}}{\text{sec}} \quad \omega_5 = 0.192 \cdot \frac{\text{Grads}}{\text{sec}}$$

Hence, the time response to the Short Voltage Pulse is:

$$y_{\text{pulse}}(t) := \begin{cases} y_{1vp}(t) & \text{if } \zeta \neq \omega_5 \\ y_{2vp}(t) & \text{otherwise} \end{cases} \quad (5.2.2.23)$$

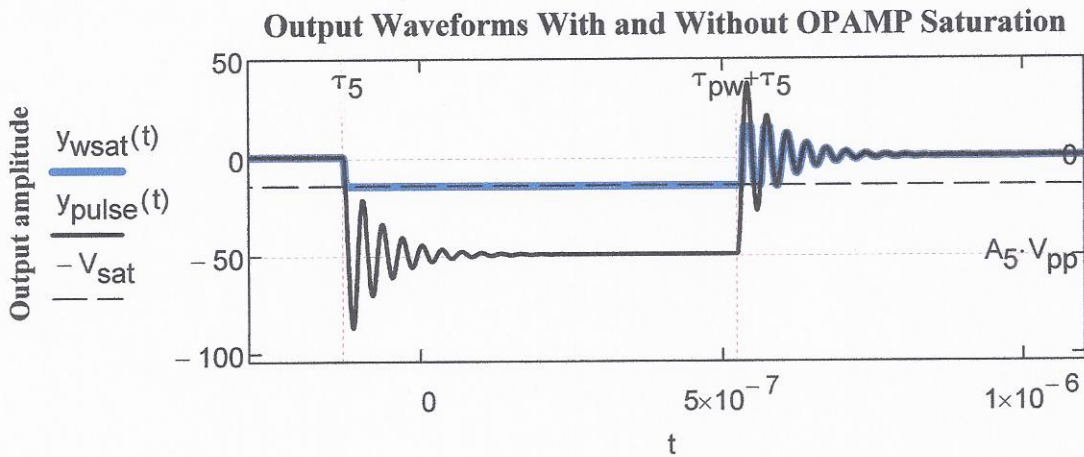
Taking into account the op. amp. saturation voltage, the output is truncated:

$$y_{\text{wsat}}(t) := \text{if}(-V_{\text{sat}} \leq y_{\text{pulse}}(t) \leq V_{\text{sat}}, y_{\text{pulse}}(t), \text{if}(y_{\text{pulse}}(t) \leq 0.0 \cdot \text{volt}, -V_{\text{sat}}, V_{\text{sat}}))$$

Graph controls

$$y_{\text{wsat}}(\tau_{pw} + \tau_5) = -15V \quad t := -4 \cdot \tau_{pw}, -4 \cdot \tau_{pw} + \frac{6 \cdot \tau_{pw}}{10000} \dots 2 \cdot \tau_{pw} \quad A_5 \cdot V_{pp} = -50V$$

$$\zeta = 0.018 \cdot \frac{\text{Grads}}{\text{sec}} \quad \omega_5 = 0.192 \cdot \frac{\text{Grads}}{\text{sec}}$$



time as multiple of  $\tau$   
fig.:5.2.2.3

$$\text{Dimensionless Output} \quad v_p(t) := \frac{y_{\text{wsat}}(t)}{V}$$

### Analog filter Output sampling.

Consider now the same signal repeated periodically, with period  $T_{vp} := 2 \cdot (\tau_{pw} + \tau_5)$ , in such a way that it is possible to calculate the bandwidth using BCSA:

Description of the program's parameters:

BCSA(Dimensionless signal name, relative error, polynomial degree, start time, signal period)  
BCSA stands for "Bandwidth Calculation and Signal Analysis"

$$rt_{gd} = 1.0\% \quad Sb_{vp} := \text{BCSA}(v_p, rt_{gd}, 50, 0.0 \cdot \text{sec}, T_{vp}) \quad (5.2.2.24)$$

Bandwidth Calculation

$$\text{Signal bandwidth: } B_{vp} = 0.046 \cdot \text{GHz}$$

$$f_{\text{test}} = 0.061 \cdot \text{GHz}$$

$$\text{Parseval}_{vp} = 251.043V^2$$

$$\text{Average}_{vp} = -7.51V$$

$$\text{RMS}_{vp} = 11.24V$$

Sampling frequency:

$$f_{\text{samp}} = \frac{1}{T_{\text{samp}}} \geq 2 \cdot f_1$$

Chosen sampling frequency  $f_{\text{svp}} := 2 \cdot B_{vp}$

$$f_{\text{svp}} = 0.091 \cdot \text{GHz}$$

$$\frac{N0}{f_{\text{svp}}} \cdot \frac{1}{T_{\text{test}}} = 170.667 \quad T_{\text{svp}} := \frac{1}{f_{\text{svp}}} \quad n_{\text{svp}_k} := \frac{k}{f_{\text{svp}}} + \tau_5 \quad (5.2.2.25)$$

Output sampling considering Op Amp saturation:

$$V_{pk} := vp(n_{\text{svp}_k}) \quad (5.2.2.26)$$

Output sampling without considering Op Amp saturation:

$$y_{\text{win}_k} := \frac{y_{\text{pulse}}(n_{\text{svp}_k})}{\text{volt}} \quad (5.2.2.27)$$

$$N0 = 256$$

$$Q_5 = 5.4$$

$$A_5 = -10$$

Dimensionless output sampling considering Op Amp saturation:

$$A_5 \cdot V_{pp} = -50V \quad y_{\text{wins}_k} := \frac{y_{\text{wsat}}(n_{\text{svp}_k})}{\text{volt}} \quad (5.2.2.28)$$

Sampling Of The Output Waveform With And Without OPAMP Saturation

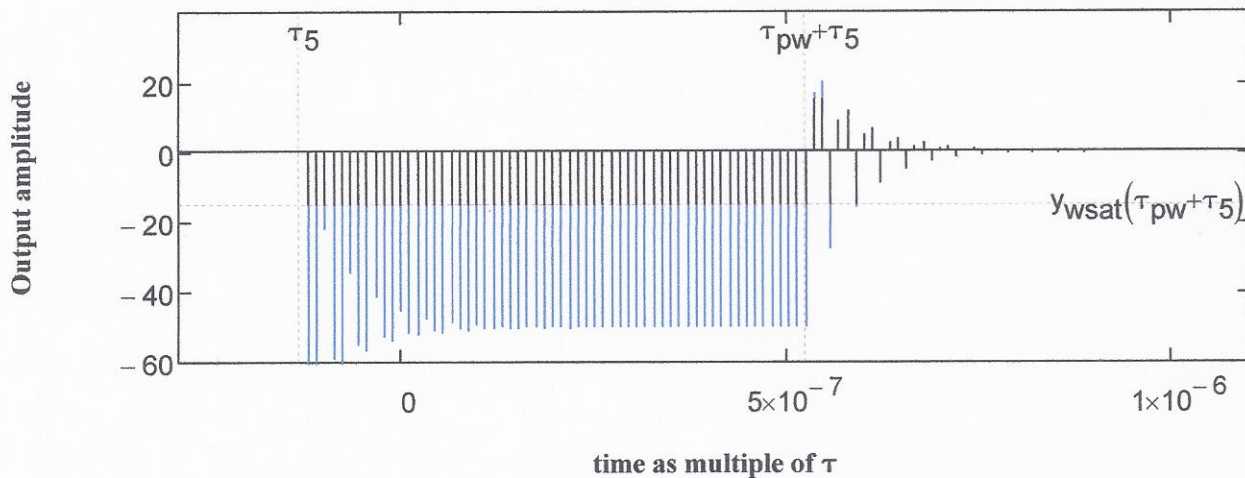


fig.:5.2.2.4

## 5.2 ANALOG FILTER OUTPUT ANALYSIS

### 5.2.3 Sawtooth response.

Input signal defined in "Test Signal.xmcd": 
$$V_{sw}(t) := \frac{v1_{sw}(t, T_{\text{test}}, V_{pp})}{\text{volt}} \quad (5.2.3.1)$$

$$V_{pp} = 5V$$



### Bipolar Sawtooth Waveform with positive slope

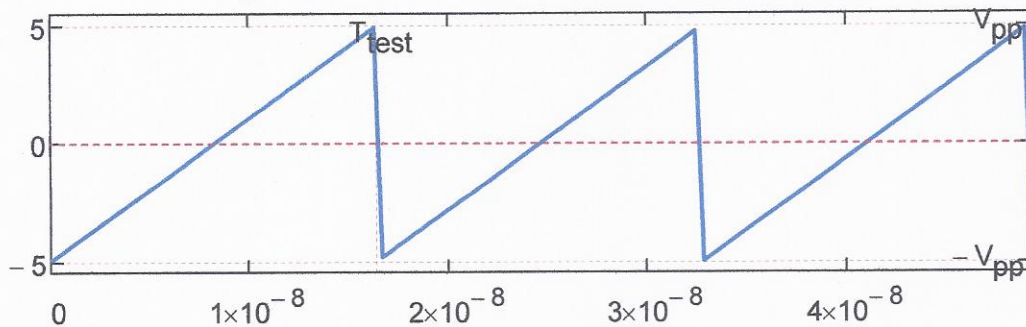


fig.:5.2.3.1

For a correct sampling one must know the signal bandwidth.

**Numerical search of the signal bandwidth.** All harmonics with amplitude less than  $rt_{gd} = 1\%$  of the fundamental one, are neglected. To do that it is used the function  $BCSA(\dots)$  defined in "Fourier Series.xmcd".

Description of the program's parameters:  
 $BCSA(\textit{Dimensionless signal name, relative error, polynomial degree, start time, signal period})$   
*BCSA stands for "Bandwidth Calculation and Signal Analysis"*

$$N = 50 \quad Sb_{sw} := BCSA(V_{sw}, rt_{gd}, N, 0.0 \cdot \text{sec}, T_{test}) \quad (5.2.3.2)$$

Bandwidth Calculation

Signal frequency:  $f_{sw} = 60.997 \cdot \text{MHz}$      $\omega_{sw} := 2 \cdot \pi \cdot f_{sw}$     Signal bandwidth:  $B_{sw} = 2.928 \cdot \text{GHz}$

Parseval<sub>sw</sub> = 16.466

Average1.volt = 0V

RMS1.volt = 2.887V

$$j70 := 0 \dots \text{rows}(Sb_{sw}^{(1)}) - 1$$

$$T_{test} = 16.394 \cdot \text{ns}$$

$$X_{sw_{fs}} := \max \left[ \sqrt{(Sb_{sw}^{(1)})^2 + (Sb_{sw}^{(2)})^2} \right]$$

$$mx_{sw_{fs}} := \frac{\sqrt{\left[ (Sb_{sw}^{(1)})_{j_{sw}} \right]^2 + \left[ (Sb_{sw}^{(2)})_{j_{sw}} \right]^2}}{X_{sw_{fs}}}$$

$$\omega_{sw} = 383.254 \cdot \frac{\text{Mrads}}{\text{sec}}$$

### Sawtooth Frequency Spectrum

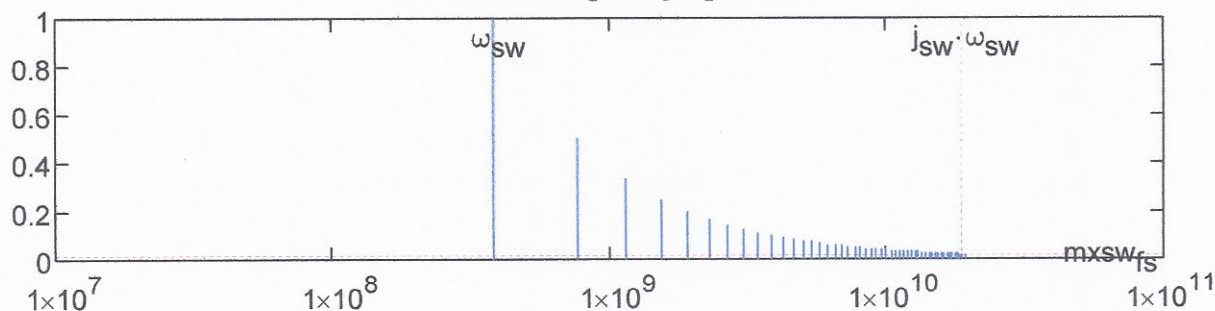


fig.:5.2.3.2



(Min) sampling frequency (Nyquist rate)  $f_{ssw} := 2 \cdot B_{sw}$        $f_{ssw} = 5.856 \cdot \text{GHz}$        $T_{ssw} := \frac{1}{f_{ssw}}$

sampling time step:  $n_{sw_k} := \frac{k}{f_{ssw}}$  (5.2.3.3)

$\text{rows}(n_{sw}) = 256$        $\frac{1}{T_{test}} = 60.997 \cdot \text{MHz}$        $\frac{N0}{f_{ssw}} \cdot \frac{1}{T_{test}} = 2.667$

$\text{RMS1} = 2.887$       Sampled signal:  $u10_k := v1_{sw}(n_{sw_k}, T_{test}, V_{pp})$  (5.2.3.4)

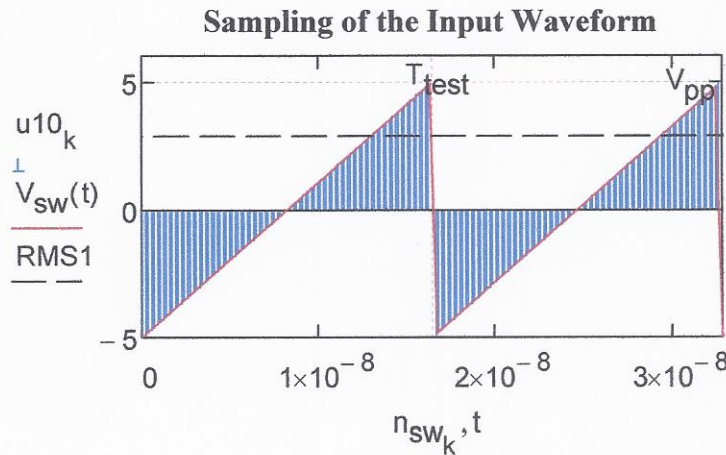


fig.:5.2.3.3

**Approximate signal reconstruction according to the Shannon sampling theorem:**

$$\omega_{s2} := 2 \cdot \pi \cdot B_{sw} \quad sh3(t) := \left[ \sum_{n=0}^{N0-1} (u10_n \cdot \text{sinc}(\omega_{s2} \cdot t - n \cdot \pi)) \right] \quad (5.2.3.5)$$

$N0 - 1 = 255$

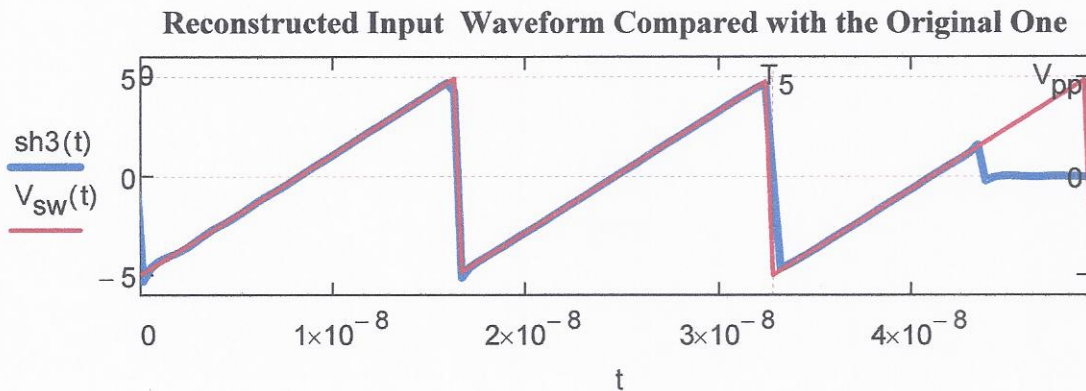


fig.:5.2.3.4

$rt_{gd} = 1.0\%$

**Search of the filter's transient response**

Given the signal:

$$v1_{sw}(t, T_{test}, V_{pp}) := 2 \cdot \frac{V_{pp}}{T_{test}} \cdot \sum_{k=0}^{\infty} \left[ (t - k \cdot T_{test}) \cdot \text{rect1}(t - k \cdot T_{test}, 0.0 \cdot T_{test}, T_{test}) \right] - V_{pp}$$

or:

(5.2.3.6)

$$v1_{sw0}(t, T_{test}, V_{pp}) := 2 \cdot \frac{V_{pp}}{T_{test}} \cdot \sum_{k=0}^{20} \left[ \left( (t - k \cdot T_{test}) \cdot \Phi(t - k \cdot T_{test}) \dots \right. \right. \\ \left. \left. + (-1) \cdot \left[ (t - k \cdot T_{test}) \cdot \Phi[t - T_{test} \cdot (k + 1)] \right] \right) \right] - V_{pp}$$

$$RMS1 = 2.887$$

$$(5.2.3.7)$$

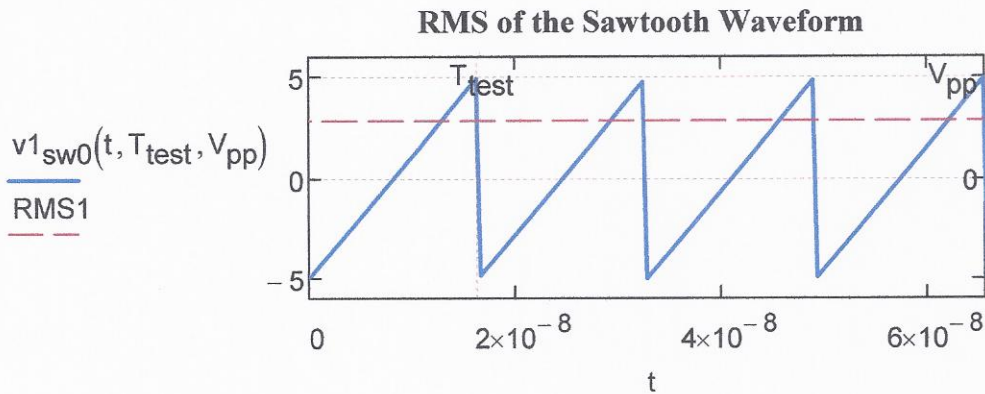


fig.:5.2.3.5

► Laplace Transform calculation of the Output Signal .

Hence, the compact laplace transform of the (21) is:

$$\mathcal{L}(v1_{sw}(t, T_{test}, V_{pp})) = 2 \cdot \frac{V_{pp}}{T_{test}} \cdot \frac{1}{s} \cdot \left[ \frac{1}{s} - \left( T_{test} + \frac{1}{s} \right) \cdot e^{-s \cdot T_{test}} \right] \cdot \sum_{k=0}^{\infty} \left( e^{-T_{test} \cdot k \cdot s} \right) - \frac{V_{pp}}{s},$$

or

$$\mathcal{L}(v1_{sw}(t, T_{test}, V_{pp})) = \frac{V_{pp}}{s} \cdot \left( \frac{2}{T_{test} \cdot s} - \coth\left(\frac{T_{test} \cdot s}{2}\right) \right) \quad (5.2.3.10)$$

Search of the corresponding output waveform of the filter.

Given the transfer function:  $W_{lp}(s) :=$

$\frac{A_5 \cdot \omega_5^2}{s^2 + 2 \cdot \zeta \cdot s + \omega_5^2} \quad \text{if } \zeta \neq \omega_5$	(5.2.3.11)
$A_5 \cdot \frac{\omega_5^2}{(s + \omega_5)^2} \quad \text{otherwise}$	

the Laplace transform of the filter's output is:  $V_{osw}(s) = W_{lp}(s) \cdot \mathcal{L}(v1_{sw}(t, T_{test}, V_{pp}))$

**First case**  $\zeta \neq \omega_5$ :

$$V_{osw}(s) = \frac{V_{pp} \cdot A_5 \cdot \omega_5^2}{s^2 + 2 \cdot \zeta \cdot s + \omega_5^2} \cdot \frac{1}{s} \cdot \left( \frac{2}{T_{test} \cdot s} - \coth\left(\frac{T_{test} \cdot s}{2}\right) \right) \quad (5.2.3.12)$$

**Second case**  $\zeta = \omega_5$ :

$$V_{osw}(s) = \frac{V_{pp} \cdot A_5 \cdot \omega_5^2}{(s + \omega_5)^2} \cdot \frac{1}{s} \cdot \left( \frac{2}{T_{test} \cdot s} - \coth\left(\frac{T_{test} \cdot s}{2}\right) \right) \quad (5.2.3.13)$$

Now apply the following theorems:

**Initial value theorem:**  $\lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} (s \cdot F(s))$ ,

**Final value theorem:**  $\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} (s \cdot F(s))$

$T_{\text{test}} := T_{\text{test}} \quad \omega_5 := \omega_5 \quad \zeta := \zeta \quad A_5 := A_5 \quad V_{\text{pp}} := V_{\text{pp}} \quad s := s$

Final value of the output voltage:

$\zeta \neq \omega_5$

$$V_{\text{ofin}} := \lim_{s \rightarrow 0} \left[ s \cdot \left[ \frac{V_{\text{pp}} \cdot A_5 \cdot \omega_5^2 \cdot \left( 2 - T_{\text{test}} \cdot s \cdot \coth\left(\frac{T_{\text{test}} \cdot s}{2}\right) \right)}{T_{\text{test}} \cdot s^2 \cdot (s^2 + 2 \cdot \zeta \cdot s + \omega_5^2)} \right] \right] \quad \left| \begin{array}{l} \text{assume, } (T_{\text{test}} > 0.0) \rightarrow 0 \\ \text{simplify} \end{array} \right.$$

$\zeta = \omega_5$

$$V_{\text{ofin1}} := \lim_{s \rightarrow 0} \left[ s \cdot \left[ \frac{V_{\text{pp}} \cdot A_5 \cdot \omega_5^2 \cdot \left( 2 - T_{\text{test}} \cdot s \cdot \coth\left(\frac{T_{\text{test}} \cdot s}{2}\right) \right)}{T_{\text{test}} \cdot s^2 \cdot (s + \omega_5)^2} \right] \right] \quad \left| \begin{array}{l} \text{assume, } (T_{\text{test}} > 0.0) \rightarrow 0 \\ \text{simplify} \end{array} \right.$$

Initial value of the output voltage:

$T_{\text{test}} := T_{\text{test}} \quad \omega_5 := \omega_5 \quad A_5 := A_5 \quad V_{\text{pp}} := V_{\text{pp}}$

$\zeta \neq \omega_5$

$$\lim_{s \rightarrow \infty} \left[ s \cdot \left[ \frac{V_{\text{pp}} \cdot A_5 \cdot \omega_5^2 \cdot \left( 2 - T_{\text{test}} \cdot s \cdot \coth\left(\frac{T_{\text{test}} \cdot s}{2}\right) \right)}{T_{\text{test}} \cdot s^2 \cdot (s^2 + 2 \cdot \zeta \cdot s + \omega_5^2)} \right] \right] \quad \left| \begin{array}{l} \text{assume, } (T_{\text{test}} > 0.0) \rightarrow \\ \text{simplify} \end{array} \right. \quad \blacksquare$$

$\zeta = \omega_5$

$$\lim_{s \rightarrow \infty} \left[ s \cdot \left[ \frac{V_{\text{pp}} \cdot A_5 \cdot \omega_5^2 \cdot \left( 2 - T_{\text{test}} \cdot s \cdot \coth\left(\frac{T_{\text{test}} \cdot s}{2}\right) \right)}{T_{\text{test}} \cdot s^2 \cdot (s + \omega_5)^2} \right] \right] \quad \left| \begin{array}{l} \text{assume, } (T_{\text{test}} > 0.0) \rightarrow \\ \text{simplify} \end{array} \right. \quad \blacksquare$$

it results that:  $V_{\text{ofin}} = 0 \cdot V$

**Transient response calculation:**

As seen, the filter output is:

**First case**  $\zeta \neq \omega_5$ :

$$v_{\text{osw}}(t) = \frac{V_{\text{pp}} \cdot A_5 \cdot \omega_5^2}{T_{\text{test}}} \cdot \mathcal{L}^{-1} \left[ \frac{2 - T_{\text{test}} \cdot s \cdot \coth\left(\frac{T_{\text{test}} \cdot s}{2}\right)}{s^2 \cdot (s^2 + 2 \cdot \zeta \cdot s + \omega_5^2)} \right] \quad (5.2.3.14)$$

**Second case**  $\zeta = \omega_5$ :

$$v_{\text{osw}}(t) = \frac{V_{\text{pp}} \cdot A_5 \cdot \omega_5^2}{T_{\text{test}}} \cdot \mathcal{L}^{-1} \left[ \frac{2 - T_{\text{test}} \cdot s \cdot \coth\left(\frac{T_{\text{test}} \cdot s}{2}\right)}{s^2 \cdot (s + \omega_5)^2} \right] \quad (5.2.3.15)$$

$$\frac{V_{pp} \cdot A_5 \cdot \omega_5^2}{T_{test}} = -0.112 \cdot \frac{V}{ns^3}$$

▶ Calculation of the Sawtooth time response

$$N = 50 \quad t := 0 \cdot T_{test}, 0 \cdot T_{test} + \frac{20 \cdot T_{test} - 0 \cdot T_{test}}{1000} .. 20 \cdot T_{test} \quad A_5 = -10$$

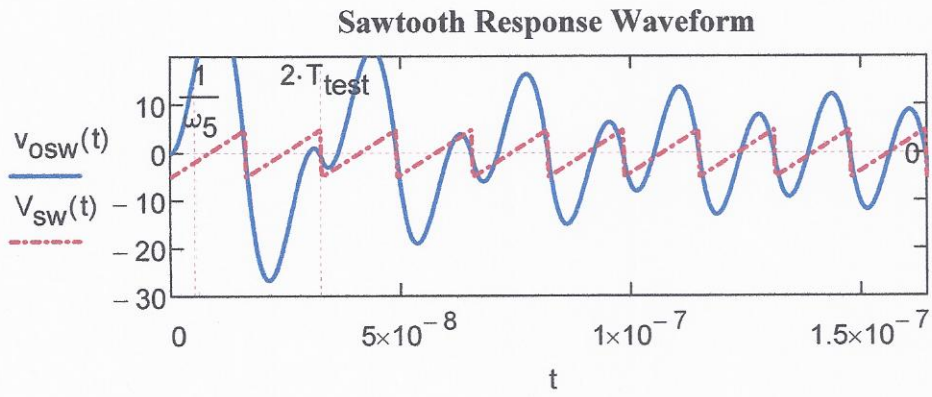


fig.:5.2.3.6



## Convolution:

The second method is to calculate **the time domain convolution product** between the signal and the impulse response.

Sawtooth function. The number of sawtooth is limited to forty one. The function rect1 is defined in "Signal list.xmcd".

$$fs1(t) := \frac{2 \cdot V_{pp}}{T_{test}} \cdot \sum_{k=0}^{40} \left[ (t - k \cdot T_{test}) \cdot \text{rect1}(t, k \cdot T_{test}, T_{test}) \right] - V_{pp} \quad (5.2.3.16)$$

Convolution: 
$$v_{osw}(t) = \int_0^t V_{sw}(\tau) \cdot w(t - \tau) d\tau = \int_0^t V_{sw}(t - \tau) \cdot w(\tau) d\tau$$

Output: 
$$v_{oswconv}(t) := \int_0^t w(t - \sigma) \cdot fs1(\sigma) d\sigma \quad (5.2.3.17)$$

$$t := -1 \cdot T_{test}, -1 \cdot T_{test} + \frac{5 \cdot T_5 + 1 \cdot T_{test}}{100} .. 5 \cdot T_5$$

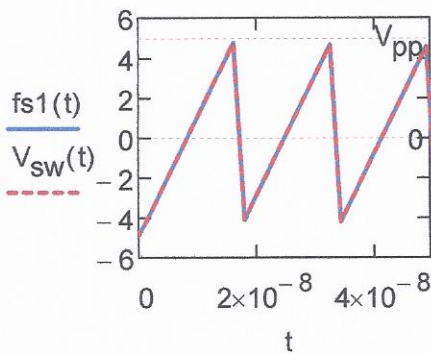


fig.:5.2.3.7

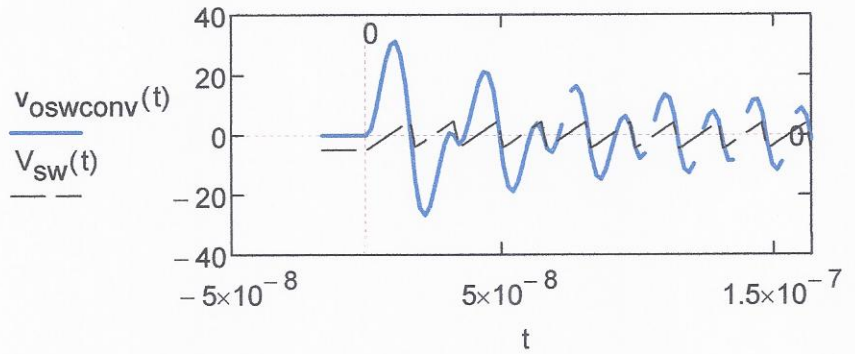


fig.:5.2.3.8

Now the output signal will be sampled. To do that correctly, one must know the signal bandwidth. The following program (BCSA) will do that calculation.

Dimensionless output signal: 
$$V_{osw}(t) := \frac{v_{osw}(t)}{V} \quad (5.2.3.18)$$

Description of the program's parameters:

$BCSA(\text{Dimensionless signal name, relative error, polynomial degree, start time, signal period})$   
 BCSA stands for "Bandwidth Calculation and Signal Analysis"

$N = 50$        $Sb_{osw} := BCSA(V_{osw}, rt_{gd}, N, 0.0 \cdot \text{sec}, T_{test}) \quad (5.2.3.19)$

▢ Bandwidth Calculation

Output signal frequency:  $f_{osw} = 60.997 \cdot \text{MHz}$

Output signal pulsation:  $\omega_{osw} := 2 \cdot \pi \cdot f_{osw}$

Output signal bandwidth:  $B_{osw} = 2.928 \cdot \text{GHz}$

Parseval<sub>osw</sub> = 861.113

Average1o.volt = 17.656V

RMS1o.volt = 20.75V

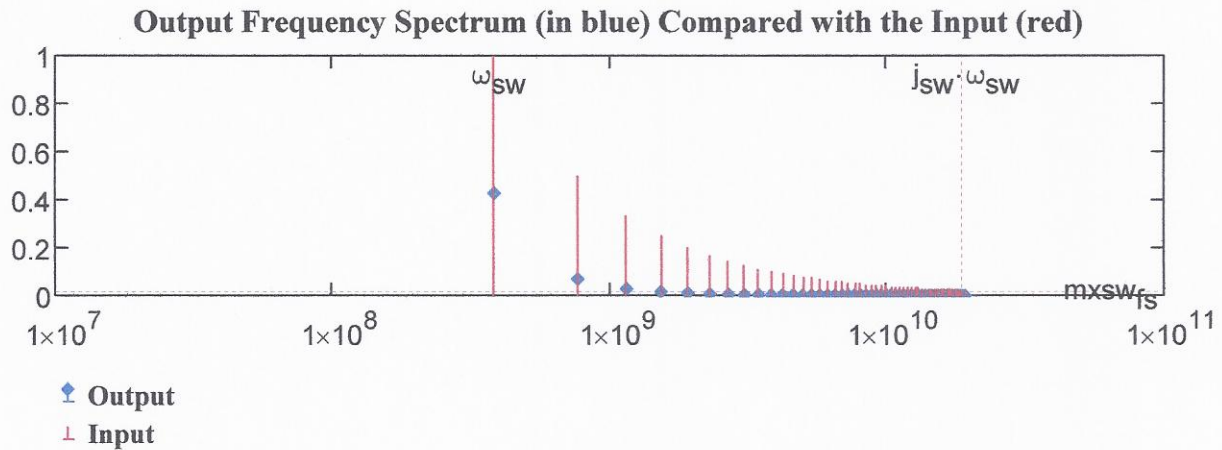
$j70 := 0 .. \text{rows}(Sb_{osw} \langle 1 \rangle) - 1$

$T_{test} = 16.394 \cdot \text{ns}$

$$X_{sw_{f_{SO}}} := \max \left[ \sqrt{\left( Sb_{osw}^{(1)} \right)^2 + \left( Sb_{osw}^{(2)} \right)^2} \right]$$

$$mx_{sw_{f_{SO}}} := \frac{\sqrt{\left[ \left( Sb_{osw}^{(1)} \right)_{j_{sw}} \right]^2 + \left[ \left( Sb_{osw}^{(2)} \right)_{j_{sw}} \right]^2}}{X_{sw_{f_{SO}}}}$$

$$\omega_{osw} = 383.254 \cdot \frac{\text{Mrads}}{\text{sec}}$$



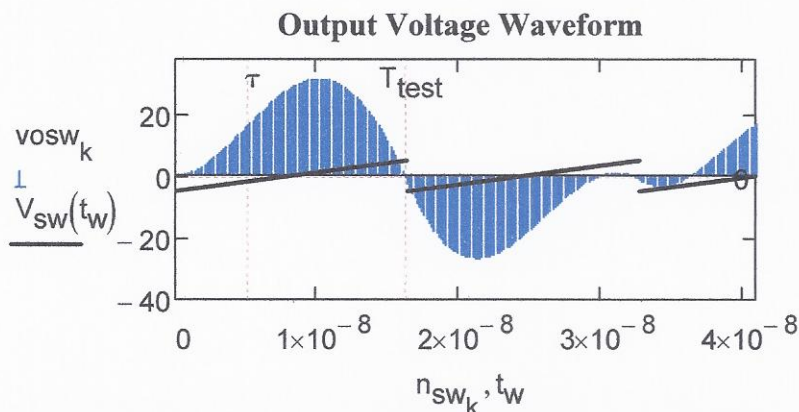
(Min) sampling frequency (Nyquist rate)  $f_{sswo} := 2 \cdot B_{osw}$      $f_{sswo} = 5.856 \cdot \text{GHz}$      $T_{sswo} := \frac{1}{f_{sswo}}$

sampling time step:  $n_{sw_{0k}} := \frac{k}{f_{sswo}}$     (5.2.3.20)

$T_{\text{test}} = 16.394 \cdot \text{ns}$     rows( $n_{sw_{0k}}$ ) = 256     $|A_5 \cdot V_{pp}| = 50 \text{V}$

$\frac{1}{T_{\text{test}}} = 60.997 \cdot \text{MHz}$      $\frac{N_0}{f_{sswo}} \cdot \frac{1}{T_{\text{test}}} = 2.667$

$v_{osw_k} := v_{osw}(n_{sw_{0k}})$     (5.2.3.21)



**Approximate output signal reconstruction according to the Shannon sampling theorem:**

$$\omega_{s2o} := 2 \cdot \pi \cdot B_{osw} \quad sh3o(t) := \left[ \sum_{n=0}^{N0-1} (vosw_n \cdot \text{sinc}(\omega_{s2o} \cdot t - n \cdot \pi)) \right] \quad (5.2.3.22)$$

$$N0 - 1 = 255$$

$$rt_{gd} = 1\%$$

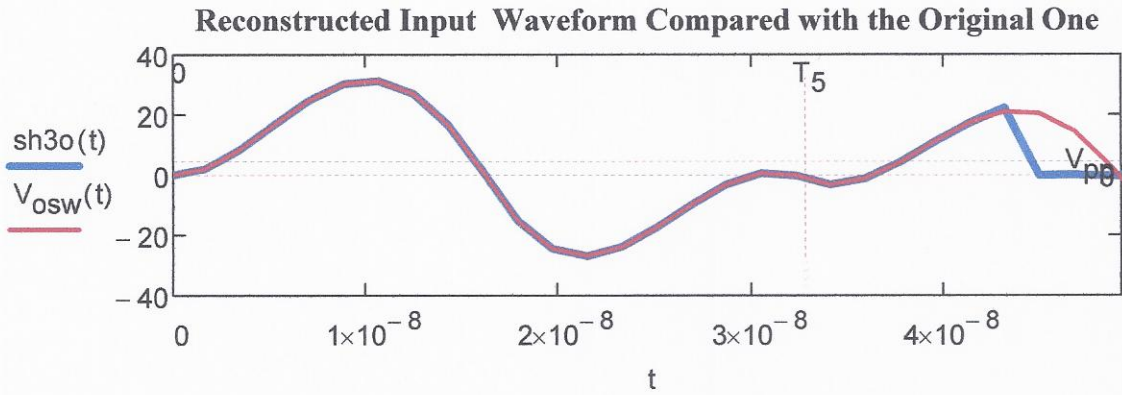


fig.:5.2.3.11

**Fourier Transform of the Test signal**

$$f_{test} = 0.061 \cdot \text{GHz} \quad \frac{f_{sswo}}{f_{test}} = 96 \quad \frac{N0}{f_{sswo}} \cdot \frac{1}{T_{test}} = 2.667$$

$$Osw := \text{fft}(vosw) \quad (5.2.3.23)$$

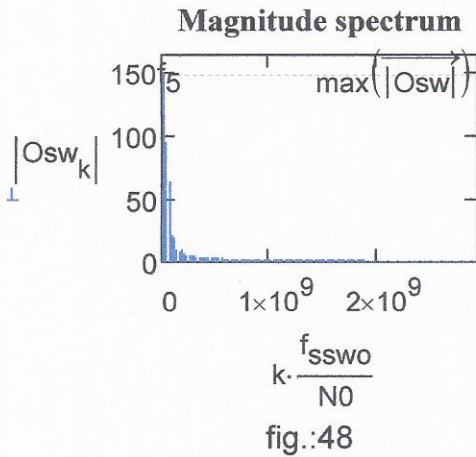


fig.:48

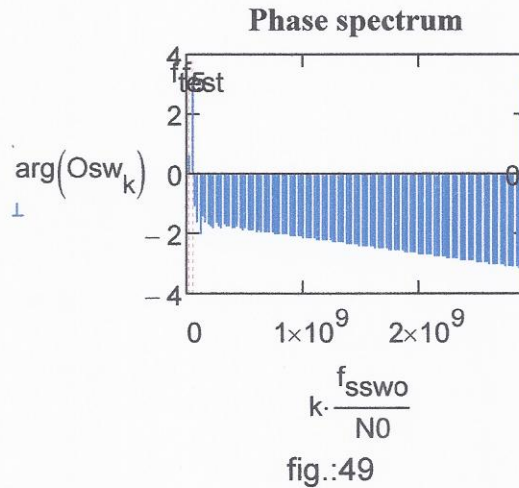


fig.:49



## 5.2 ANALOG FILTER OUTPUT ANALYSIS

### 5.2.4 Bipolar Square Wave response.

Signal Bandwidth calculation using the Fourier series' harmonics. Are excluded all harmonics with amplitude less than  $rt_{gd} = 1\%$  of the fundamental.

$$V_{pp} = 5 \times 10^3 \cdot \text{mV} \quad V_{sqwb}(t) := \frac{v_{sqw}(t, T_{test}, V_{pp})}{\text{volt}} \quad (5.2.4.1)$$

$$T_{test} = 1.639 \times 10^{-8} \text{ s}$$

Signal bandwidth:

Description of the program's parameters:

*BCSA(Dimensionless signal name, relative error, polynomial degree, start time, signal period)*

*BCSA stands for "Bandwidth Calculation and Signal Analysis"*

$$rt_{gd} = 1\% \quad Sb_{sqw} := \text{BCSA}(V_{sqwb}, rt_{gd}, 50, 0.0 \cdot \text{sec}, T_{test}) \quad (5.2.4.2)$$

▶ Bandwidth Calculation

The function returns a three columns matrix.

The first column contains:

- pos. 0: relative error,
- pos. 1: bandwidth (Dimensionless),
- pos. 2: the nth. harmonic number corresponding to the give relative error,
- pos. 3: temporary variable,
- pos. 4: Parseval,
- pos. 5: signal average,
- pos. 6: signal rms.

The second column contains the coefficients  $a_k$  of the Fourier series,  
the third column contains the coefficients  $b_k$  of the Fourier series.

$$\text{Signal bandwidth: } B_{sqw} = 2.928 \cdot \text{GHz}$$

$$\text{Parseval}_{sqwb} = 49.595 \text{ V}^2$$

$$\text{Average2} = 0 \text{ V}$$

$$\text{RMS2} = 2.887 \text{ V}$$

$$\text{sampling frequency (Nyquist rate)} f_{ssqw} := 2 \cdot B_{sqw} \quad f_{ssqw} = 5.856 \cdot \frac{\text{Grads}}{\text{sec}} \quad (5.2.4.3)$$

$$n_{sqw_k} := \frac{k}{f_{ssqw}} \quad T_{ssqw} := \frac{1}{f_{ssqw}} \quad (5.2.4.4)$$

$$sqw_k := V_{sqwb}(n_{sqw_k}) \quad \frac{N0}{f_{ssqw}} \cdot \frac{1}{T_{test}} = 2.667 \quad (5.2.4.5)$$



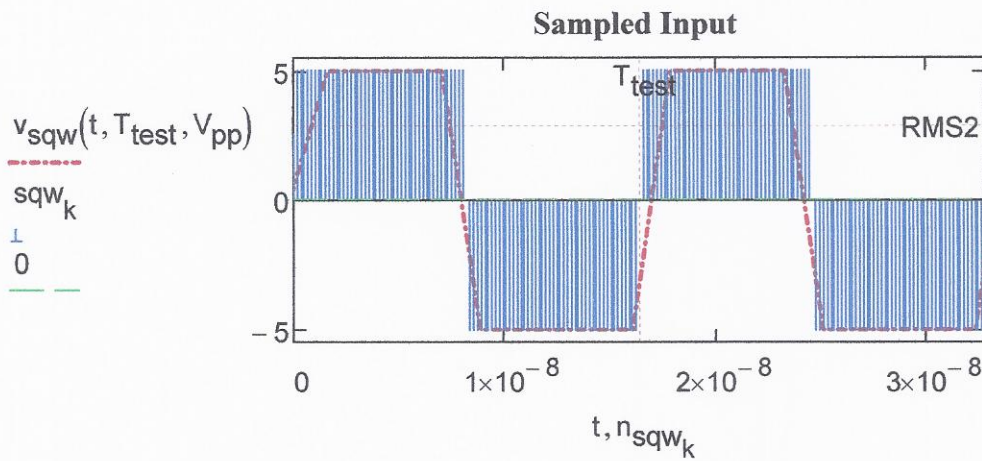


fig.:5.2.4.1

**Approximate signal reconstruction according to the Shannon sampling theorem:**

$$\omega_{s4} := 2 \cdot \pi \cdot B_{sqw} \quad sh4(t) := \left[ \sum_{n=0}^{N0-1} (sqw_n \cdot \text{sinc}(\omega_{s4} \cdot t - n \cdot \pi)) \right] \quad (5.2.4.6)$$

$$t := 0 \cdot T_{test}, 0 \cdot T_{test} + \frac{5 \cdot T_{test} - 0 \cdot T_{test}}{1000} .. 5 \cdot T_{test} \quad rt_{gd} = 1 \cdot \%$$

**Reconstructed Waveform Compared to the Original One**

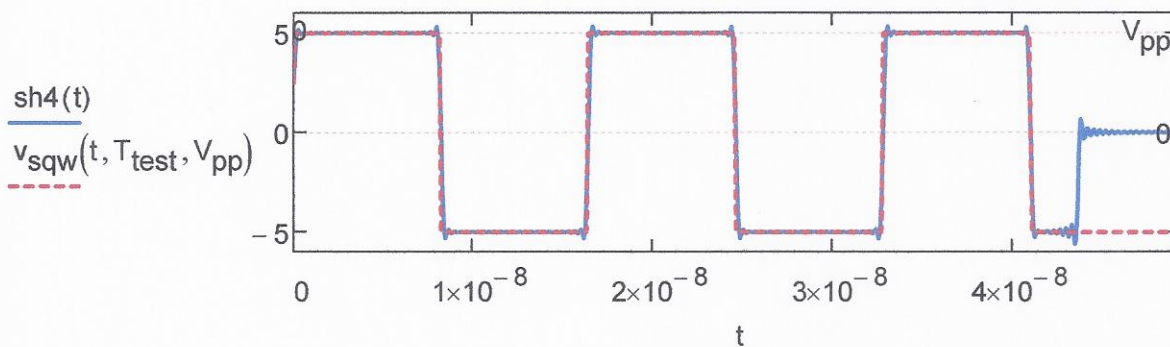


fig.:5.2.4.2

Output calculation using the Laplace transform of the input and the filter's transfer function.

$$\text{Input L. t.} \quad \mathcal{L}(v_i(t, T_{test}, V_{pp})) = \frac{V_{pp}}{s} \cdot \tanh\left(\frac{T_{test} \cdot s}{4}\right) \quad (5.2.4.7)$$

First case  $\zeta \neq \omega_5$ :

$$\text{Output inverse L. t.:} \quad V_{opt}(t) = \mathcal{L}^{-1}\left(\frac{V_{pp}}{s} \cdot \tanh\left(\frac{T_{test} \cdot s}{4}\right) \cdot \frac{A_5 \cdot \omega_5^2}{s^2 + 2 \cdot \zeta \cdot s + \omega_5^2}\right) \quad (5.2.4.8)$$

Second case  $\zeta = \omega_5$ :

$$\text{Output inverse L. t.:} \quad V_{opt}(t) = \mathcal{L}^{-1}\left[\frac{V_{pp}}{s} \cdot \tanh\left(\frac{T_{test} \cdot s}{4}\right) \cdot \frac{A_5 \cdot \omega_5^2}{(s + \omega_5)^2}\right] \quad (5.2.4.9)$$

Output final value:

$$\text{First case } \zeta \neq \omega_5: \quad \lim_{s \rightarrow 0} \left( \frac{V_{pp} \cdot s}{s} \cdot \tanh\left(\frac{T_{\text{test}} \cdot s}{4}\right) \cdot \frac{A_5 \cdot \omega_5^2}{s^2 + 2 \cdot \zeta \cdot s + \omega_5^2} \right) \rightarrow 0$$

$$\text{Second case } \zeta = \omega_5: \quad \lim_{s \rightarrow 0} \left[ \frac{V_{pp} \cdot s}{s} \cdot \tanh\left(\frac{T_{\text{test}} \cdot s}{4}\right) \cdot \frac{A_5 \cdot \omega_5^2}{(s + \omega_5)^2} \right] \rightarrow 0$$

How to calculate the Laplace transform of the input signal:

Shift theorem  $\mathcal{L}(f(t-a)) = e^{-a \cdot s} \cdot F(s)$

$$V_i(s) = \frac{V_{pp}}{s} \cdot \sum_{k=0}^N \left[ e^{-k \cdot T_{\text{test}} \cdot s} - 2 \cdot e^{-\frac{2 \cdot k+1}{2} \cdot T_{\text{test}} \cdot s} + e^{-(k+1) \cdot T_{\text{test}} \cdot s} \right] \quad (5.2.4.10)$$

$$\mathcal{L} \left( \sum_{k=0}^{\infty} f(t-k \cdot T) \right) = \frac{F(s)}{1 - e^{-T \cdot s}}$$

Filter's output signal calculation:

$$\text{First case } \zeta \neq \omega_5: \quad V_o(s) = V_i(s) \cdot \frac{A_5 \cdot \omega_5^2}{s^2 + 2 \cdot \zeta \cdot s + \omega_5^2} \quad (5.2.4.11)$$

$$\text{Second case } \zeta = \omega_5: \quad V_o(s) = V_i(s) \cdot \frac{A_5 \cdot \omega_5^2}{(s + \omega_5)^2} \quad (5.2.4.12)$$

**First case**  $\zeta \neq \omega_5$ .

► First case: Output calculation

Once defined the function:

$$f1_{\text{sq}}(t) := A_5 \cdot \left[ 1 - e^{-\zeta \cdot t} \cdot \left( \cosh\left(t \cdot \sqrt{\zeta^2 - \omega_5^2}\right) + \frac{\zeta \cdot \sinh\left(t \cdot \sqrt{\zeta^2 - \omega_5^2}\right)}{\sqrt{\zeta^2 - \omega_5^2}} \right) \right] \cdot \Phi(t) \quad (5.2.4.13)$$

the output is given by the sum:

$$V_{o1}(t) := V_{pp} \cdot \left[ \sum_{k=0}^N \left[ f1_{\text{sq}}\left(t - k \cdot T_{\text{test}}\right) - 2 \cdot f1_{\text{sq}}\left(t - \frac{2 \cdot k+1}{2} \cdot T_{\text{test}}\right) + f1_{\text{sq}}\left[t - (k+1) \cdot T_{\text{test}}\right] \right] \right] \quad (5.2.4.13')$$

**Second case**  $\zeta = \omega_5$ :

► Second case: Output calculation

$$f2_{\text{sq}}(t) := A_5 \cdot \left[ 1 - e^{-t \cdot \omega_5} \cdot (1 + t \cdot \omega_5) \right] \cdot \Phi(t) \quad (5.2.4.14)$$

$$V_{o2}(t) := V_{pp} \cdot \left[ \sum_{k=0}^N \left[ f_{2sq}(t - k \cdot T_{test}) - 2 \cdot f_{2sq}\left(t - \frac{2 \cdot k + 1}{2} \cdot T_{test}\right) + f_{2sq}[t - (k + 1) \cdot T_{test}] \right] \right] \quad (5.2.4.15)$$

$$\zeta = 0.018 \cdot \frac{\text{Grads}}{\text{sec}} \quad \frac{1}{T_{test}} = 0.061 \cdot \text{GHz}$$

$$N = 50$$

### Output signal

$$V_{sqw}(t) := \begin{cases} V_{o1}(t) & \text{if } \zeta \neq \omega_5 \\ V_{o2}(t) & \text{otherwise} \end{cases} \quad (5.2.4.16)$$

$$V_{sqw}\left(\frac{T_{test}}{1.3}\right) = -45.135V$$

Graph of the bipolar Square Wave response considering the Op Amp saturation:

$$V_{osqw}(t) := \text{if}(-V_{sat} \leq V_{sqw}(t) \leq V_{sat}, V_{sqw}(t), \text{if}(V_{sqw}(t) \leq 0.0 \cdot \text{volt}, -V_{sat}, V_{sat})) \quad (5.2.4.17)$$

$$A_5 = -10$$

$$T_{test} = 16.394 \cdot \text{ns}$$

### Bipolar Square Wave response.

$$f_{test} = 0.061 \cdot \text{GHz}$$

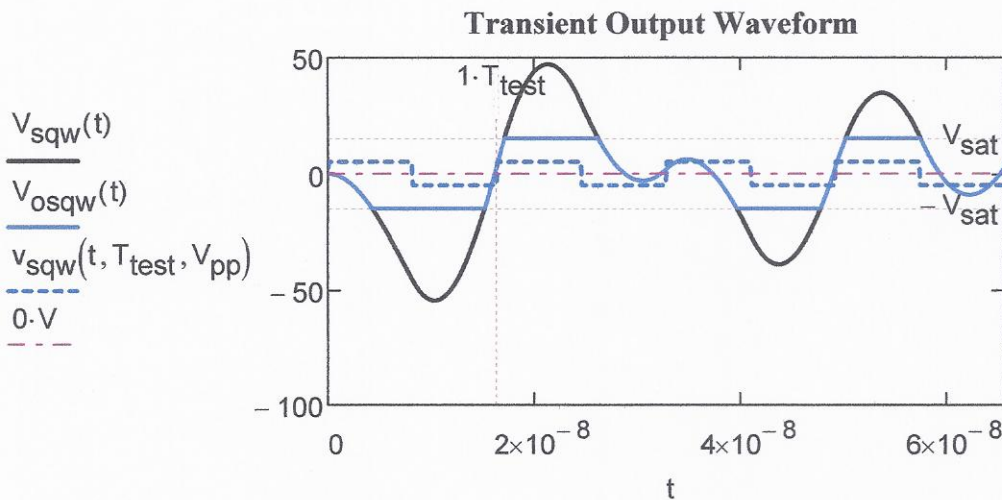


fig.:5.2.4.3

$$vosqw_k := V_{sqw}(n_{sqw_k}) \quad (5.2.4.18)$$

**Approximate output signal reconstruction according to the Shannon sampling theorem:**

$$\omega_{s5} := 2 \cdot \pi \cdot B_{sqw} \quad sh5(t) := \left[ \sum_{n=0}^{N0-1} \left( vosqw_n \cdot \text{sinc}(\omega_{s5} \cdot t - n \cdot \pi) \right) \right] \quad (5.2.4.19)$$



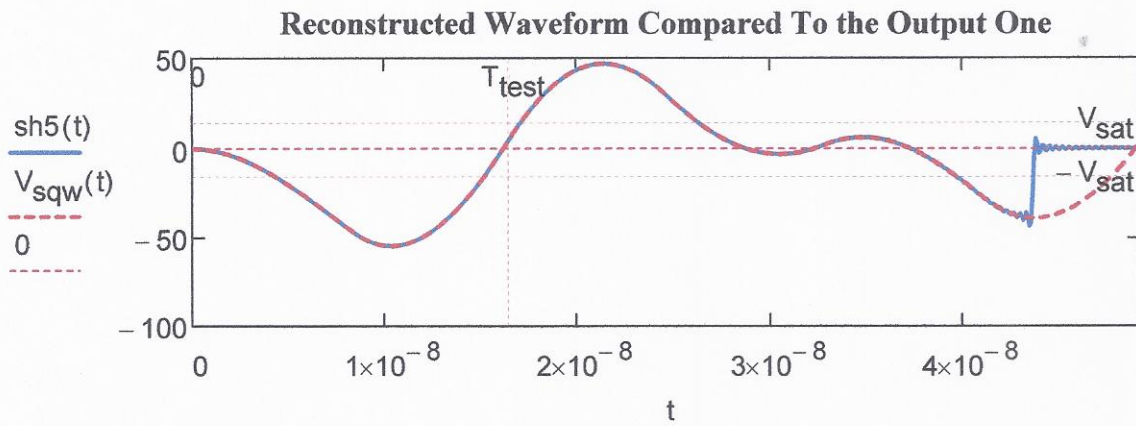


fig.:5.2.4.5

$$vopt_k := V_{osqw}(n_{sqw_k}) \quad (5.2.4.20)$$

**Approximate reconstruction of the output signal (with Op Amp saturation) according to the Shannon sampling theorem:**

$$\omega_{s5} := 2 \cdot \pi \cdot B_{sqw} \quad sh5(t) := \left[ \sum_{n=0}^{N0-1} (vopt_n \cdot \text{sinc}(\omega_{s5} \cdot t - n \cdot \pi)) \right] \quad (5.2.4.21)$$

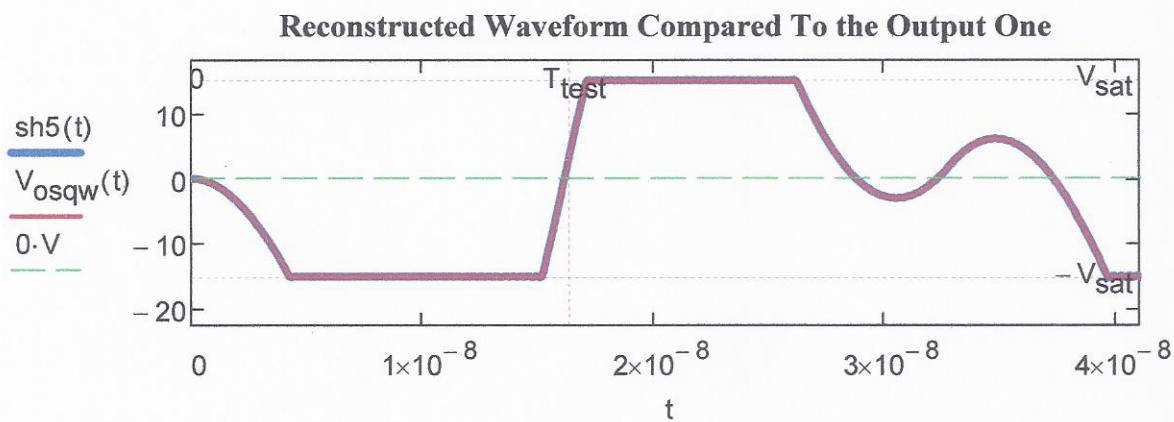


fig.:5.2.4.6

$$\frac{N0}{f_{ssqw}} \cdot \frac{1}{T_{test}} = 2.667$$

$$Opt := \text{fft}(vopt) \quad (5.2.4.22)$$

**Magnitude spectrum**

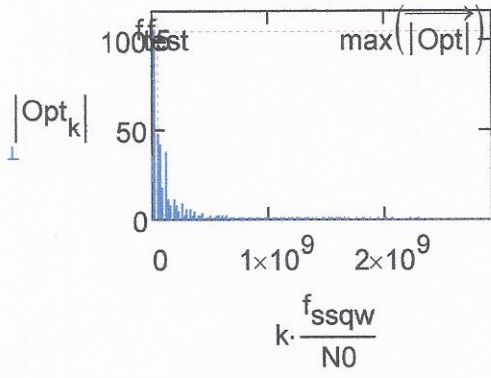


fig.:5.2.4.7

**Phase spectrum**

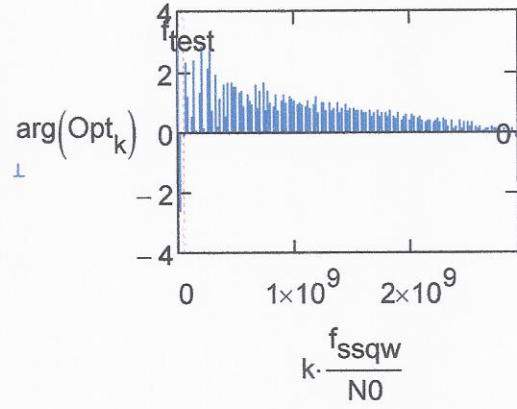


fig.:5.2.4.8

## Convolution:

The second method is to calculate *the time domain convolution product* between the signal and the impulse response:

$$\begin{aligned} t_w &:= -1 \cdot T_{\text{test}}, -1 \cdot T_{\text{test}} + \frac{4 \cdot T_{\text{test}} + 1 \cdot T_{\text{test}}}{100} \dots 4 \cdot T_{\text{test}} \\ v_{\text{osqconv}}(t) &:= \int_0^t w(t-\sigma) \cdot v_{\text{sqw}}(\sigma, T_{\text{test}}, V_{\text{pp}}) d\sigma \end{aligned} \quad (5.2.4.23)$$

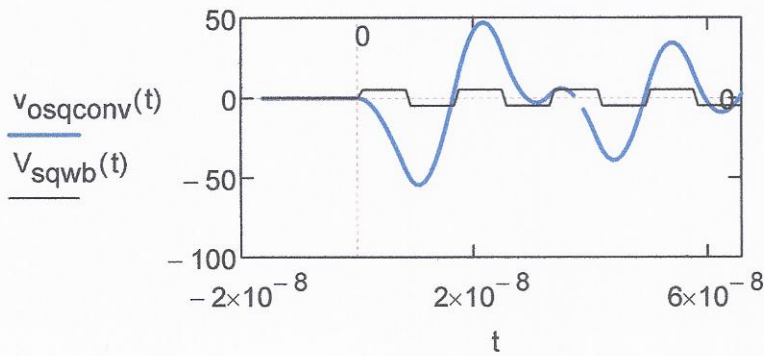


fig.:5.2.4.9



## 5.2 ANALOG FILTER OUTPUT ANALYSIS

### 5.2.5 (single tone) AM Signal response.

$$A1 = 10V \quad \omega_c := 1 \cdot \omega_{\text{test}} \quad f_c := \frac{\omega_c}{2 \cdot \pi} \quad T_c := \frac{1}{f_c} \quad \omega_{\text{mam}} := \frac{1}{5} \cdot \omega_5 \quad f_{\text{mam}} := \frac{\omega_{\text{mam}}}{2 \cdot \pi}$$

$$B1 = 5.5V \quad f_c = 60.997 \cdot \text{MHz}$$

$$T_{\text{mam}} := \frac{1}{f_{\text{mam}}} \quad f_{\text{mam}} = 6.1 \cdot \text{MHz} \quad \text{modulation index: } m_{\text{am}} = 55\%$$

$$\omega_{\text{mam}} = 0.038 \cdot \frac{\text{Grads}}{\text{sec}} \quad \omega_c = 0.383 \cdot \frac{\text{Grads}}{\text{sec}} \quad Bw_{\text{am}} := f_c + 2 \cdot f_{\text{mam}}$$

$$\text{sampling frequency (Nyquist rate): } f_{\text{sam}} := 2 \cdot Bw_{\text{am}}, \quad f_{\text{sam}} = 146.392 \cdot \text{MHz}$$

$$\text{sampling angular frequency: } \omega_{\text{sam}} := 2 \cdot \pi \cdot f_{\text{sam}}, \quad \omega_{\text{sam}} = 0.92 \cdot \frac{\text{Grads}}{\text{sec}}$$

$$\text{sampling period: } T_{\text{sam}} := \frac{1}{f_{\text{sam}}}, \quad T_{\text{sam}} = 6.831 \times 10^{-3} \cdot \mu\text{s}$$

$$\text{sampling time step: } \text{nam}_k := \frac{k}{f_{\text{sam}}},$$

$$N0 \cdot \frac{T_{\text{sam}}}{T_{\text{test}}} = 106.667$$

$$\text{nam}^T = \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & \\ \hline 0 & 0 & 6.831 \cdot 10^{-3} & 0.014 & 0.02 & & \dots \end{array} \cdot \mu\text{s}$$

$$\frac{\omega_c}{\omega_{\text{mam}}} = 10$$

$$V2_i(t) := v2_i(t, \omega_{\text{mam}}, \omega_c, A1, B1) \quad (5.2.5.1)$$

$$u_{7k} := \frac{V2_i(\text{nam}_k)}{\text{volt}} \quad (5.2.5.2)$$

$$t := 0 \cdot T_c, 0 \cdot T_c + \frac{20 \cdot T_c}{1000} \dots 20 \cdot T_c \quad \frac{N0}{f_{\text{sam}}} \cdot \frac{1}{T_{\text{test}}} = 106.667$$

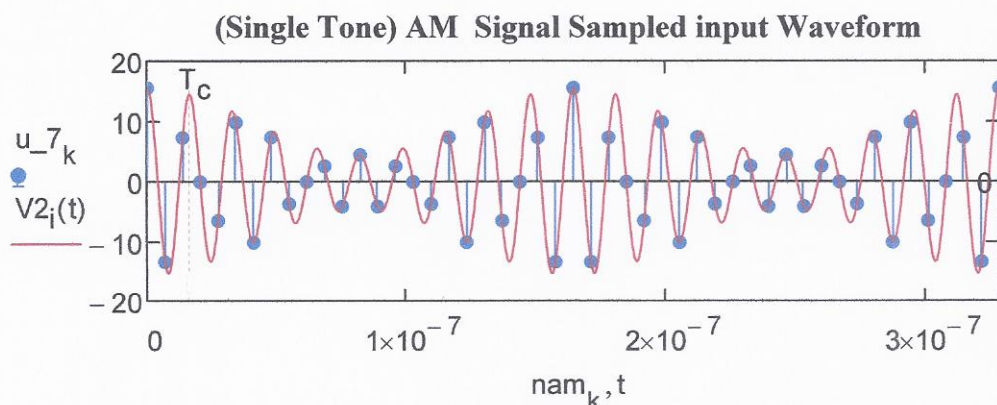


fig.:5.2.5.1

**Approximate signal reconstruction according to the Shannon sampling theorem:**

$$\omega_{sh6} := 2 \cdot \pi \cdot Bw_{am} \quad sh6(t) := \left[ \sum_{n=0}^{N0-1} (u\_7n \cdot \text{sinc}(\omega_{sh6} \cdot t - n \cdot \pi)) \right] \quad (5.2.5.3)$$

$$rt_{gd} = 1.0\%$$

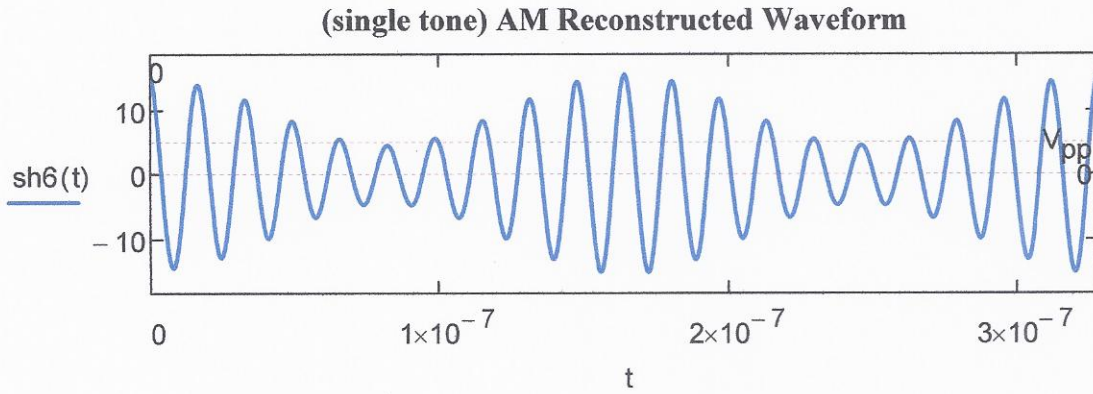


fig.:5.2.5.2

$$Spec\_7 := \text{fft}(u\_7)$$

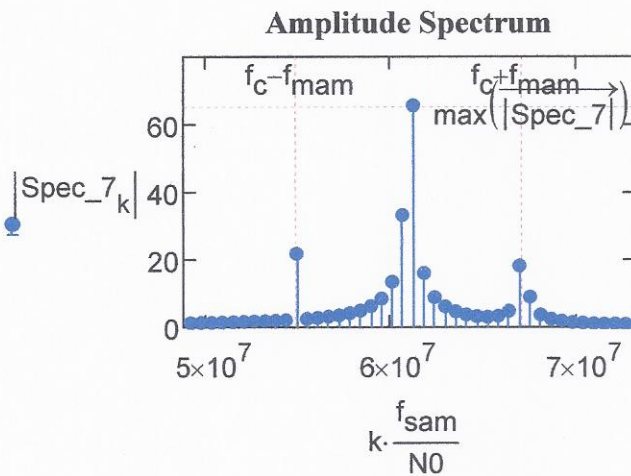


fig.:5.2.5.3

$$V_{pp} = 5V$$

$$\text{Exact output: } v_{oam}(t) := \int_0^t w(t - \sigma) \cdot V2_i(\sigma) d\sigma \quad (5.2.5.4)$$

$$m_{am} = 55.0\%$$

$$t_w := 0 \cdot T_c, 0 \cdot T_c + \frac{40 \cdot T_c}{1000} \dots 40 \cdot T_c$$

$$T_c = 0.016 \cdot \mu s$$

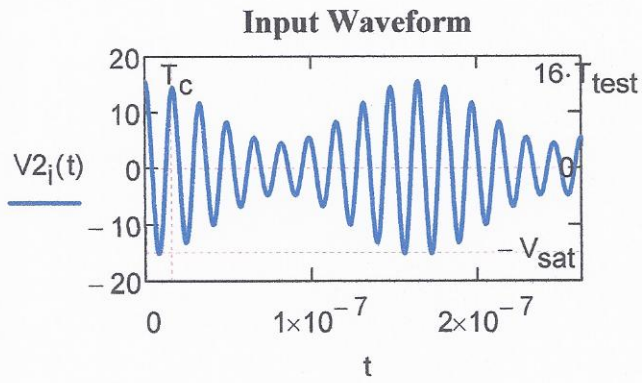


fig.:5.2.5.4

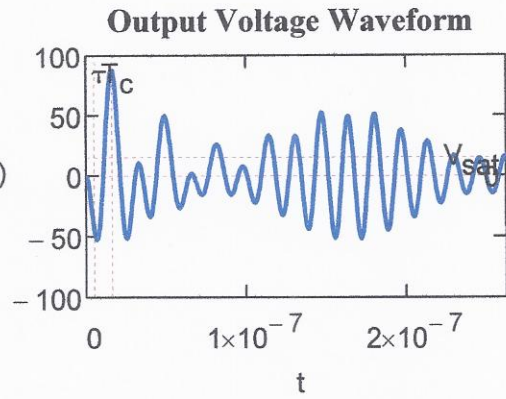


fig.:5.2.5.5

Output sampling:  $V_{am_k} := v_{oam}(nam_k)$

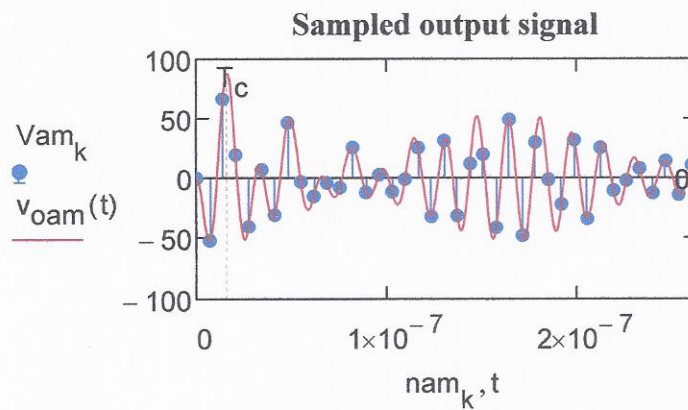


fig.:5.2.5.6

Fourier Transform of the output signal

$$f_c = 0.061 \cdot \text{GHz} \quad \frac{f_{\text{sam}}}{f_c} = 2.4$$

$\text{Spec}_{am} := \text{fft}(V_{am})$

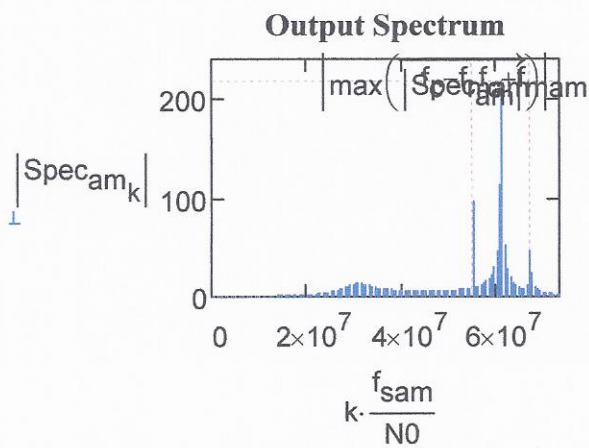


fig.:5.2.5.7

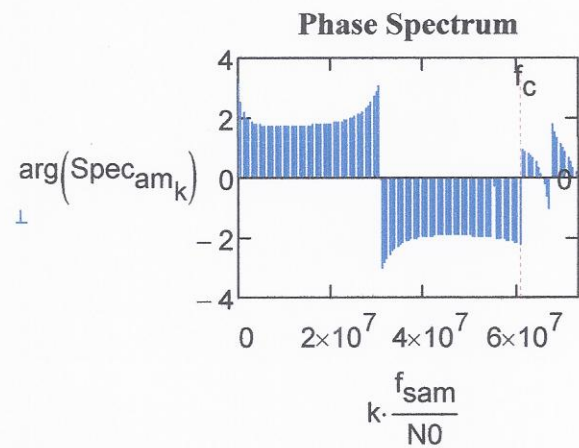


fig.:5.2.5.8

Magnitude of  $W(j\omega)$  in dB

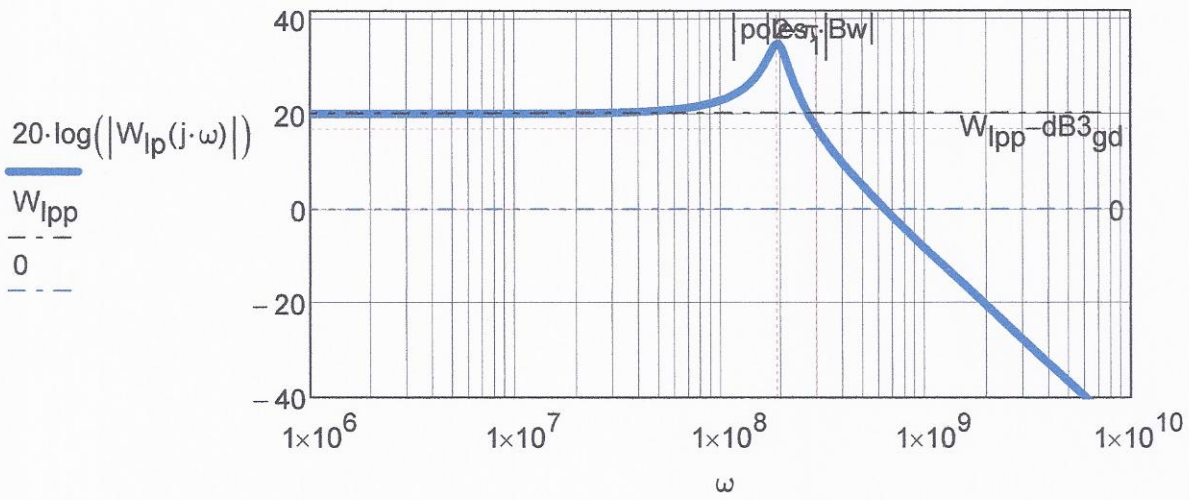


fig.:5.2.5.9

Phase of  $W(j\omega)$

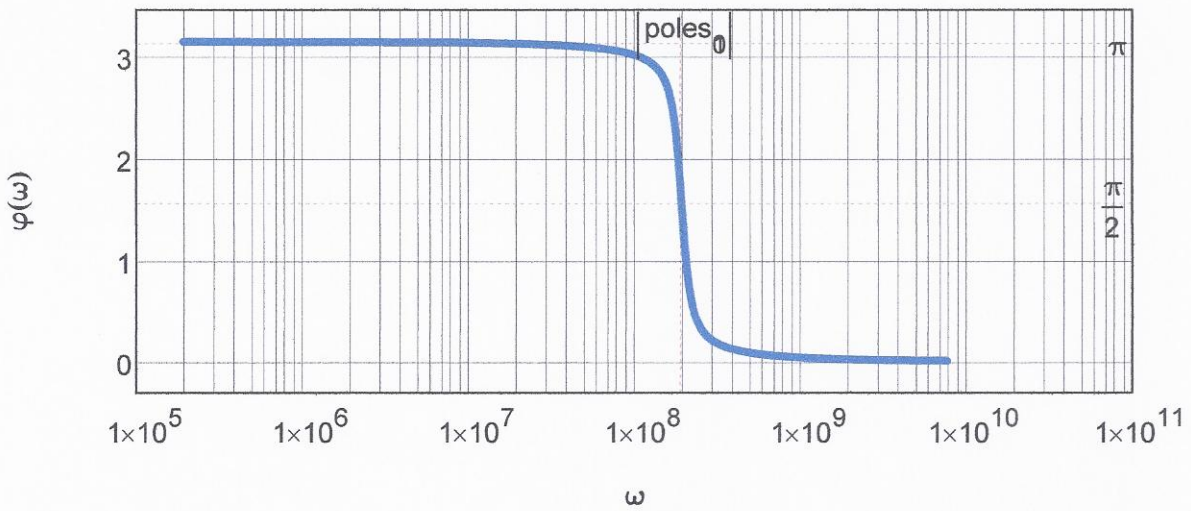


fig.:5.2.5.10



## 5.2 ANALOG FILTER OUTPUT ANALYSIS

### 5.2.6 (single tone) Frequency Modulated carrier response.

$$\begin{aligned} \omega_{cfm} &:= 1 \cdot \omega_{test} & f_{cfm} &:= \frac{\omega_{cfm}}{2 \cdot \pi} \\ T_{cfm} &:= \frac{1}{f_{cfm}} & \omega_{mfm} &:= \frac{\omega_c}{20} \\ f_{mfm} &:= \frac{\omega_{mfm}}{2 \cdot \pi} & T_{mfm} &:= \frac{1}{f_{mfm}} & T_{mfm} &= 0.328 \cdot \mu\text{s} \\ T_{cfm} &= 0.016 \cdot \mu\text{s} \end{aligned}$$

$$\text{frequency modulation index: } m_f := \frac{2 \cdot K_{st} \cdot \pi \cdot B}{\omega_m} \quad m_f := 8 \quad (5.2.6.1)$$

$$\text{Carson bandwidth: } \text{Cars1} := 2 \cdot \omega_{mfm} \cdot (m_f + 1) \quad \text{Cars1} = 0.345 \cdot \frac{\text{Grads}}{\text{sec}} \quad (5.2.6.2)$$

$$\text{sampling frequency (Nyquist rate): } f_{sfm} := 2 \cdot \text{Cars1}, \quad f_{sfm} = 0.69 \cdot \text{GHz} \quad (5.2.6.3)$$

$$\text{sampling angular frequency: } \omega_{sfm} := 2 \cdot \pi \cdot f_{sfm}, \quad \omega_{sfm} = 4.335 \cdot \frac{\text{Grads}}{\text{sec}},$$

$$\text{sampling period: } T_{sfm} := \frac{1}{f_{sfm}}, \quad T_{sfm} = 1.45 \times 10^{-3} \cdot \mu\text{s},$$

$$\begin{aligned} \text{sampling time step: } n_{fmk} &:= \frac{k}{f_{sfm}}, & (5.2.6.4) \\ \frac{N_0}{f_{sfm}} \cdot \frac{1}{T_{test}} &= 22.635 \end{aligned}$$

$$n_{fm}^T = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 0 & 0 & 1.45 \cdot 10^{-3} & 2.899 \cdot 10^{-3} & 4.349 \cdot 10^{-3} & 5.798 \cdot 10^{-3} & 7.248 \cdot 10^{-3} & \dots \\ \hline \end{array} \cdot \mu\text{s}$$

$$\frac{\omega_{cfm}}{\omega_{mfm}} = 20 \quad V_{fm}(t) := v_{fmsl}(t, \omega_{cfm}, \omega_{mfm}, A_{fm}, m_f) \quad (5.2.6.5)$$

$$A_{fm} = 0.02\text{V} \quad u_{8k} := \frac{V_{fm}(n_{fmk})}{\text{volt}} \quad V_{pp} = 5\text{V} \quad (5.2.6.6)$$

**Approximate signal reconstruction according to the Shannon sampling theorem:**

$$\omega_{sh7} := 2 \cdot \pi \cdot \text{Cars1} \quad \text{sh7}(t) := \left[ \sum_{n=0}^{N_0-1} (u_{8n} \cdot \text{sinc}(\omega_{sh7} \cdot t - n \cdot \pi)) \right] \quad (5.2.6.7)$$

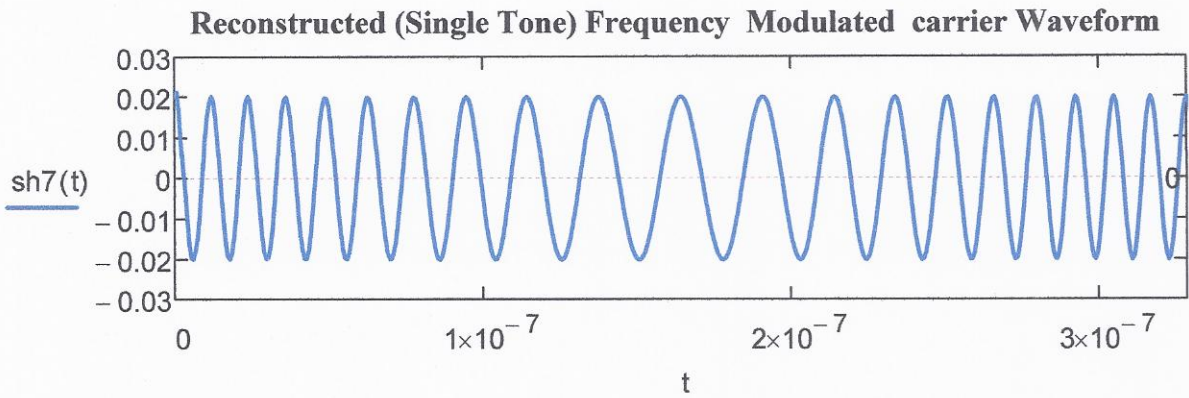


fig.:5.2.6.1

Exact output:

$$f_{2fm}(t, \sigma, \omega_5, \zeta, \omega_{cfm}, \omega_{mfm}, m_f) := \sin\left[(t - \sigma) \cdot \sqrt{\omega_5^2 - \zeta^2}\right] \cdot e^{\zeta \cdot \sigma} \cdot \cos(\omega_{cfm} \cdot \sigma + m_f \cdot \sin(\omega_{mfm} \cdot \sigma))$$

$$v_{ofm}(t) := A_{fm} \cdot A_5 \cdot \omega_5^2 \cdot \begin{cases} \left( \frac{e^{-\zeta \cdot t}}{\sqrt{\omega_5^2 - \zeta^2}} \cdot \int_0^t f_{2fm}(t, \sigma, \omega_5, \zeta, \omega_{cfm}, \omega_{mfm}, m_f) d\sigma \right) & \text{if } \zeta \neq \omega_5 \\ \int_0^t (t - \sigma) \cdot e^{-(t-\sigma) \cdot \omega_5} \cdot \cos(\omega_{cfm} \cdot \sigma + m_f \cdot \sin(\omega_{mfm} \cdot \sigma)) d\sigma & \text{otherwise} \end{cases}$$

(5.2.6.8)

$$v_{ofm}(t) := \int_0^t w(t - \sigma) \cdot V_{fm}(\sigma) d\sigma$$

(5.2.6.9)

$T_{test} = 0.016 \cdot \mu s$

$$t_{fm} := \frac{T_{mfm} \cdot 0}{100}, \frac{T_{mfm} \cdot 0}{100} + \frac{1 \cdot T_{mfm} - \frac{T_{mfm} \cdot 0}{100}}{100} \dots 1 \cdot T_{mfm}$$

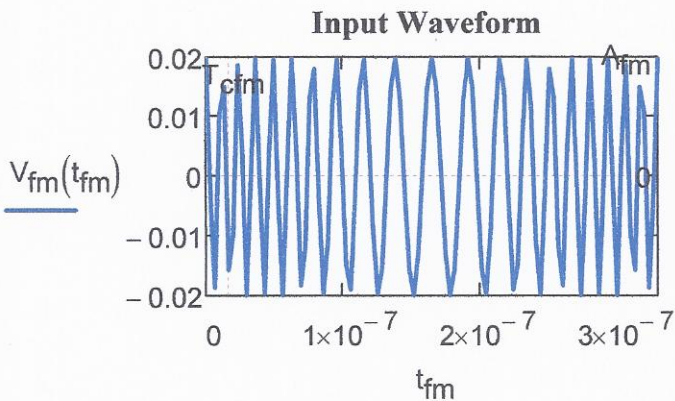


fig.:5.2.6.2

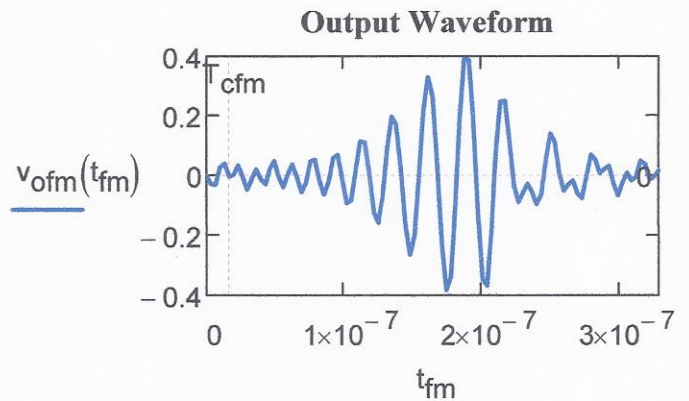


fig.:5.2.6.3

$$v_{ofm}(nfm_{100}) = -0.199V \quad \text{Output sampling: } Ofm_k := v_{ofm}(nfm_k) \quad (5.2.6.10)$$

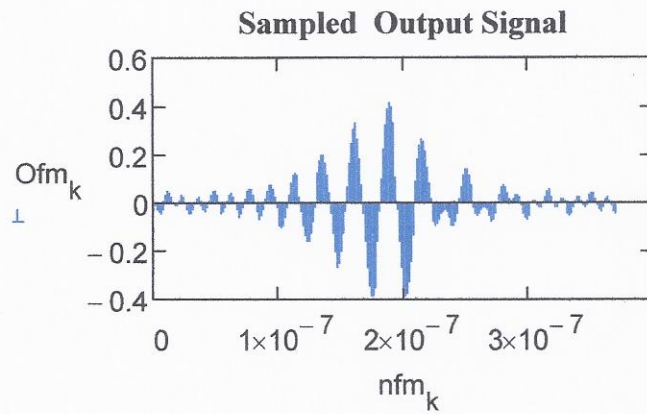


fig.:5.2.6.4

Fourier Transform of the Test signal

$$m_f = 8$$

$$f_c = 0.061 \cdot \text{GHz}$$

$$\frac{f_{\text{sfm}}}{f_c} = 11.31$$

$$\frac{N_0}{f_{\text{sfm}}} \cdot \frac{1}{T_{\text{test}}} = 22.635$$

$$\omega_{\text{mfm}} = 0.019 \cdot \frac{\text{Grads}}{\text{sec}}$$

$$\text{OSpecfm} := \text{fft}(\text{Ofm})$$

$$(5.2.6.11)$$

FM Test Signal spectrum

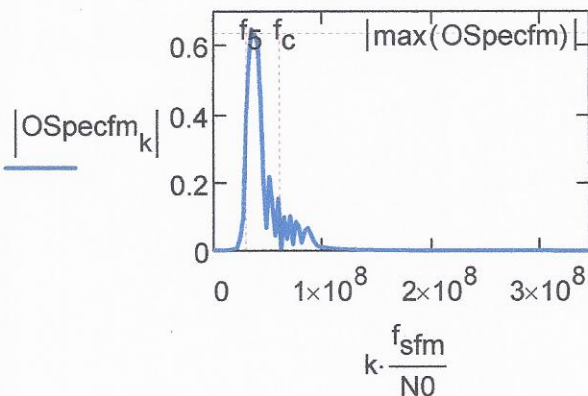


fig.:5.2.6.5

Phase spectrum

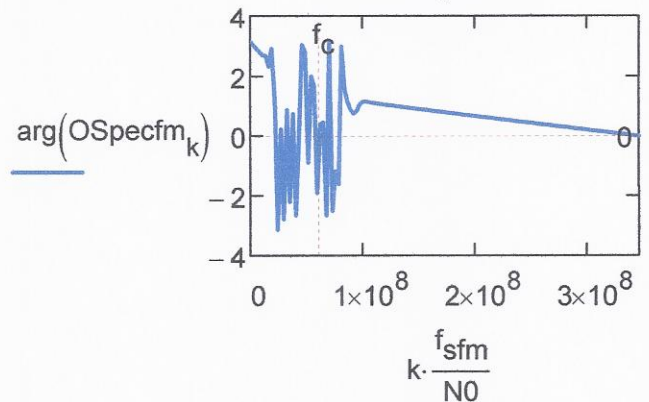


fig.:5.2.6.6

On the other hand if the carrier frequency is located in the passing band, the filter response is :

$$\omega_{3c} := \frac{\omega_c}{100}$$

$$\omega_{3m} := \frac{\omega_{\text{mfm}}}{100}$$

$$T_{3c} := \frac{2 \cdot \pi}{\omega_{3c}}$$

$$T_{3m} := \frac{2 \cdot \pi}{\omega_{3m}}$$

$$\text{Carson bandwidth: } \text{Cars3} := 2 \cdot \omega_{3m} \cdot (m_f + 1) \quad \text{Cars3} = 3.449 \cdot \frac{\text{Mrads}}{\text{sec}} \quad (5.2.6.12)$$

$$\text{Carrier frequency: } f_{3c} := \frac{\omega_{3c}}{2 \cdot \pi}, \quad f_{3c} = 6.1 \times 10^{-4} \cdot \text{GHz}$$

$$\text{sampling frequency (Nyquist rate): } f_{3\text{sfm}} := 2 \cdot \text{Cars3}, \quad f_{3\text{sfm}} = 6.899 \times 10^{-3} \cdot \text{GHz}$$

$$\text{sampling angular frequency: } \omega_{3\text{sfm}} := 2 \cdot \pi \cdot f_{3\text{sfm}}, \quad \omega_{3\text{sfm}} = 0.043 \cdot \frac{\text{Grads}}{\text{sec}}$$

$$\text{sampling period: } T_{3\text{sfm}} := \frac{1}{f_{3\text{sfm}}}, \quad T_{3\text{sfm}} = 0.145 \cdot \mu\text{s}$$



sampling time step:  $n3fm_k := \frac{k}{f3_{sfm}}$ ,

$$\frac{N0}{f3_{sfm}} \cdot \frac{1}{T_{test}} = 2.264 \times 10^3$$

$$A_{fm} = 0.02V \quad V1_{fm}(t) := v_{fmsl}(t, \omega3_c, \omega3_m, A_{fm}, m_f) \quad (5.2.6.13)$$

$$u8b_k := \frac{V1_{fm}(n3fm_k)}{\text{volt}} \quad (5.2.6.14)$$

**Approximate signal reconstruction according to the Shannon sampling theorem:**

$$\omega_{s7} := 2 \cdot \pi \cdot Cars1 \quad sh7b(t) := \left[ \sum_{n=0}^{N0-1} (u8b_n \cdot \text{sinc}(\omega_{s7} \cdot t - n \cdot \pi)) \right] \quad (5.2.6.15)$$

$$rt_{gd} = 1\%$$

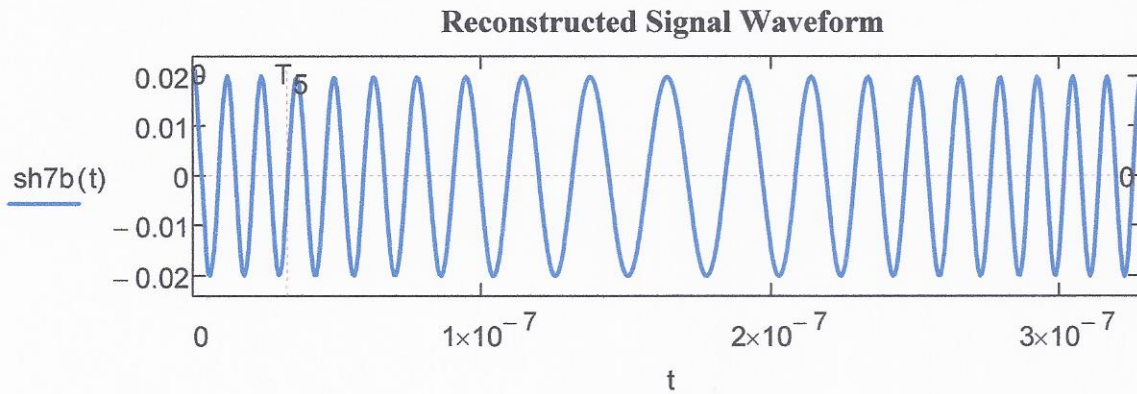


fig.:5.2.6.7

Exact output:

$$v2_{ofm}(t) := A_{fm} \cdot A_5 \cdot \omega_5^2 \cdot \begin{cases} \frac{e^{-\zeta \cdot t}}{\sqrt{\omega_5^2 - \zeta^2}} \cdot \int_0^t f2_{fm}(t, \sigma, \omega_5, \zeta, \omega3_c, \omega3_m, m_f) d\sigma & \text{if } \zeta \neq \omega_5 \\ \int_0^t (t - \sigma) \cdot e^{-(t-\sigma) \cdot \omega_5} \cdot \cos(\omega3_c \cdot \sigma + m_f \cdot \sin(\omega3_m \cdot \sigma)) d\sigma & \text{otherwise} \end{cases} \quad (5.2.6.16)$$

$$v2_{ofm}(t) := \int_0^t w(t - \sigma) \cdot V1_{fm}(\sigma) d\sigma \quad (5.2.6.17)$$

$$T_{test} = 0.016 \cdot \mu s \quad T1_c = 2 \times 10^3 \cdot \mu s \quad \frac{1}{f3_{sfm}} = 144.957 \cdot ns \quad V_{pp} = 5V$$



$$t1_{fm} := T_{mfm} \cdot 0, T_{mfm} \cdot 0 + \frac{100 \cdot T_{mfm} - T_{mfm} \cdot 0}{400} .. 100 \cdot T_{mfm}$$

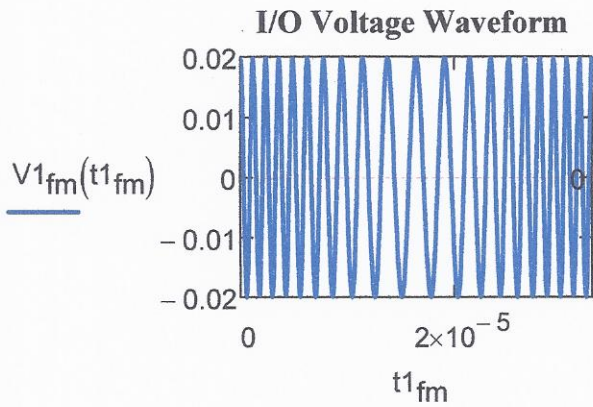


fig.:5.2.6.8

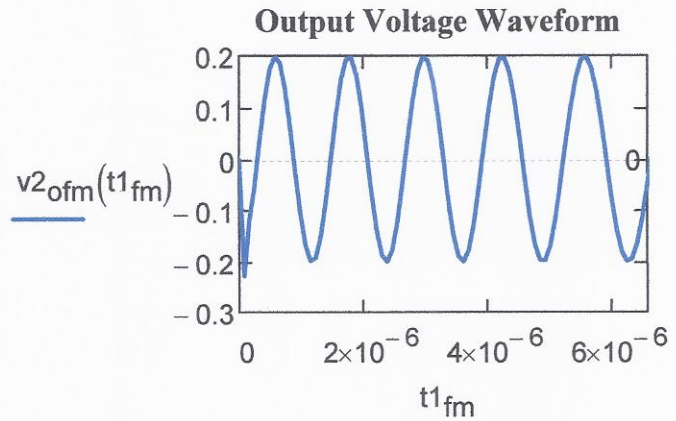


fig.:5.2.6.9

$$n3fm_k := \frac{k}{N0} \cdot T3_c$$

Output sampling:  $Ofm_k := v2_{ofm}(n3fm_k)$

(5.2.6.18)

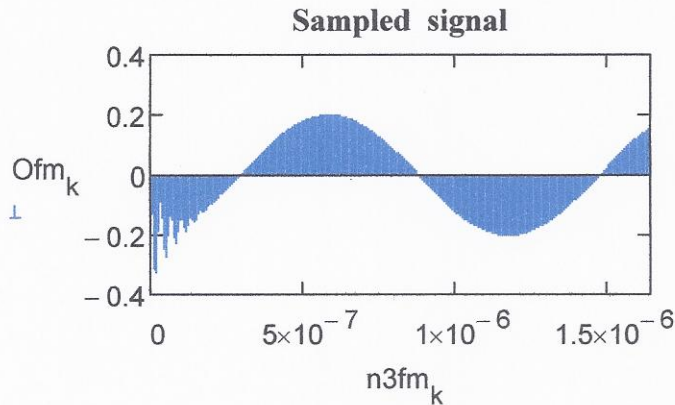


fig.:5.2.6.10

Fourier Transform of the Test signal  $f3_c = 6.1 \times 10^{-4} \cdot \text{GHz}$   $\frac{f3_{sfm}}{f3_c} = 11.31$   $m_f = 8$

$$\omega_{mfm} = 0.019 \cdot \frac{\text{Grads}}{\text{sec}}$$

OSpecfm3 := fft(Ofm) (5.2.6.19)

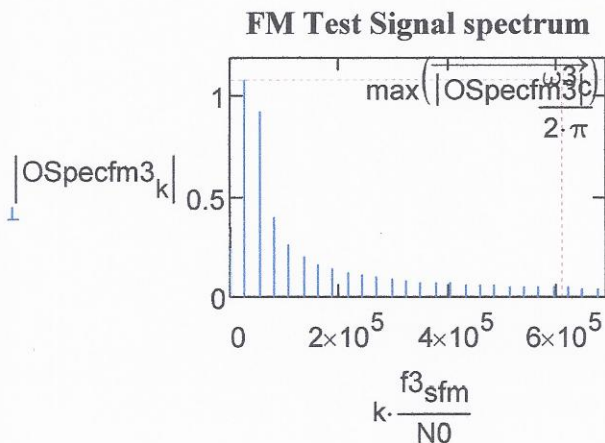


fig.:5.2.6.11

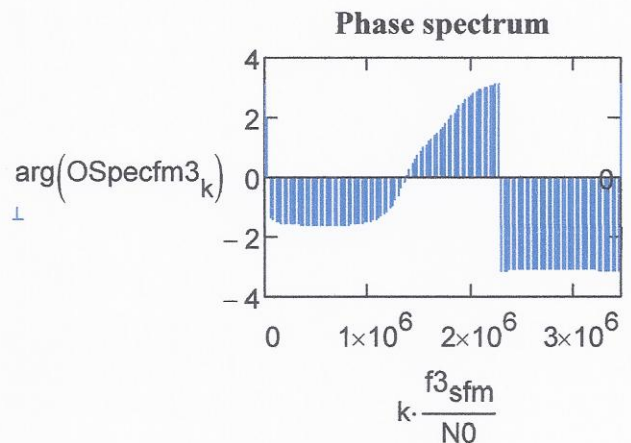


fig.:5.2.6.12

$$\omega_c := \frac{\omega_5}{20 \cdot U}, \frac{\omega_5}{20 \cdot U} + \frac{\omega_5 \cdot U - \frac{\omega_5}{20 \cdot U}}{U^2} \dots 10 \cdot U \cdot \omega_5$$

$$Wdb\omega_{1c} := 20 \cdot \log(|W_{lp}(j \cdot \omega_{1c})|) \quad (5.2.6.20)$$

$$Wdb\omega_{1c} = 20 \cdot \text{dB}$$

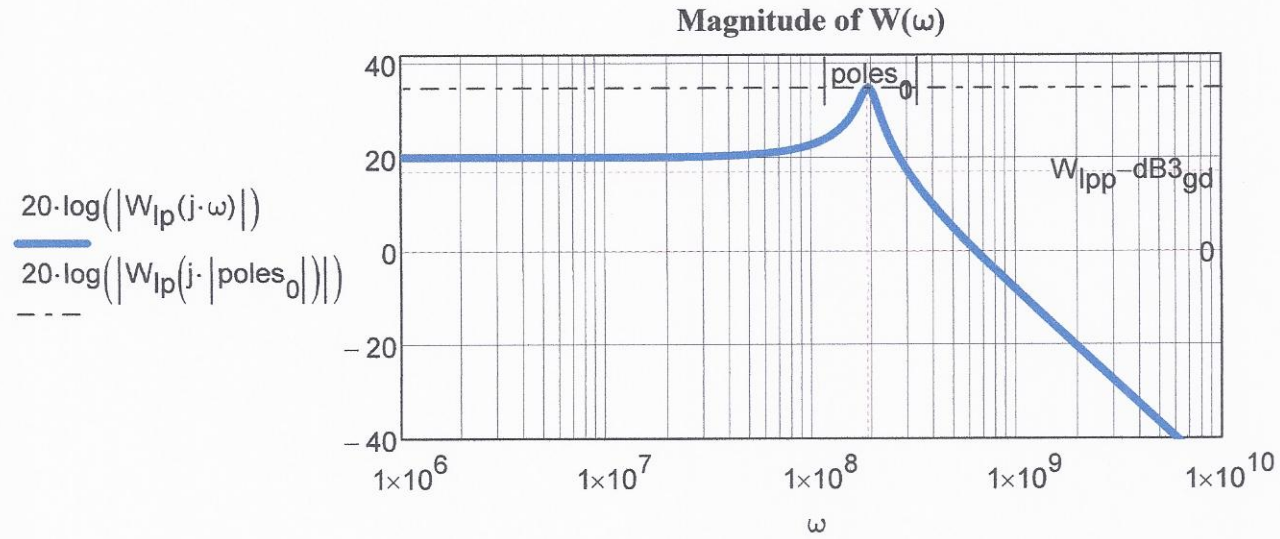


fig.:5.2.6.13

## 5.2 ANALOG FILTER OUTPUT ANALYSIS

### 5.2.7 (single tone) Phase Modulated carrier response.

$$\omega_{\text{cpm}} := 4 \cdot \omega_{\text{test}} \quad f_{\text{cpm}} := \frac{\omega_{\text{cpm}}}{2 \cdot \pi} \quad T_{\text{cpm}} := \frac{1}{f_{\text{cpm}}} \quad (5.2.7.1)$$

$$\omega_{\text{mpm}} := \frac{\omega_{\text{cpm}}}{20} \quad f_{\text{mpm}} := \frac{\omega_{\text{mpm}}}{2 \cdot \pi} \quad T_{\text{mpm}} := \frac{1}{f_{\text{mpm}}} \quad (5.2.7.2)$$

$$T_{\text{cpm}} = 4.099 \times 10^{-3} \cdot \mu\text{s} \quad T_{\text{mpm}} = 0.082 \cdot \mu\text{s}$$

$$m_p := 8$$

$$\text{Carson bandwidth: } \text{Cars4} := 2 \cdot \omega_{\text{mpm}} \cdot (m_p + 1) \quad (5.2.7.3)$$

$$\text{sampling frequency (Nyquist rate): } f_{\text{spm}} := 2 \cdot \text{Cars4}, \quad f_{\text{spm}} = 2.759 \cdot \text{GHz} \quad (5.2.7.4)$$

$$\text{sampling angular frequency: } \omega_{\text{spm}} := 2 \cdot \pi \cdot f_{\text{spm}}, \quad \omega_{\text{spm}} = 17.338 \cdot \frac{\text{Grads}}{\text{sec}},$$

$$\text{sampling period: } T_{\text{spm}} := \frac{1}{f_{\text{spm}}}, \quad T_{\text{spm}} = 3.624 \times 10^{-4} \cdot \mu\text{s},$$

$$\text{sampling time step: } \text{npm}_k := \frac{k}{f_{\text{spm}}}, \quad (5.2.7.5)$$

$$\frac{N0}{f_{\text{spm}}} \cdot \frac{1}{T_{\text{test}}} = 5.659 \quad (5.2.7)$$

$$\text{npm}^T = \begin{array}{|c|c|c|c|c|c|} \hline & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 0 & 3.624 \cdot 10^{-4} & 7.248 \cdot 10^{-4} & 1.087 \cdot 10^{-3} & \dots \\ \hline \end{array} \cdot \mu\text{s}$$

$$A_{\text{pm}} = 0.02\text{V} \quad V_{\text{pm}}(t) := v_{\text{pm}}(t, \omega_{\text{cpm}}, \omega_{\text{mpm}}, A_{\text{pm}}, m_p) \quad (5.2.7.7)$$

$$u_{9k} := \frac{V_{\text{pm}}(\text{npm}_k)}{\text{volt}} \quad (5.2.7.8)$$

Sampled PM Waveform

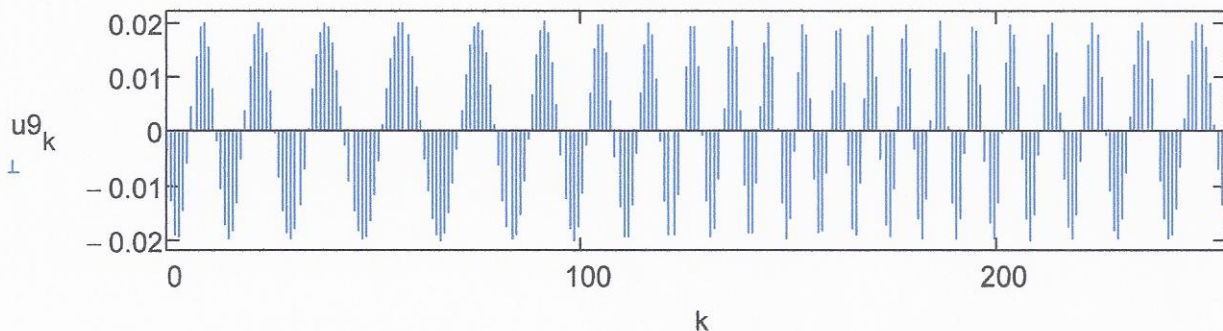


fig.:5.2.7.1



**Approximate signal reconstruction according to the Shannon sampling theorem:**

$$\omega_{ts8} := 2 \cdot \pi \cdot Cars4 \quad sh8(t) := \left[ \sum_{n=0}^{N0-1} (u9n \cdot \text{sinc}(\omega_{ts8} \cdot t - n \cdot \pi)) \right] \quad (5.2.7.9)$$

$$t_{pm} := 0 \cdot T_{cpm}, 0 \cdot T_{cpm} + \frac{40 \cdot T_{cpm}}{10000} .. 40 \cdot T_{cpm} \quad rt_{gd} = 1 \cdot \%$$

**Waveform of The Reconstructed Signal**

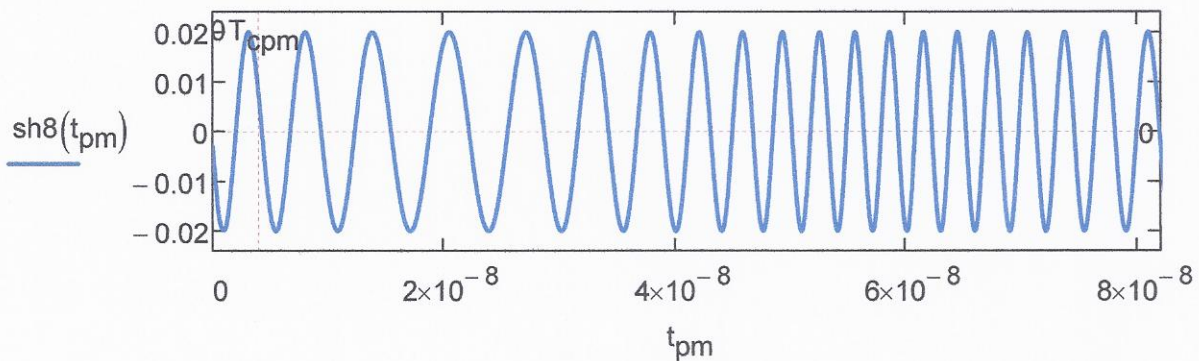


fig.:5.2.7.2

$$V_{pp} = 5V$$

$$\text{Exact output: } v_{opm}(t) := \int_0^t w(t-\sigma) \cdot V_{pm}(\sigma) d\sigma \quad (5.2.7.10)$$

$$T_{cpm} = 4.099 \times 10^{-3} \cdot \mu s \quad T_{mpm} = 0.082 \cdot \mu s$$

**Output Voltage Waveform**

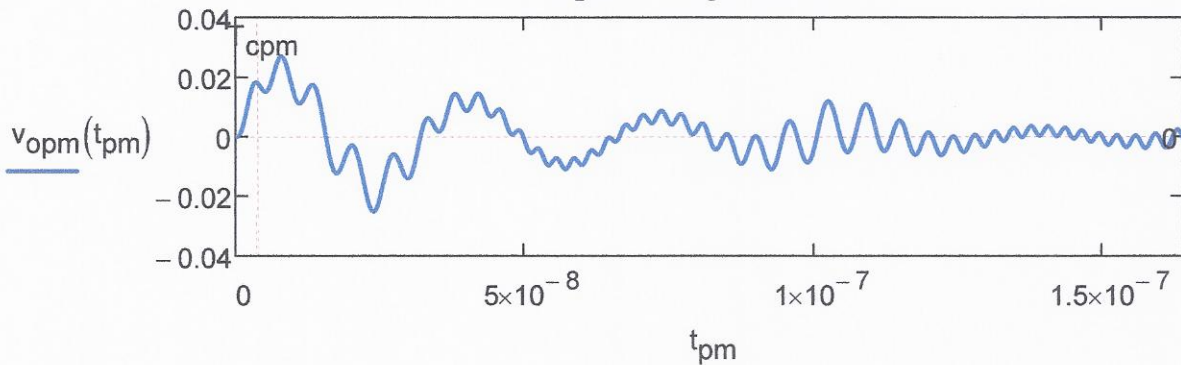


fig.:5.2.7.3

$$Opm_k := \frac{v_{opm}(n_{pmk})}{\text{volt}} \quad (5.2.7.11)$$

Sampled PM signal Waveform

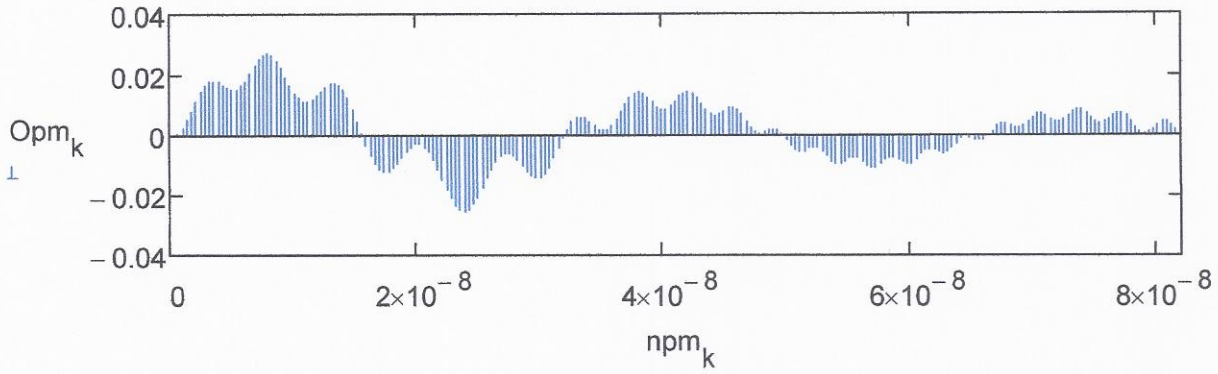


fig.:5.2.7.4

Fourier Transform of the Test signal

$$f_{\text{cpm}} = 0.244 \cdot \text{GHz} \quad \frac{f_{\text{spm}}}{f_{\text{cpm}}} = 11.31$$

$$m_p = 8$$

$$\omega_{\text{mpm}} = 0.077 \cdot \frac{\text{Grads}}{\text{sec}}$$

$$\text{OSpecpm} := \text{fft}(\text{Opm})$$

$$(5.2.7.12)$$

Output signal spectrum

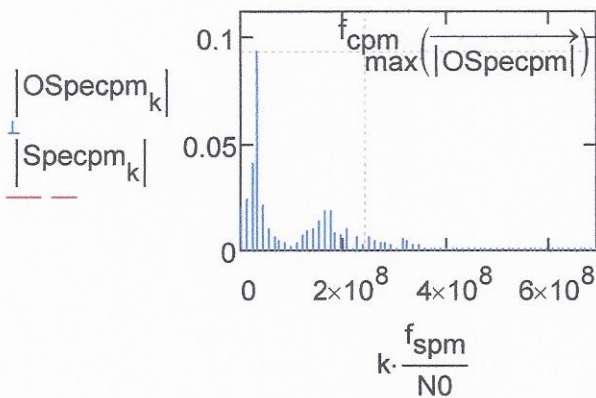


fig.:5.2.7.5

Phase spectrum

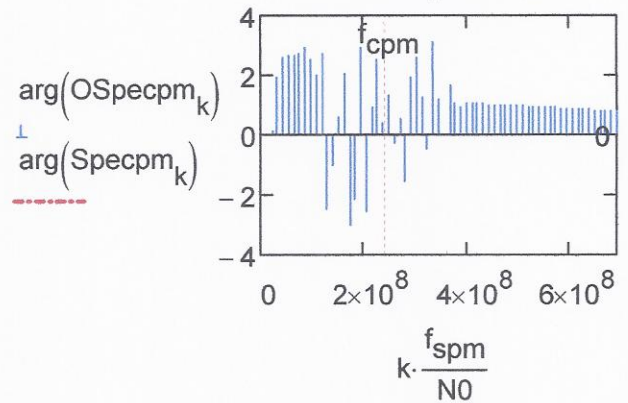


fig.:5.2.7.6

$$W_{\text{db}\omega_c} := 20 \cdot \log(|W_{\text{lp}}(j \cdot \omega_c)|)$$

$\omega_c$  is the angular frequency of the carrier

$$W_{\text{db}\omega_c} = 10.392 \cdot \text{dB}$$

### Magnitude of $W(\omega)$

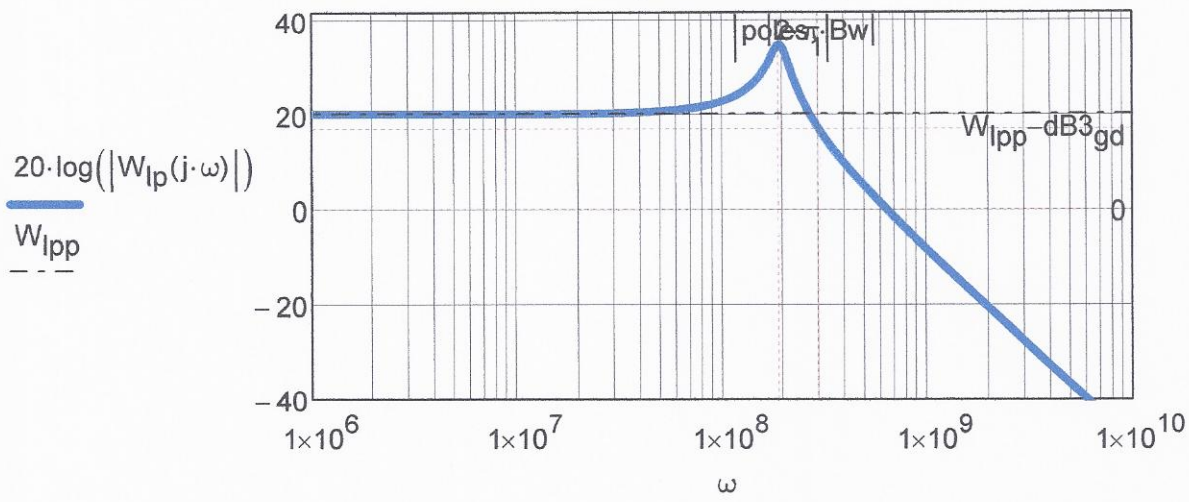


fig.:5.2.7.7



## 5.3

# Analog-Equivalent Digital Low Pass II°order Filter

### 5.3.1 Z-transfer function of the II° Order Low Pass Digital Filter.

Now the previous analog results will be compared with the digital one

Consider the first order approximation with the change of variable:  $s = \frac{1-z^{-1}}{T_s}$  (5.3.1.1)

(see file" 1)DIGITAL FILTERS EQUIVALENT TO LINEAR CLASSICS - BASICS.xmcd, § 1.1.1)

From Laplace transform to Z transform),

and the following substitution into the transfer function

$$\begin{array}{l}
 A_5 := A_5 \quad \omega_5 \\
 \text{If } \zeta \neq \omega_5
 \end{array}
 \quad
 H_{lp}(z) := \frac{A_5 \cdot \omega_5^2}{s^2 + 2 \cdot \zeta \cdot s + \omega_5^2}
 \quad
 \left.
 \begin{array}{l}
 \text{substitute, } s = \frac{1-z^{-1}}{T_s} \\
 \text{collect, } z
 \end{array}
 \right\}
 \rightarrow$$

$$\text{otherwise } x=z^{-1} : H_{lp}(x) := A_5 \cdot \frac{\omega_5^2}{(s + \omega_5)^2}
 \quad
 \left.
 \begin{array}{l}
 \text{substitute, } s = \frac{1-x}{T_s} \\
 \text{collect, } x \\
 \text{factor}
 \end{array}
 \right\}
 \rightarrow
 \frac{A_5 \cdot T_s^2 \cdot \omega_5^2}{(T_s \cdot \omega_5 - x + 1)^2}$$

the result is:

$$H_{lp}(z) = \left\{ \begin{array}{l}
 \frac{A_5 \cdot T_s^2 \cdot \omega_5^2}{z^{-2} - z^{-1} \cdot 2 \cdot (T_s \cdot \zeta + 1) + 2 \cdot T_s \cdot \zeta + T_s^2 \cdot \omega_5^2 + 1} \quad \text{if } \zeta \neq \omega_5 \\
 \frac{A_5 \cdot T_s^2 \cdot \omega_5^2}{(-z^{-1} + T_s \cdot \omega_5 + 1)^2} \quad \text{otherwise}
 \end{array} \right. \quad (5.3.1.2)$$

To simplify define the following parameters:

Place the sampling period  $T_s := T_{s\text{stp}}$  which is the one defined for the step function. In addition define he constants:

$$A0 := A_5 \cdot \omega_5^2 \cdot T_s^2 \quad B0 := 2 \cdot (1 + \zeta \cdot T_s) \quad C0 := T_s \cdot (\omega_5^2 \cdot T_s + 2 \cdot \zeta) + 1 \quad D0 := T_s \cdot \omega_5 + 1$$

$$A0 = -0.98696044 \quad B0 = 2.0581776417 \quad C0 = 1.1568736857 \quad D0 = 1.314$$

you get the following result for the t. f. as a function of z:

$$T_s = 1.639 \times 10^{-6} \cdot \text{ms}$$

$$H_{lp}(z) := \begin{cases} \frac{A_0}{z^{-2} - B_0 \cdot z^{-1} + C_0} & \text{if } \zeta \neq \omega_5 \\ \frac{A_0}{(D_0 - z^{-1})^2} & \text{otherwise} \end{cases} \quad (5.3.1.3)$$

$$W_{1lp} := 20 \cdot \log \left( \left| \frac{A_5 \cdot \omega_5}{2 \cdot \zeta} \right| \right)$$

## BODE PLOTS (Low Pass (II° order)):

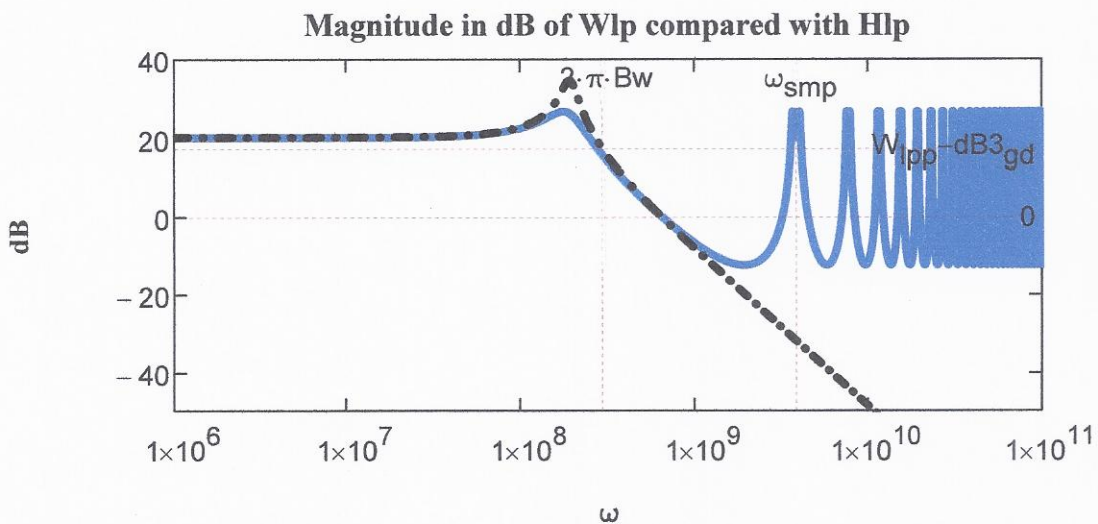


fig.:5.3.1.1

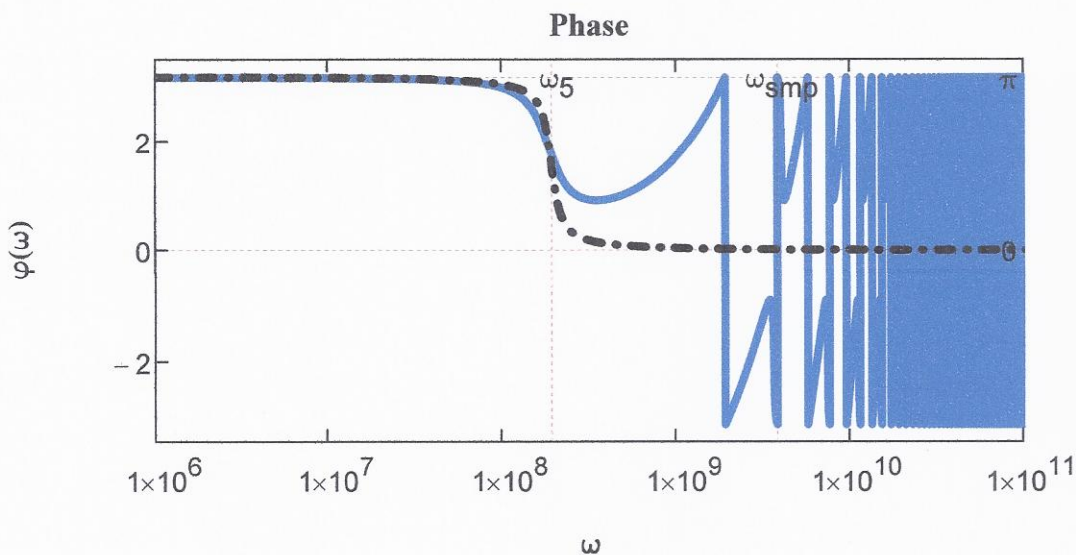


fig.:5.3.1.2

### 5.3 Equivalent Digital Low Pass II<sup>o</sup>order Filter

#### 5.3.2 Difference equations Low Pass II<sup>o</sup>order filter. Canonical form.

$$\text{Given the transfer function: } H_{lp}(z) = \begin{cases} \frac{A0}{z^{-2} - B0 \cdot z^{-1} + C0} & \text{if } \zeta \neq \omega_5 \\ \frac{A0}{(D0 - z^{-1})^2} & \text{otherwise} \end{cases} \quad (5.3.2.1)$$

Multiply and divide its definition for the same function  $G(z)$ , so that

$$H_{lp}(z) = \frac{Y(z)}{X(z)} = \frac{Y(z)}{G(z)} \cdot \frac{G(z)}{X(z)} \quad (5.3.2.2)$$

$$\frac{Y(z)}{G(z)} = A0$$

$$Y(z) = A0 \cdot G(z)$$

$$\boxed{y(\nu) = A0 \cdot g(\nu)} \quad (5.3.2.3)$$

$$\frac{G(z)}{X(z)} = \begin{cases} \frac{1}{z^{-2} - B0 \cdot z^{-1} + C0} & \text{if } \zeta \neq \omega_5 \\ \frac{1}{(D0 - z^{-1})^2} & \text{otherwise} \end{cases} \quad (5.3.2.4)$$

$$X(z) = \begin{cases} [(z^{-2} - B0 \cdot z^{-1} + C0) \cdot G(z)] & \text{if } \zeta \neq \omega_5 \\ (D0 - z^{-1})^2 \cdot G(z) & \text{otherwise} \end{cases} \quad (5.3.2.5)$$

$$X(z) = \begin{cases} (C0 \cdot G(z) - B0 \cdot z^{-1} \cdot G(z) + z^{-2} \cdot G(z)) & \text{if } \zeta \neq \omega_5 \\ G(z) \cdot D0^2 - 2 \cdot G(z) \cdot D0 \cdot z^{-1} + G(z) \cdot z^{-2} & \text{otherwise} \end{cases} \quad (5.3.2.6)$$

$$\boxed{x(\nu) = \begin{cases} (C0 \cdot g(\nu) - B0 \cdot g(\nu - 1) + g(\nu - 2)) & \text{if } \zeta \neq \omega_5 \\ g(\nu) \cdot D0^2 - 2 \cdot g(\nu - 1) \cdot D0 + g(\nu - 2) & \text{otherwise} \end{cases}} \quad (5.3.2.7)$$

The corresponding set of difference equations is:

$$1) \quad g(\nu) = \begin{cases} \frac{x(\nu) + B0 \cdot g(\nu - 1) - g(\nu - 2)}{C0} & \text{if } \zeta \neq \omega_5 \\ \frac{x(\nu) + 2 \cdot D0 \cdot g(\nu - 1) - g(\nu - 2)}{D0^2} & \text{otherwise} \end{cases} \quad (5.3.2.8)$$

2)  $y(v) = A0 \cdot g(v)$

(5.3.2.9)

$A0 := A0$      $B0 := B0$      $C0 := C0$

Z T. Initial value theorem:  $\lim_{z \rightarrow \infty} \left( \frac{A0}{z^{-2} - B0 \cdot z^{-1} + C0} \right) \rightarrow \frac{A0}{C0}$

Z T. Final value theorem:  $\lim_{z \rightarrow 0} \left( \frac{A0}{z^{-2} - B0 \cdot z^{-1} + C0} \right) \rightarrow 0$

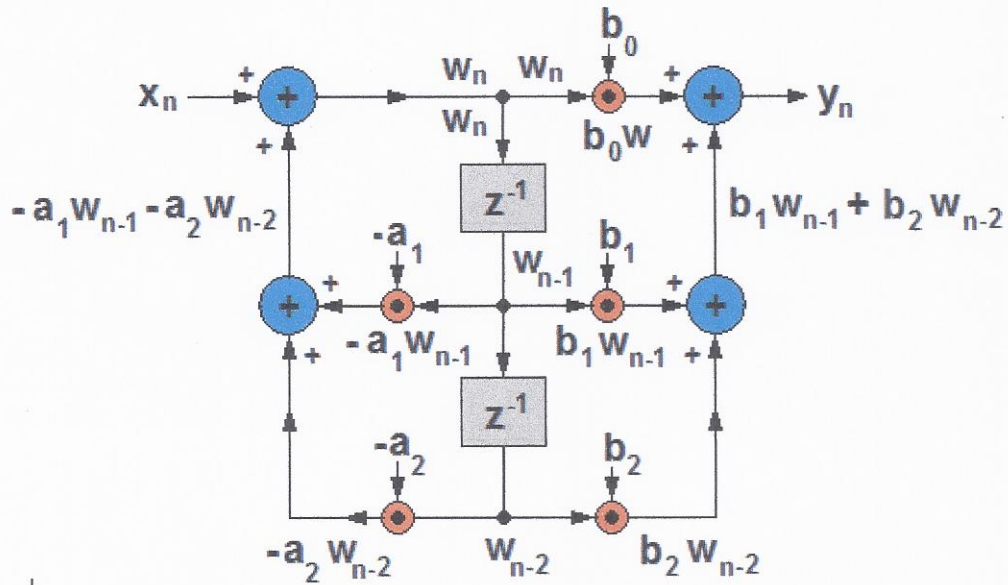


fig.:5.3.2.1



Recurrence relations:

$$\begin{aligned}
 1) \quad g(\nu) := & \left| \begin{array}{l} \text{if } \nu > 1 \\ \left| \begin{array}{l} \frac{v_i(\nu) + B0 \cdot g(\nu - 1) - g(\nu - 2)}{C0} \quad \text{if } \zeta \neq \omega_5 \\ \frac{v_i(\nu) + 2 \cdot D0 \cdot g(\nu - 1) - g(\nu - 2)}{D0^2} \quad \text{otherwise} \end{array} \right. \\ \text{if } \nu = 0 \\ \left| \begin{array}{l} \frac{v_i(0)}{C0} \quad \text{if } \zeta \neq \omega_5 \\ \frac{v_i(0)}{D0^2} \quad \text{otherwise} \end{array} \right. \\ \text{if } \nu = 1 \\ \left| \begin{array}{l} \frac{v_i(1) + B0 \cdot g(0)}{C0} \quad \text{if } \zeta \neq \omega_5 \\ \frac{v_i(1) + 2 \cdot D0 \cdot g(0)}{D0^2} \quad \text{otherwise} \end{array} \right. \end{array} \right. \quad \blacksquare \quad (5.3.2.10)
 \end{aligned}$$

$$2) \quad y(\nu) := \left| \begin{array}{l} A0 \cdot g(\nu) \quad \text{if } \nu > 0 \\ 0 \quad \text{otherwise} \end{array} \right. \quad \blacksquare \quad (5.3.2.11)$$

Vectorized:

$$\text{Glp}_k := 0 \quad \text{rows}(\text{Glp}) = 256 \quad (5.3.2.12)$$

$$\begin{aligned}
 \text{Glp}_k := & \begin{cases} \text{if } k > 1 & \begin{cases} \frac{v_i(k) + B0 \cdot \text{Glp}_{k-1} - \text{Glp}_{k-2}}{C0} & \text{if } \zeta \neq \omega_5 \\ \frac{v_i(k) + 2 \cdot D0 \cdot \text{Glp}_{k-1} - \text{Glp}_{k-2}}{D0^2} & \text{otherwise} \end{cases} \\ \text{if } k = 0 & \begin{cases} \frac{v_i(0)}{C0} & \text{if } \zeta \neq \omega_5 \\ \frac{v_i(0)}{D0^2} & \text{otherwise} \end{cases} \\ \text{if } k = 1 & \begin{cases} \frac{v_i(1) + B0 \cdot \text{Glp}_0}{C0} & \text{if } \zeta \neq \omega_5 \\ \frac{v_i(1) + 2 \cdot D0 \cdot \text{Glp}_0}{D0^2} & \text{otherwise} \end{cases} \end{cases} \quad (5.3.2.13) \\
 \text{Y22}_k := & \begin{cases} A0 \cdot \text{Glp}_k & \text{if } k > 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{rows}(\text{Y22}) = \quad (5.3.2.14)
 \end{aligned}$$

Block diagram of the difference equation algorithm for a second order system

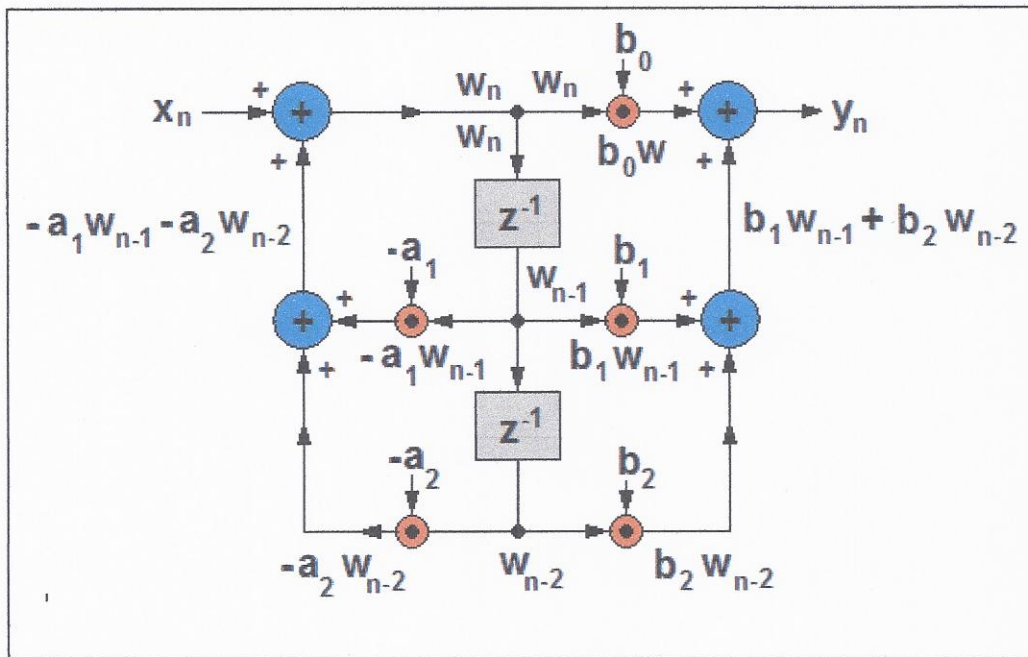


fig.:5.3.2.2

To save space when applying the previous algorithm, it is convenient to call the following program CANONIC2LP (it is the acronym of: Canonical Form Second Order Low Pass):

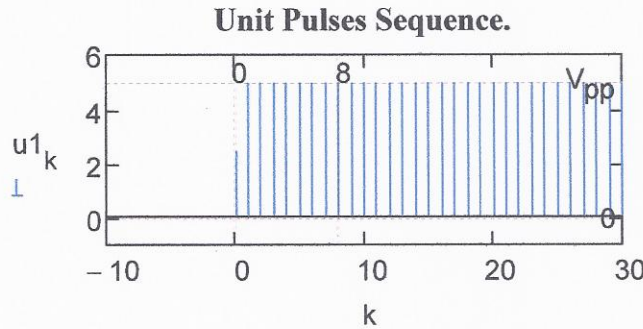
### 5.3 Equivalent Digital Low Pass Filter (II°order)

#### 5.3.2.1 Sequence of the voltage Step response.

Given the filter's input signal:  $u1_k := V_{stpsl}(nstp_k, V_{pp})$  (5.3.2.1.1)

and the filter's z transfer function, it will be calculated the output.

$$V_{pp} = 5 \times 10^3 \cdot \text{mV}$$



$$\text{rows}(u1) = 256$$

fig.:5.3.2.1.1

Transfer function coefficients:

$$A0 := A0 \quad B0 := B0 \quad C0 := C0$$

Output's Z transform Initial value theorem:  $\lim_{z \rightarrow \infty} \left[ \frac{A0}{z^{-2} - B0 \cdot z^{-1} + C0} \cdot \frac{z}{(z-1)^2} \right] \rightarrow 0$

$$v_i(v) := \frac{u1_v}{V} \quad v_i(0) = 2.5 \quad (5.3.2.1.2)$$

Numerical calculation of the filter's response to the input step function

$$\text{svsr} := \text{CANONIC2LP}(v_i, A_5, \zeta, \omega_5, T_{sstp}, N0) \quad (5.3.2.1.3)$$

$$\text{svsr} = (-0.987 \quad 2.058 \quad 1.157 \quad \{256,1\} \quad \{256,1\} \quad 1.314)$$

Calculated transfer function coefficients:

$$A01 := \text{svsr}_{0,0} \quad B01 := \text{svsr}_{0,1} \quad C01 := \text{svsr}_{0,2} \quad D01 := \text{svsr}_{0,5}$$

$$A01 = -0.98696044 \quad B01 = 2.0581776417 \quad C01 = 1.1568736857$$

Sequences of the state function and of the output:

$$\text{Glp0} := \text{svsr}_{0,3} \quad \text{Y00} := \text{svsr}_{0,4}$$

#### Sequence of the voltage Step response.

$$t_w := -1 \cdot T_{\text{test}}, -1 \cdot T_{\text{test}} + \frac{20 \cdot T_5 + 1 \cdot T_{\text{test}}}{10000} .. 20 \cdot T_5 \quad Q_5 = 5.4$$

Sequence of the state function

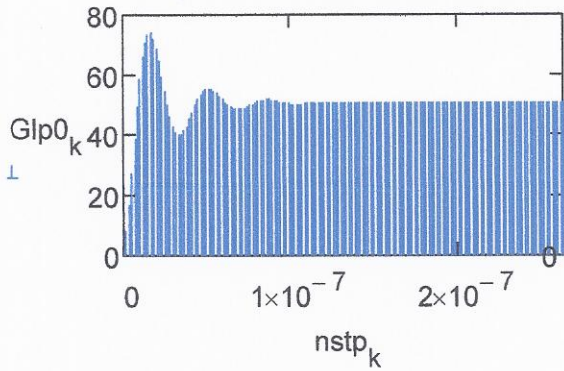


fig.:5.3.2.1.2

Spec0x := FFT(Y00)

Sequence of the response

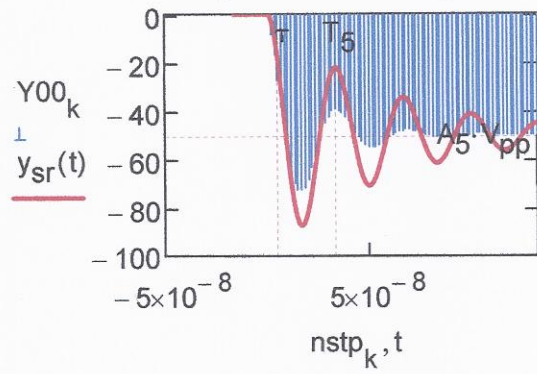


fig.:5.3.2.1.3

Amplitude Spectrum

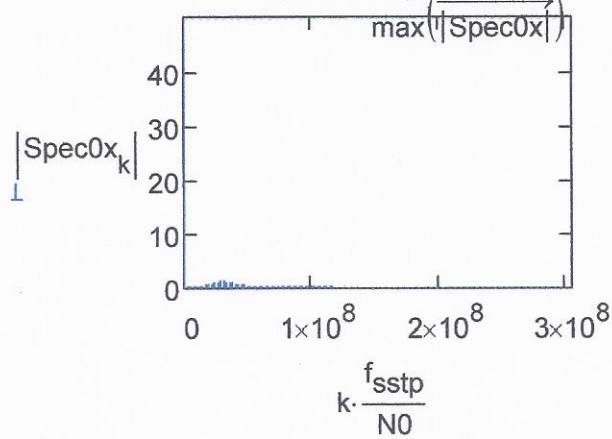


fig.:5.3.2.1.4

Phase spectrum

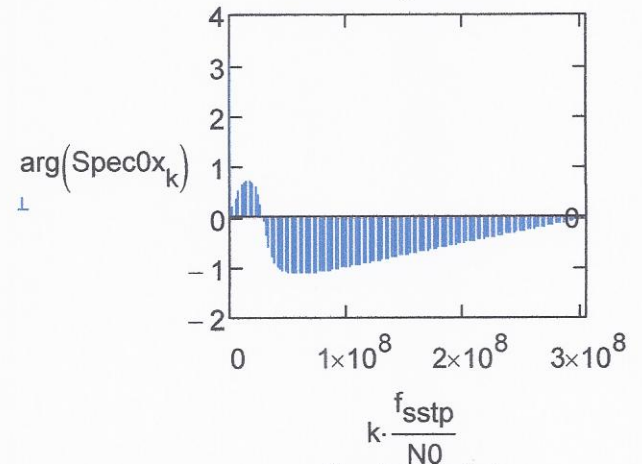


fig.:5.3.2.1.5

Bode plots of the  $\mathcal{Z}$  transfer function:

$$H_{lp}(z) := \begin{cases} \frac{A01}{z^{-2} - B01 \cdot z^{-1} + C01} & \text{if } \zeta \neq \omega_5 \\ \frac{A01}{(D01 - z^{-1})^2} & \text{otherwise} \end{cases} \quad (5.3.2.1.4)$$

Frequency Responses for sampling period  $T_{sstp}$

Magnitude in dB of Wlp compared with Hlp

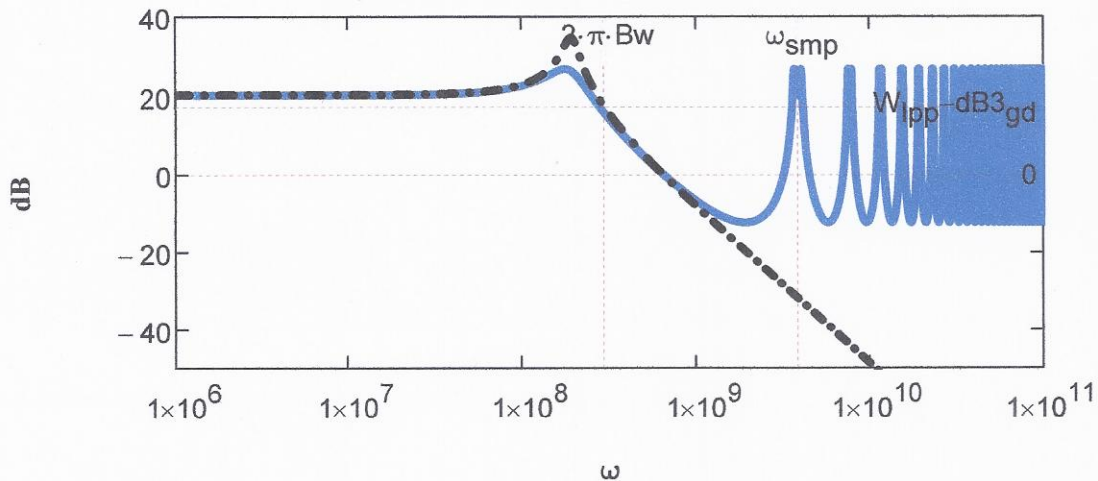


fig.:5.3.2.1.6



Phase of Wlp compared with Hlp

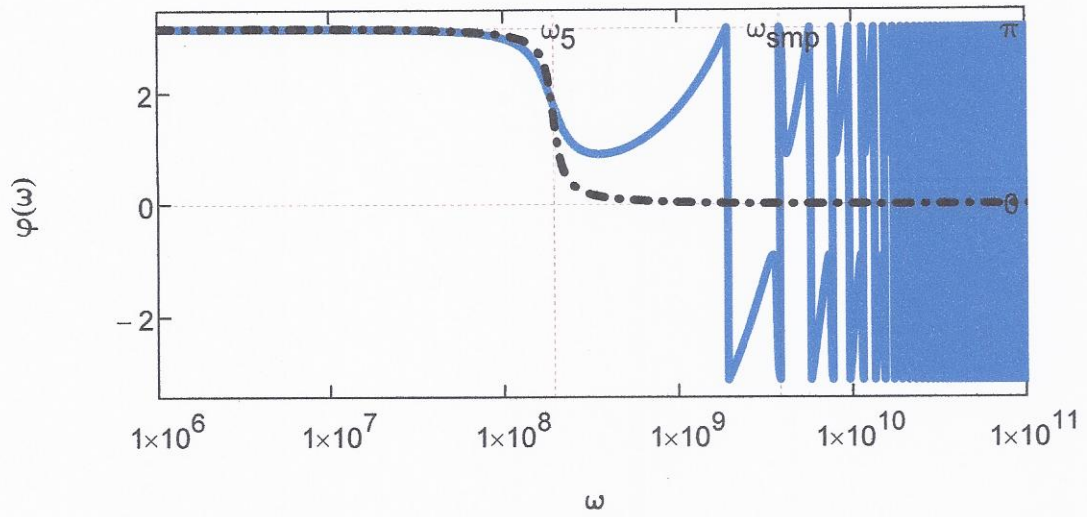


fig.:5.3.2.1.7

### 5.3 Equivalent Digital Low Pass Filter (II°order)

#### 5.3.2.2 Sequence of the Short Voltage Pulse response.

In this paragraph, given the second order low pass filter difference equations, it will be calculated numeric using the program "CANONIC2LP", the filter's response to a very short time duration voltage pulse:

$$V_{pp} = 5 \times 10^3 \cdot \text{mV} \quad t_w := -(\tau_5 + 2 \cdot \tau_{pw}), -(\tau_5 + 2 \cdot \tau_{pw}) + \frac{2 \cdot (\tau_5 + 2 \cdot \tau_{pw})}{10000} \dots \tau_5 + 2 \cdot \tau_{pw}$$

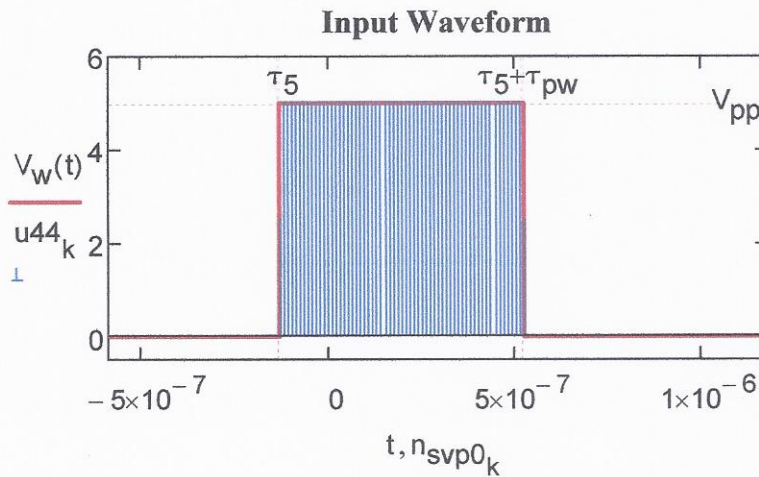


fig.:5.3.2.2.1

Sampling period of the short pulse defined earlier (5.2.2.25):  $T_{sA} := T_{svp}$

The first parameter of the program is the time discrete input function  $v_i(v) := u44_v$

The calculation's results will be stored into the following vector:

$$\begin{aligned} \text{svsr1} &:= \text{CANONIC2LP}(v_i, A_5, \zeta, \omega_5, T_s, N0) & (5.3.2.2.1) \\ \text{svsr1} &= (-43.865 \quad 2.388 \quad 5.774 \quad \{256,1\} \quad \{256,1\} \quad 3.094) \end{aligned}$$

Calculated transfer function coefficients:

$$\begin{aligned} a1 &:= \text{svsr1}_{0,0} & b1 &:= \text{svsr1}_{0,1} & c1 &:= \text{svsr1}_{0,2} & d01 &:= \text{svsr1}_{0,5} \\ a1 &= -43.86490845 & b1 &= 2.3878509449 & c1 &= 5.7743417898 & d01 &= 3.094 \end{aligned}$$

Sequences of the state function and of the output:

$$\text{Glp1} := \text{svsr1}_{0,3} \quad \text{Y01} := \text{svsr1}_{0,4}$$

Block diagram of the difference equation algorithm for a second order system

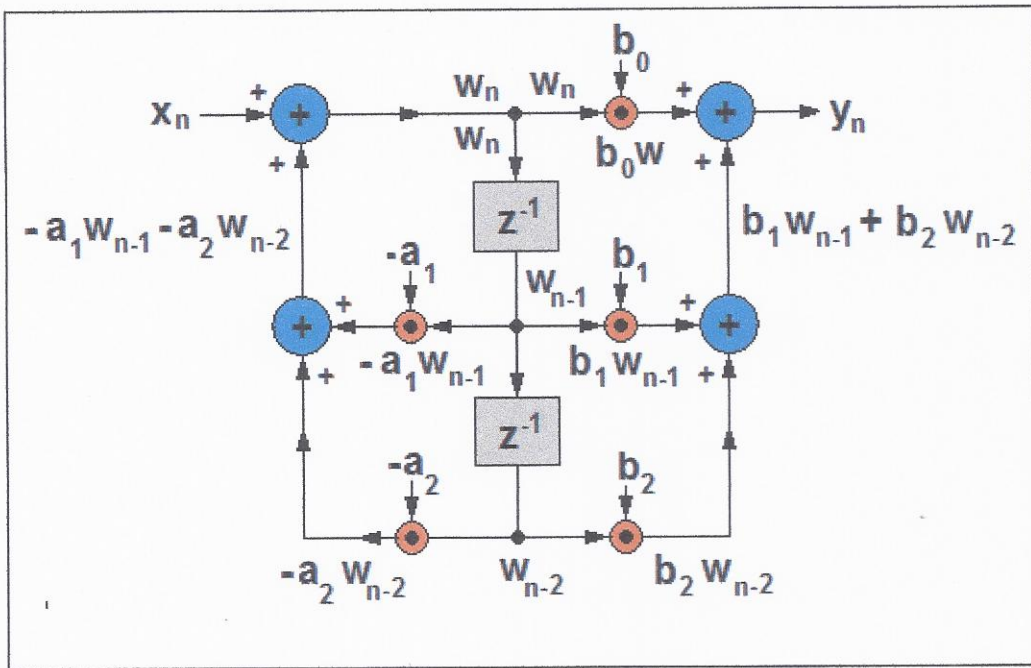


fig.:5.3.2.2.2

$$t_w := -0.1 \cdot \tau_5, -0.1 \cdot \tau_5 + \frac{\tau_5 + 8 \cdot (\tau_{pw} + \tau_5) + 0.1 \cdot \tau_5}{20000} .. \tau_5 + 8 \cdot (\tau_{pw} + \tau_5)$$

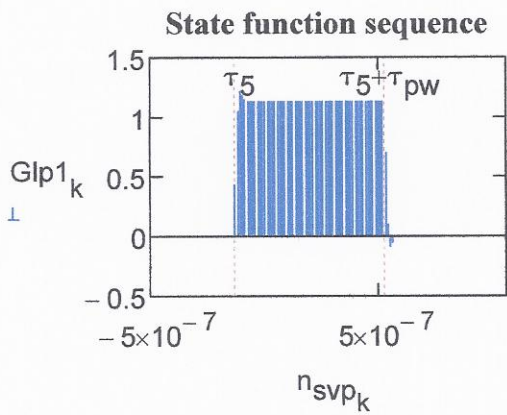


fig.:5.3.2.2.3

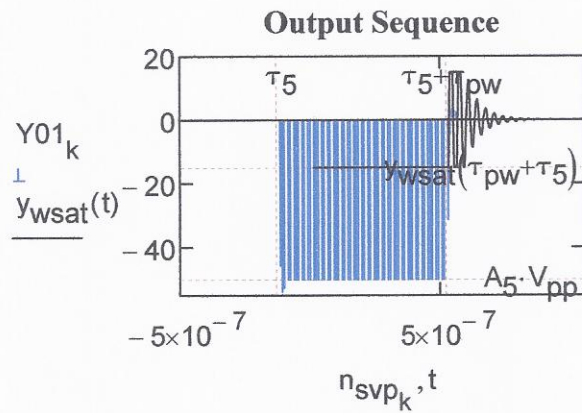


fig.:5.3.2.2.4

Sampled signal:

Spec1x := fft(Y01)

(5.3.2.2.2)

**Amplitude Spectrum**

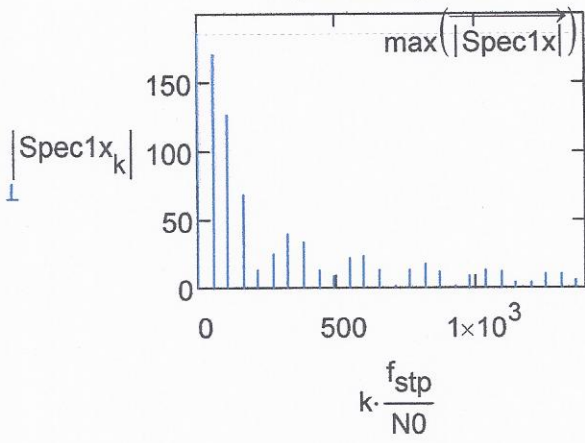


fig.:5.3.2.2.5

**Phase spectrum**

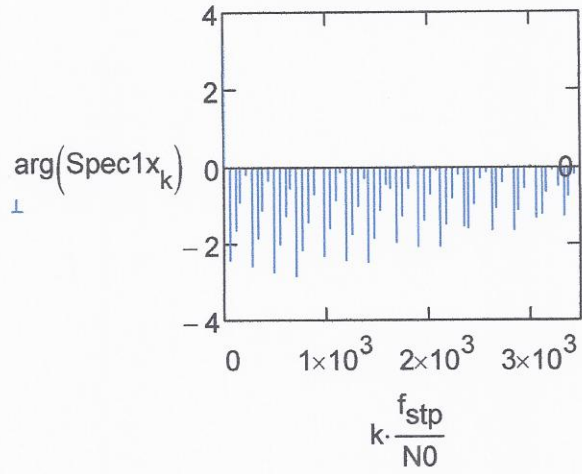


fig.:5.3.2.2.6

$$H_{lp}(z) := \begin{cases} \frac{a1}{z^{-2} - b1 \cdot z^{-1} + c1} & \text{if } \zeta \neq \omega_5 \\ \frac{a1}{(d01 - z^{-1})^2} & \text{otherwise} \end{cases} \quad (5.3.2.2.3)$$

Frequency Responses for sampling period  $T_{svp}$

**Magnitude in dB of  $W_{lp}$  compared with  $H_{lp}$**

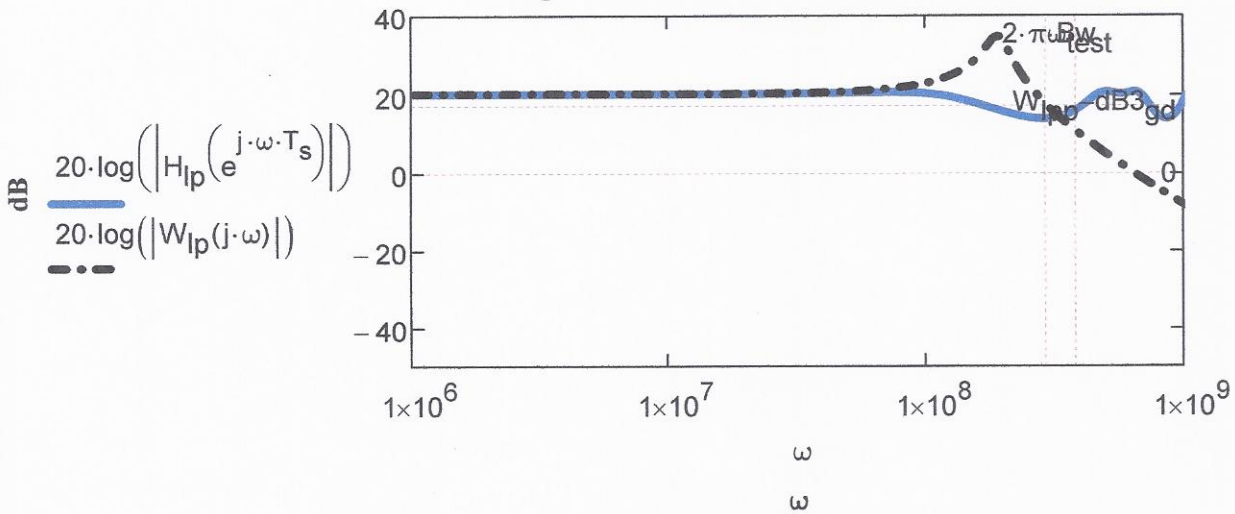


fig.:5.3.2.2.7



### 5.3 Equivalent Digital Low Pass Filter (II° order)

#### 5.3.2.3 Sequence of the Sawtooth response

$$\text{Place } T_s := T_{ssw}$$

$$T_s = 1.708 \times 10^{-4} \cdot \mu\text{s}$$

$$\omega_{\text{smp}} := \frac{2 \cdot \pi}{T_s}$$

$$\text{svsr2} := \text{CANONIC2LP}(v_i, A_5, \zeta, \omega_5, T_s, N0) \quad (5.3.2.3.1)$$

$$\text{svsr2} = (-0.011 \quad 2.006 \quad 1.007 \quad \{256,1\} \quad \{256,1\} \quad 1.033)$$

$$A_0 := \text{svsr2}_{0,0} \quad B_0 := \text{svsr2}_{0,1} \quad C_0 := \text{svsr2}_{0,2} \quad \text{Glp2} := \text{svsr2}_{0,3} \quad Y2 := \text{svsr2}_{0,4} \quad D_0 := \text{svsr2}_{0,5}$$

$$A_0 = -0.01070921 \quad B_0 = 2.006060171 \quad C_0 = 1.0071310916 \quad D_0 = 1.033$$

you get the following result for the t. f. as a function of z:

$$T_s = 1.708 \times 10^{-4} \cdot \mu\text{s} \quad H_{lp}(z) := \begin{cases} \frac{A_0}{z^{-2} - B_0 \cdot z^{-1} + C_0} & \text{if } \zeta \neq \omega_5 \\ \frac{A_0}{(D_0 - z^{-1})^2} & \text{otherwise} \end{cases} \quad (5.3.2.3.2)$$

$$\omega_{1, \text{ppm}} := 20 \cdot \log \left( \left| \frac{A_5 \cdot \omega_5}{2 \cdot \zeta} \right| \right)$$

### BODE PLOTS (Low Pass (II° order)):

$$\omega_{\text{smp}} = 36.792 \cdot \frac{\text{Grads}}{\text{sec}}$$

Frequency Responses for sampling period  $T_{ssw}$

Magnitude in dB of Wlp compared with Hlp

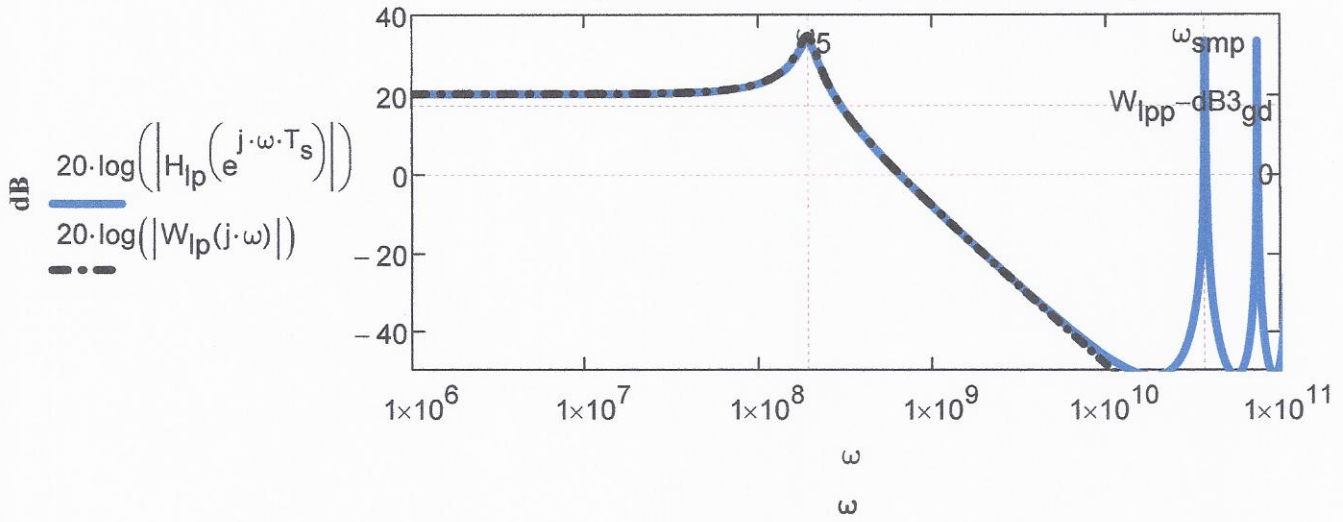


fig.:5.3.2.3.1

Phase

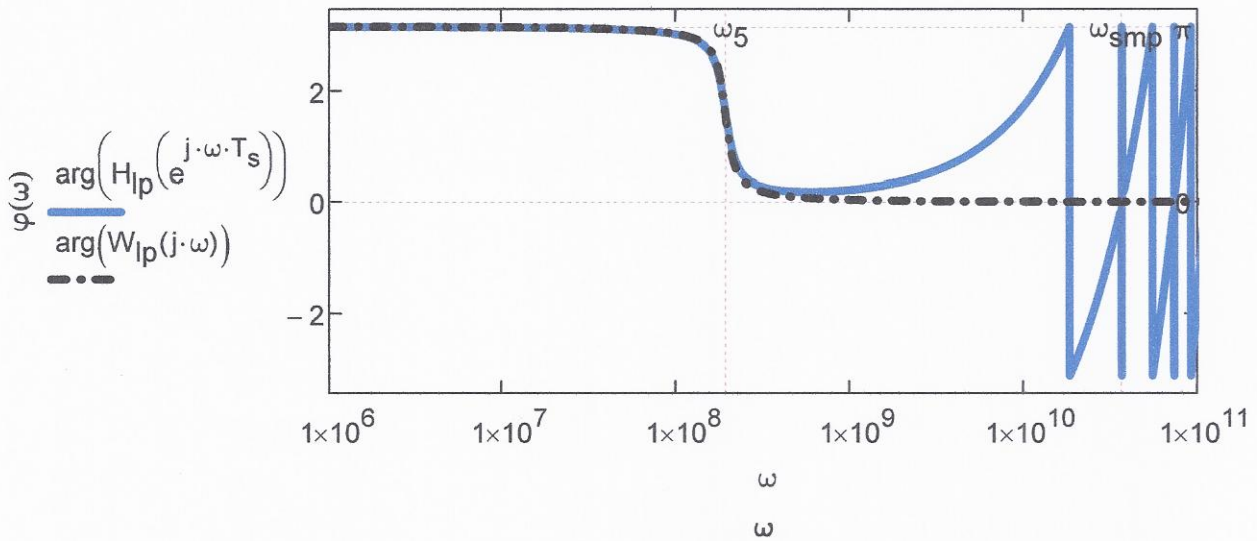


fig.:5.3.2.3.2

Digital first order Low pass filter difference equations:

$$V_{pp} = 5 \times 10^3 \cdot \text{mV}$$

$$u_{55k} := v1_{sw}(n_{sw_k}, T_{test}, V_{pp}) \quad (5.3.2.3.3)$$

Signal Waveform

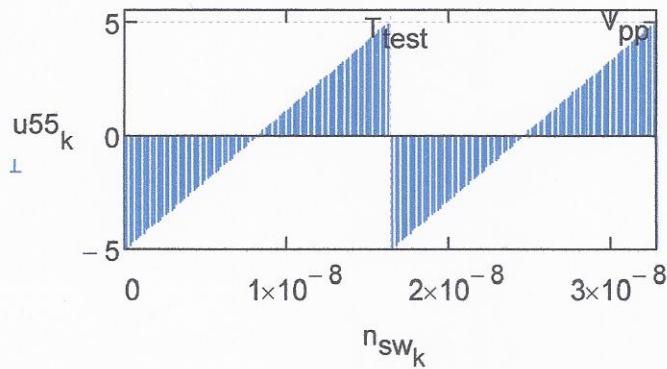


fig.:5.3.2.3.3

$$\omega_{\text{smp}} := \frac{2 \cdot \pi}{T_s} \quad v3_i(v) := \frac{u55_v}{V} \quad (5.3.2.3.4)$$

$$\text{svsr3} := \text{CANONIC2LP}(v3_i, A_5, \zeta, \omega_5, T_s, N0) \quad (5.3.2.3.5)$$

$$\text{svsr3} = (-0.011 \quad 2.006 \quad 1.007 \quad \{256,1\} \quad \{256,1\} \quad 1.033)$$

$$a3 := \text{svsr3}_{0,0} \quad b3 := \text{svsr3}_{0,1} \quad c3 := \text{svsr3}_{0,2} \quad \text{Glp3} := \text{svsr3}_{0,3} \quad Y3 := \text{svsr3}_{0,4} \quad d3 := \text{svsr3}_{0,5}$$

$$a3 = -0.01070921 \quad b3 = 2.006060171 \quad c3 = 1.0071310916 \quad d3 = 1.033$$

you get the following result for the t. f. as a function of z:

$$T_s = 1.708 \times 10^{-4} \cdot \mu\text{s} \quad H_{lp}(z) := \begin{cases} \frac{a3}{z^{-2} - b3 \cdot z^{-1} + c3} & \text{if } \zeta \neq \omega_5 \\ \frac{a3}{(d3 - z^{-1})^2} & \text{otherwise} \end{cases} \quad (5.3.2.3.6)$$

Block diagram of the difference equation algorithm for a second order system

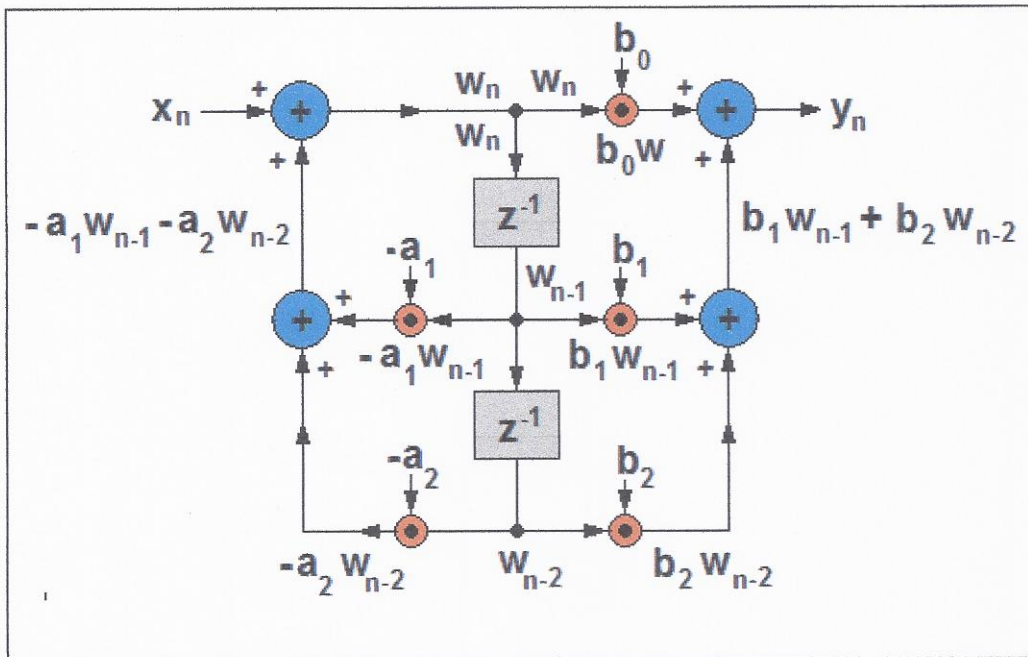


fig.:5.3.2.3.4

$$t_w := 0 \cdot T_{\text{test}}, 0 \cdot T_{\text{test}} + \frac{20 \cdot T_{\text{test}} - 0 \cdot T_{\text{test}}}{1000} .. 20 \cdot T_{\text{test}}$$

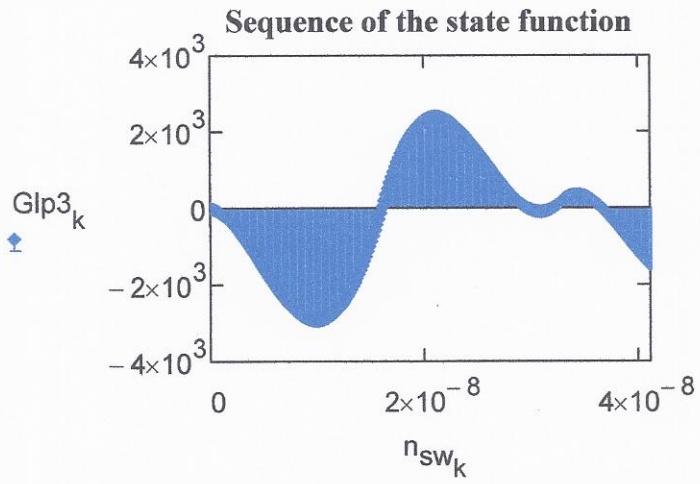


fig.:5.3.2.3.5

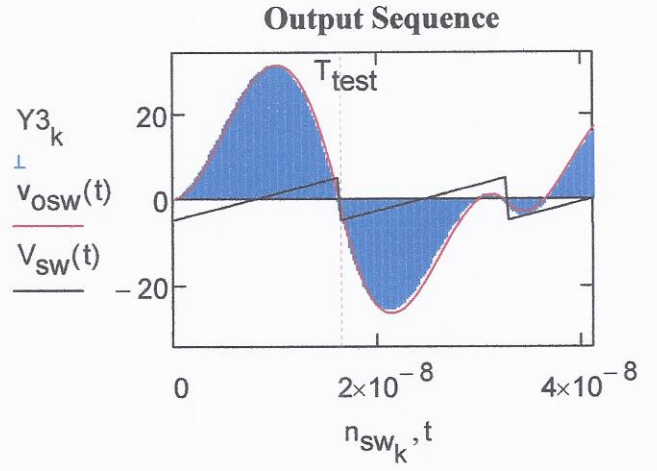


fig.:5.3.2.3.6

$$Spec3x := FFT(Y3)$$

$$(5.3.2.3.7)$$

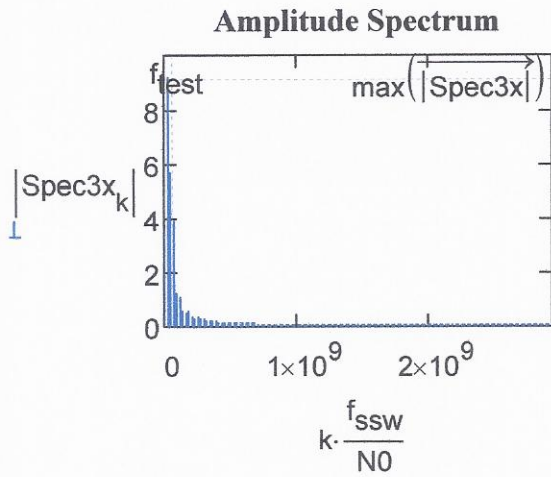


fig.:5.3.2.3.7

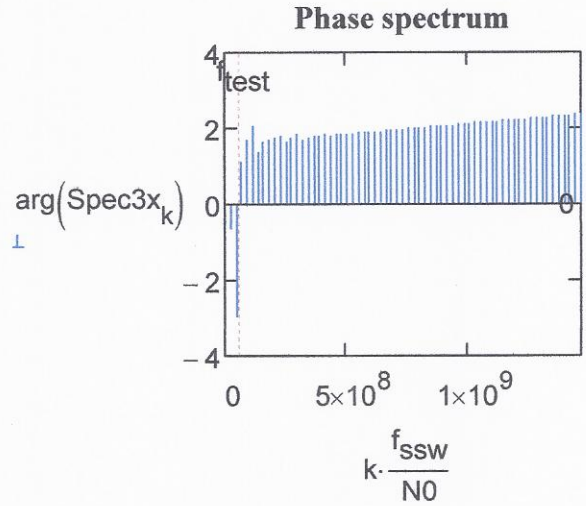


fig.:5.3.2.3.8



### 5.3 Equivalent Digital Low Pass Filter (II°order)

#### 5.3.2.4 Sequence of the Bipolar Square Wave response.

$$u66_k := V_{sqwb}(n_{sqw}_k) \quad (5.3.2.4.1)$$

$$t := 0 \cdot T_{test}, 0 \cdot T_{test} + \frac{4 \cdot T_{test} - 0 \cdot T_{test}}{1000} .. 4 \cdot T_{test}$$

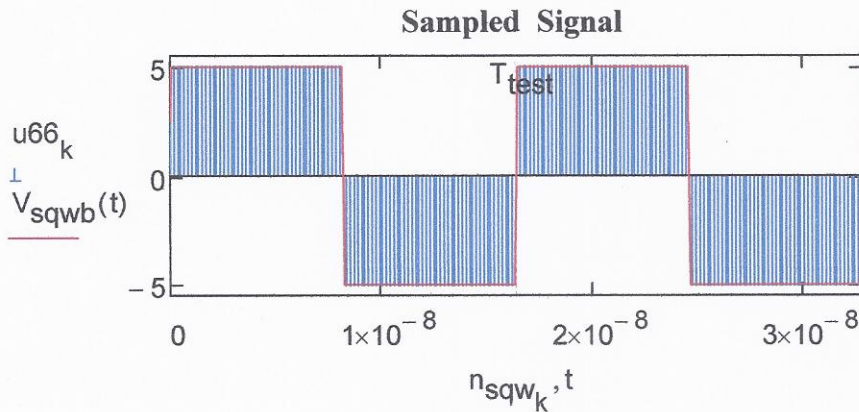


fig.:5.3.2.4.1

Place  $T_s := T_{ssqw}$

$$T_s = 0.171 \cdot ns$$

$$\omega_{smp} := \frac{2 \cdot \pi}{T_s}$$

$$v4_i(v) := u66_v \quad (5.3.2.4.2)$$

$$svsr4 := \text{CANONIC2LP}(v4_i, A_5, \zeta, \omega_5, T_s, N0) \quad (5.3.2.4.3)$$

$$svsr4 = (-0.011 \quad 2.006 \quad 1.007 \quad \{256,1\} \quad \{256,1\} \quad 1.033)$$

$$a4 := svsr4_{0,0} \quad b4 := svsr4_{0,1} \quad c4 := svsr4_{0,2} \quad Glp4 := svsr4_{0,3} \quad Y4 := svsr4_{0,4} \quad d4 := svsr4_{0,5}$$

$$a4 = -0.01070921 \quad b4 = 2.006060171 \quad c4 = 1.0071310916 \quad d4 = 1.033$$

you get the following result for the t. f. as a function of z:

$$T_s = 1.708 \times 10^{-7} \cdot ms \quad H_{lp}(z) := \begin{cases} \frac{a4}{z^{-2} - b4 \cdot z^{-1} + c4} & \text{if } \zeta \neq \omega_5 \\ \frac{a4}{(d4 - z^{-1})^2} & \text{otherwise} \end{cases} \quad (5.3.2.4.4)$$

$$W1_{pp} := 20 \cdot \log \left( \left| \frac{A_5 \cdot \omega_5}{2 \cdot \zeta} \right| \right)$$

# BODE PLOTS (Low Pass (II° order)):

Frequency Responses for sampling period  $T_{ssqw}$

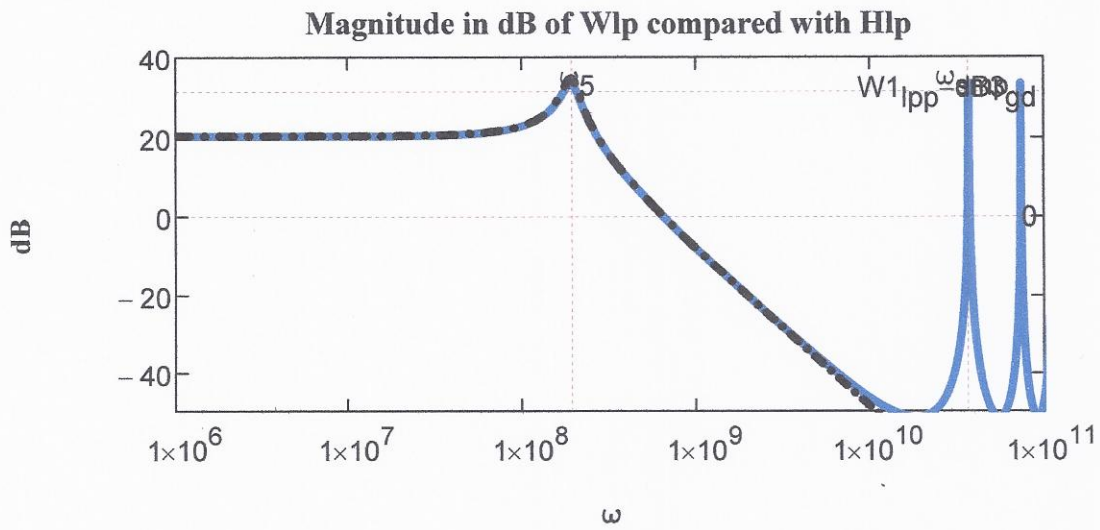


fig.:5.3.2.4.2

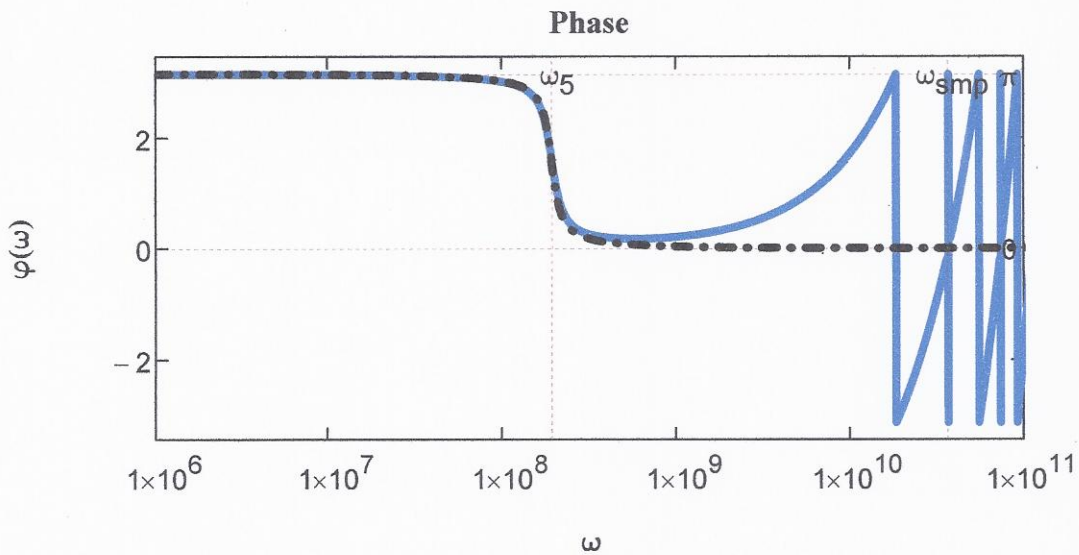


fig.:5.3.2.4.3

Block diagram of the difference equation algorithm for a second order system

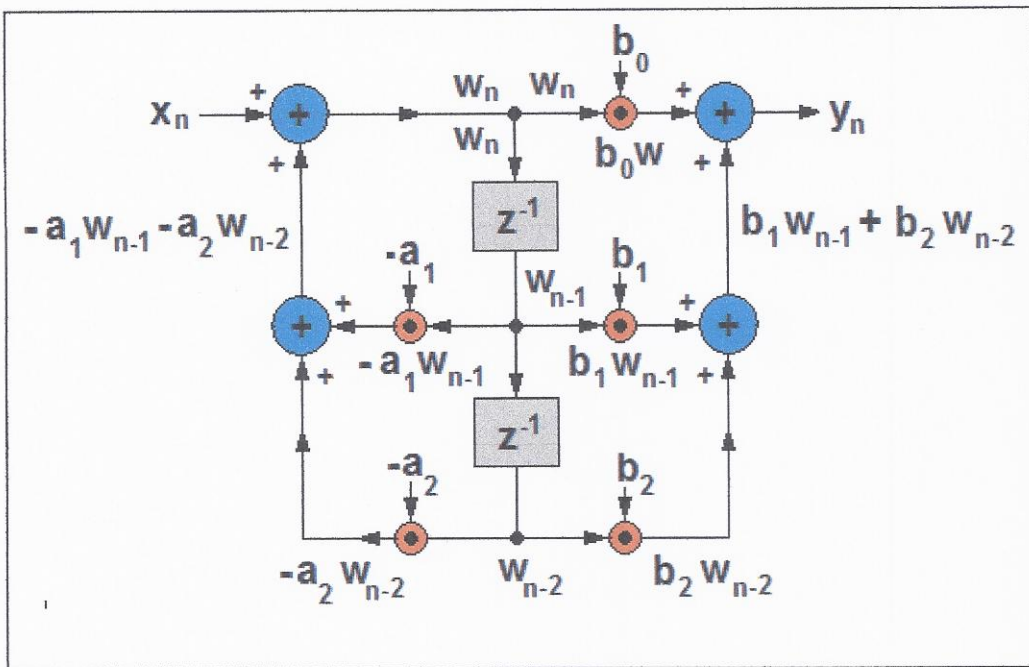


fig.:5.3.2.4.4

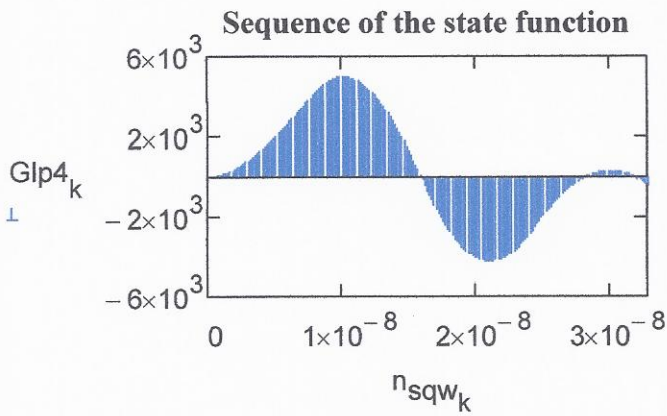


fig.:5.3.2.4.5

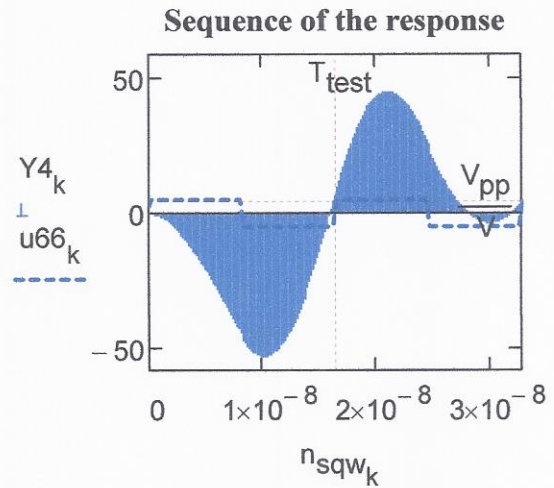


fig.:5.3.2.4.6

Spec4x := FFT(Y4)

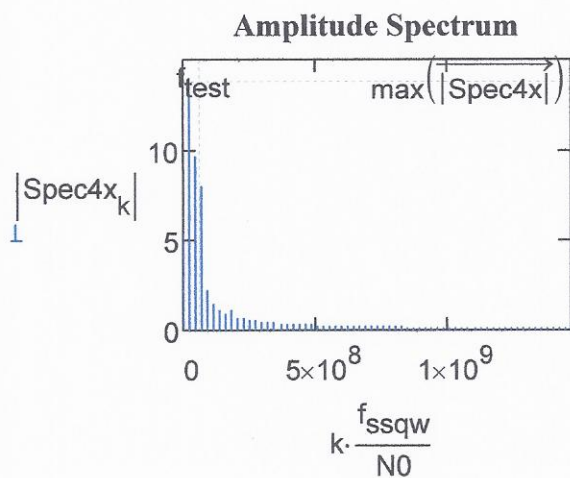


fig.:5.3.2.4.7

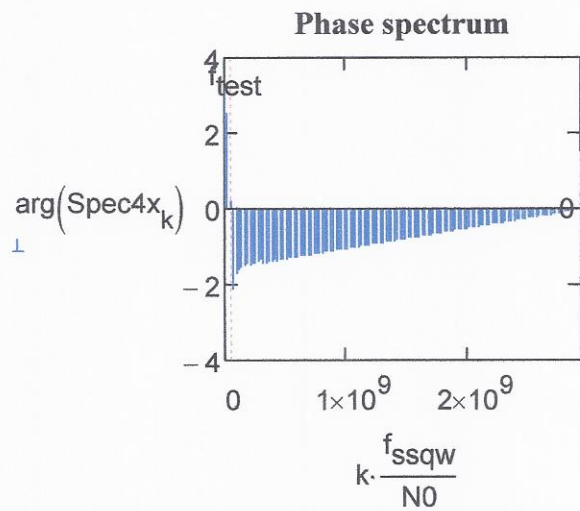


fig.:5.3.2.4.8

### 5.3 Equivalent Digital Low Pass Filter (II<sup>o</sup>order)

#### 5.3.2.5 (single tone) Sequence of the AM Signal response.

$$u77_k := v2_i(\text{nam}_k, \omega_{\text{mam}}, \omega_c, A1, B1) \quad (5.3.2.5.1)$$

$$\omega_{\text{mam}} = 0.038 \cdot \frac{\text{Grads}}{\text{sec}} \quad \omega_c = 0.383 \cdot \frac{\text{Grads}}{\text{sec}} \quad \frac{\omega_c}{\omega_{\text{mam}}} = 10 \quad \frac{N0}{f_{\text{sam}}} \cdot \frac{1}{T5} = 53.333$$

$$\text{Spec77} := \text{fft}(u77) \quad (5.3.2.4.2)$$

$$t := 0 \cdot T_{\text{mam}}, 0 \cdot T_{\text{mam}} + \frac{3 \cdot T_{\text{mam}}}{1000} .. 3 \cdot T_{\text{mam}}$$

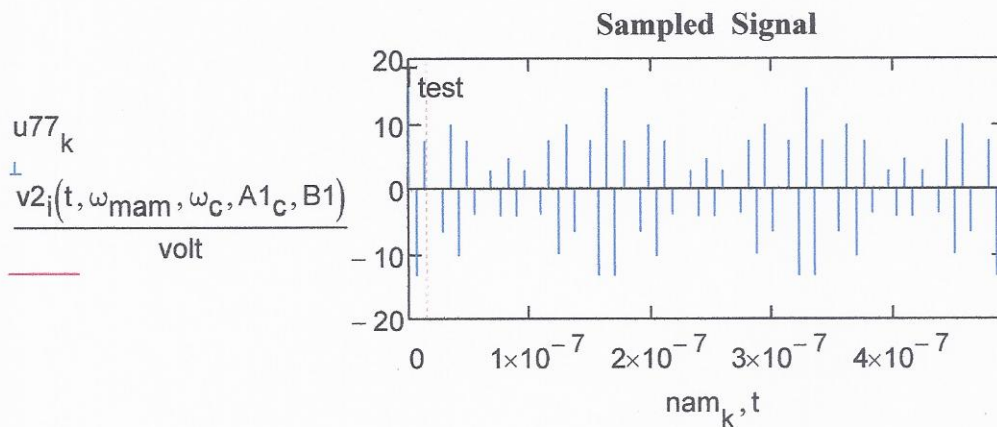


fig.:5.3.2.5.1

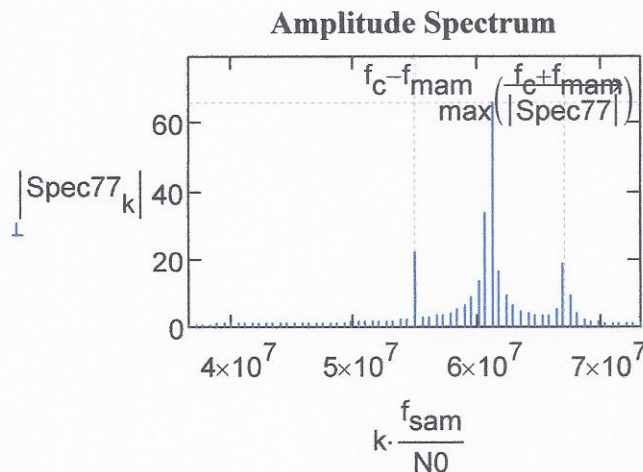


fig.:5.3.2.5.2

$$\text{Place } T_s := T_{\text{sam}}$$

$$T_s = 6.831 \cdot \text{ns}$$

$$\omega_{\text{smp}} := \frac{2 \cdot \pi}{T_s}$$

$$v5_i(v) := \frac{u77_v}{V} \quad (5.3.2.4.3)$$

$$\text{svsr5} := \text{CANONIC2LP}(v5_i, A_5, \zeta, \omega_5, T_s, N0) \quad (5.3.2.4.4)$$

$$\text{svsr5} = (-17.135 \quad 2.242 \quad 2.956 \quad \{256,1\} \quad \{256,1\} \quad 2.309)$$



$a_5 := \text{svsr50},_0$   $b_5 := \text{svsr50},_1$   $c_5 := \text{svsr50},_2$   $G_{lp5} := \text{svsr50},_3$   $Y_5 := \text{svsr50},_4$   $d_5 := \text{svsr50},_5$

$a_5 = -17.13472986$   $b_5 = 2.2424068406$   $c_5 = 2.9558798269$   $d_5 = 2.309$

you get the following result for the t. f. as a function of z:

$$T_s = 6.831 \times 10^{-6} \cdot \text{ms} \quad H_{lp}(z) := \begin{cases} \frac{a_5}{z^{-2} - b_5 \cdot z^{-1} + c_5} & \text{if } \zeta \neq \omega_5 \\ \frac{a_5}{(d_5 - z^{-1})^2} & \text{otherwise} \end{cases} \quad (5.3.2.4.5)$$

$$W_{1,lp} := 20 \cdot \log \left( \left| \frac{A_5 \cdot \omega_5}{2 \cdot \zeta} \right| \right)$$

## BODE PLOTS (Low Pass (II° order)):

Frequency Responses for sampling period  $T_{\text{sam}}$

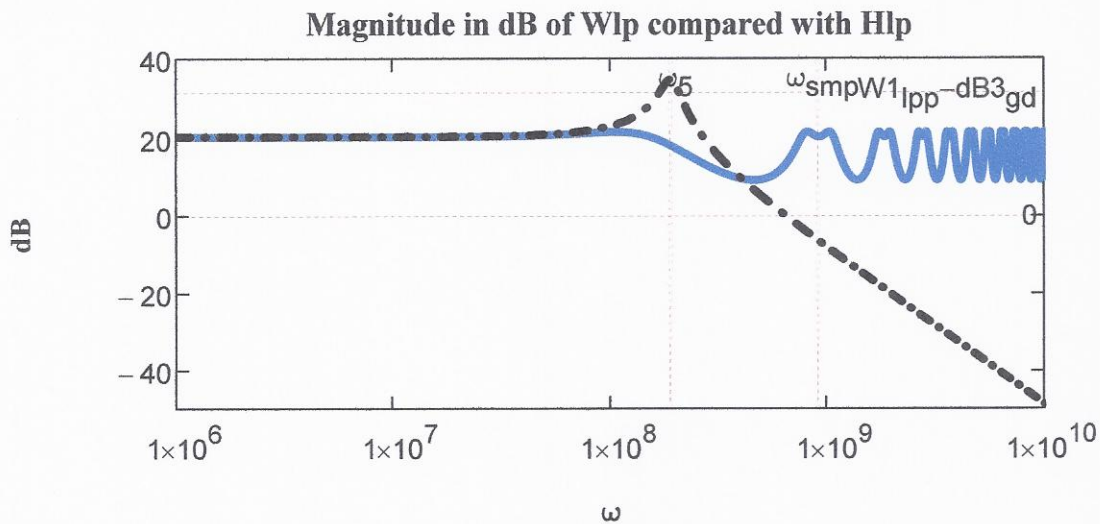


fig.:5.3.2.5.3

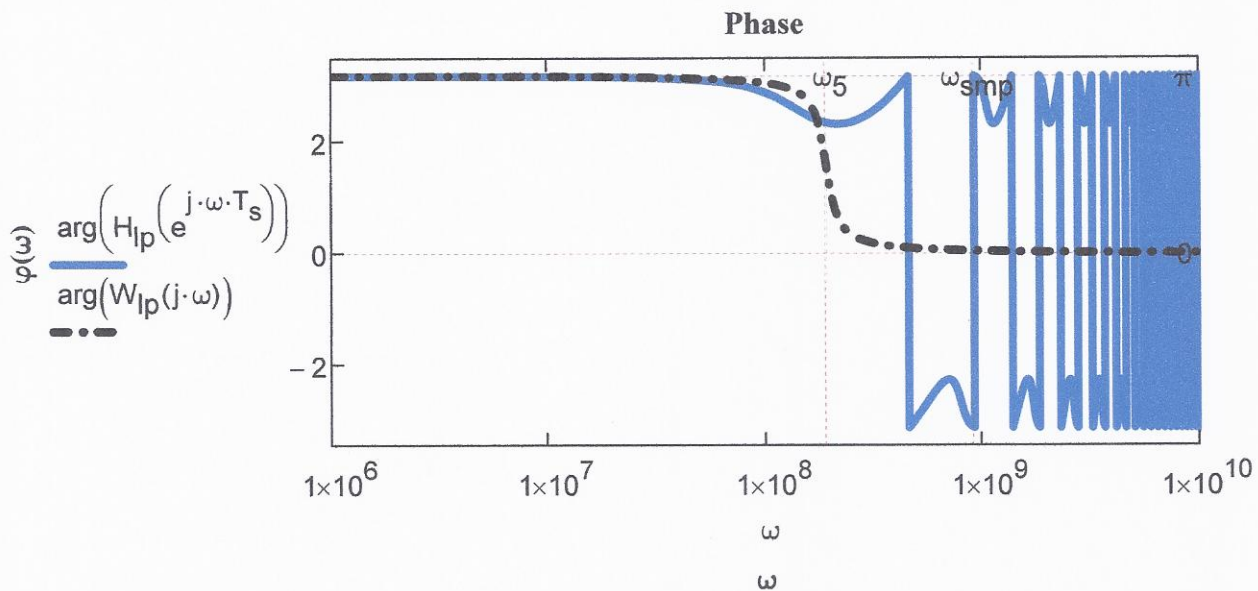


fig.:5.3.2.5.4

Block diagram of the difference equation algorithm for a second order system

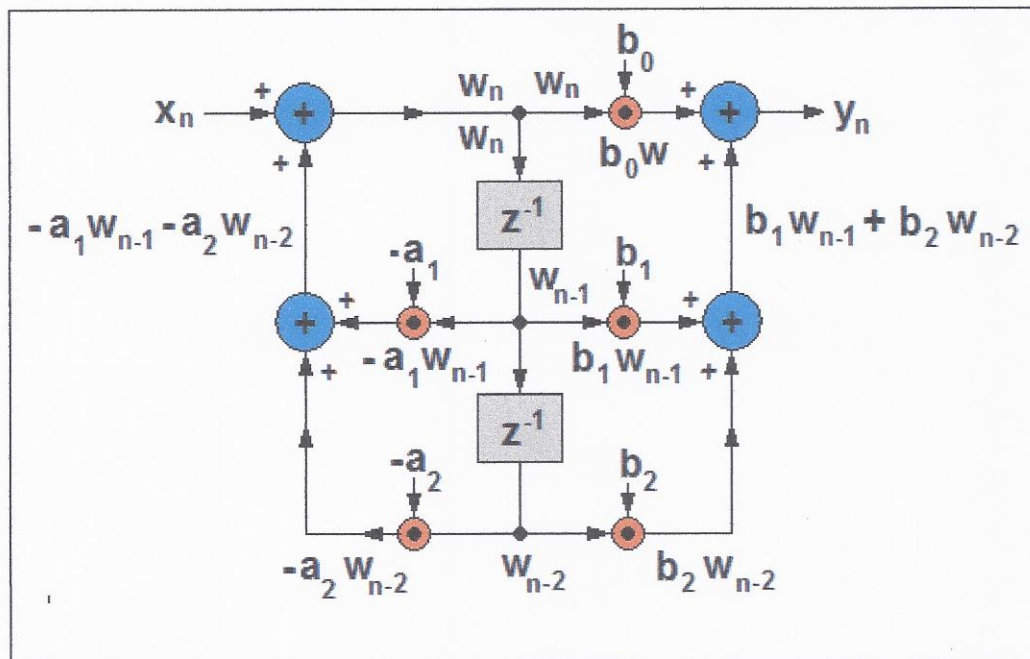


fig.:5.3.2.5.5

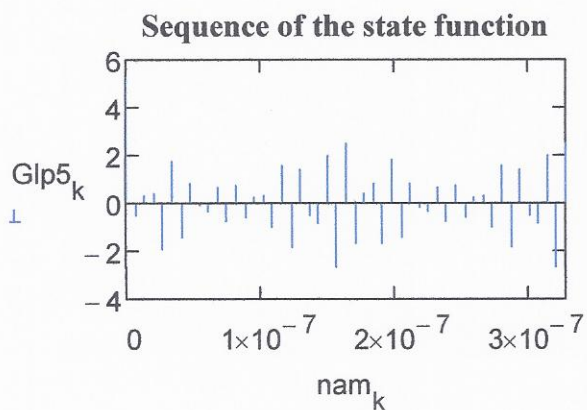


fig.:5.3.2.5.6

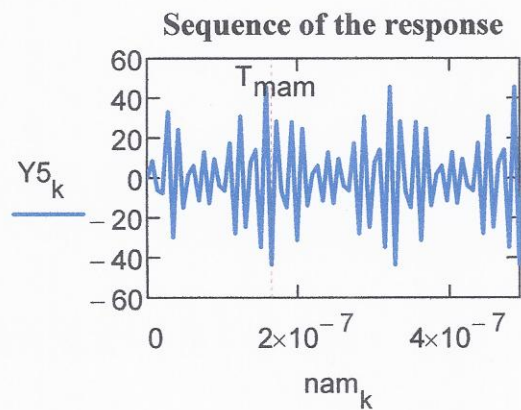


fig.:5.3.2.5.7

Spec5x := FFT(Y5)     $\max(\overrightarrow{|\text{Spec5x}|}) = 12.148$

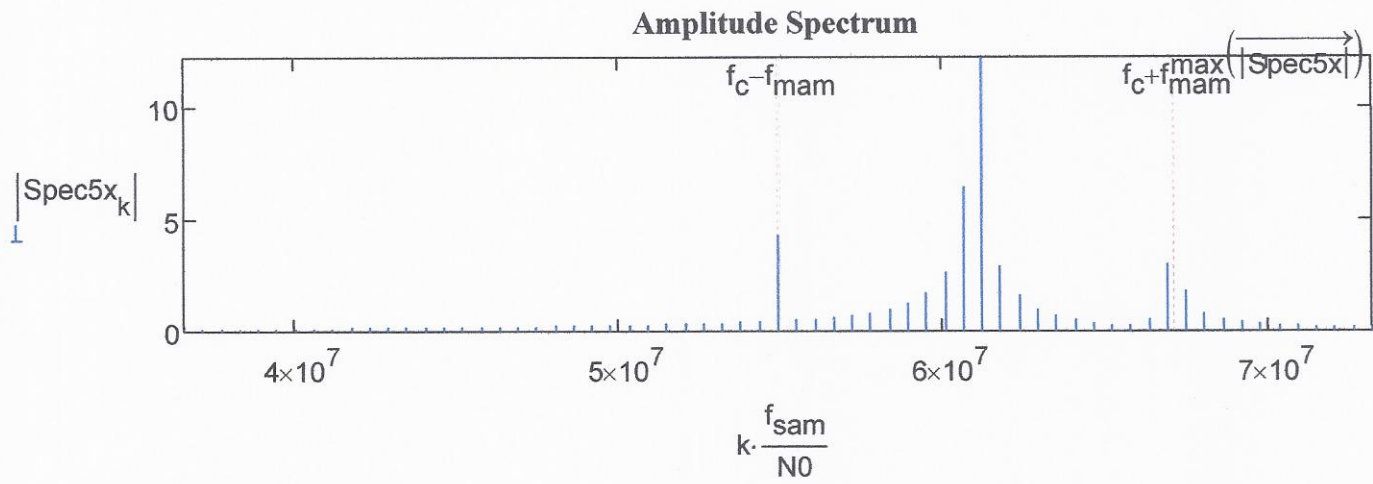


fig.:5.3.2.5.8

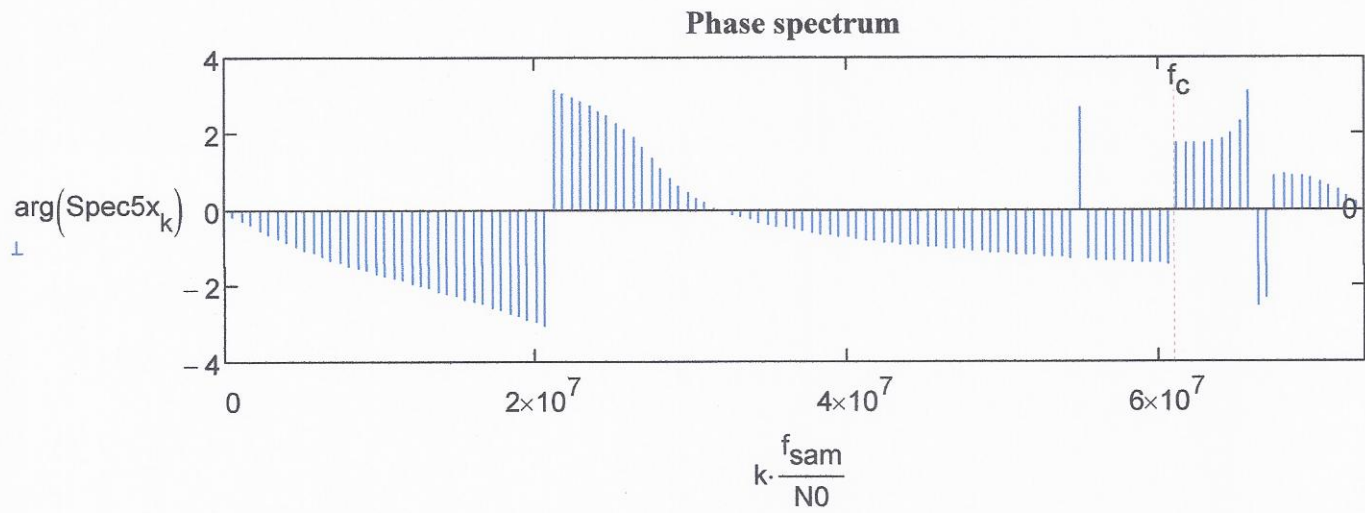


fig.:5.3.2.5.9

### 5.3 Equivalent Digital Low Pass Filter (II°order)

#### 5.3.2.6 Sequence of the (single tone) Frequency Modulated carrier response.

$$A_{fm} = 0.02V \quad m_f = 8 \quad \omega_{mfm} = 1.916 \times 10^4 \cdot \frac{\text{k rads}}{\text{sec}} \quad \omega_c = 0.383 \cdot \frac{\text{Grads}}{\text{sec}}$$

$$u_{9k} := \frac{v_{fmsl}(nfm_k, \omega_c, \omega_{mfm}, A_{fm}, m_f)}{\text{volt}} \quad (5.3.2.6.1)$$

$$\text{Spec}_9 := \text{fft}(u_9) \quad (5.3.2.6.2)$$

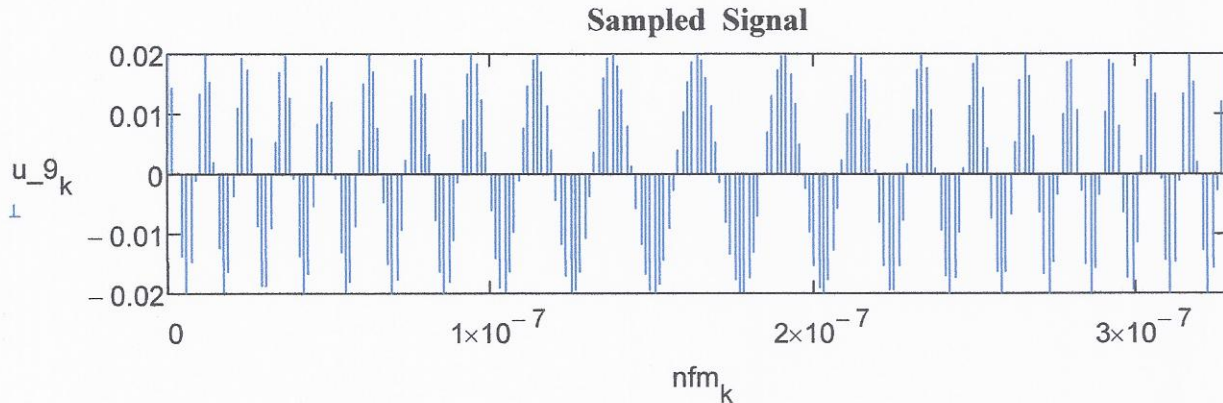


fig.:5.3.2.6.1

$$\text{Place } T_s := T_{sfm} \quad T_s = 1.45 \cdot \text{ns} \quad \omega_{smp} := \frac{2 \cdot \pi}{T_s}$$

$$v6_i(v) := u_{9v} \quad (5.3.2.6.3)$$

$$\text{svsr6} := \text{CANONIC2LP}(v6_i, A_5, \zeta, \omega_5, T_s, N0) \quad (5.3.2.6.4)$$

$$\text{svsr6} = (-0.772 \quad 2.051 \quad 1.129 \quad \{256, 1\} \quad \{256, 1\} \quad 1.278)$$

$$a6 := \text{svsr6}_{0,0} \quad b6 := \text{svsr6}_{0,1} \quad c6 := \text{svsr6}_{0,2} \quad \text{Glp6} := \text{svsr6}_{0,3} \quad Y6 := \text{svsr6}_{0,4} \quad d6 := \text{svsr6}_{0,5}$$

$$a6 = -0.77160494 \quad b6 = 2.0514403292 \quad c6 = 1.128600823 \quad d6 = 1.278$$

you get the following result for the t. f. as a function of z:

$$T_s = 1.45 \times 10^{-6} \cdot \text{ms}$$

$$H_{lp}(z) := \begin{cases} \frac{a6}{z^{-2} - b6 \cdot z^{-1} + c6} & \text{if } \zeta \neq \omega_5 \\ \frac{a6}{(d6 - z^{-1})^2} & \text{otherwise} \end{cases} \quad (5.3.2.6.5)$$

$$W1_{lpp} := 20 \cdot \log \left( \left| \frac{A_5 \cdot \omega_5}{2 \cdot \zeta} \right| \right)$$



# BODE PLOTS (Low Pass (II° order)):

Frequency Responses for sampling period  $T_{s_{fm}}$

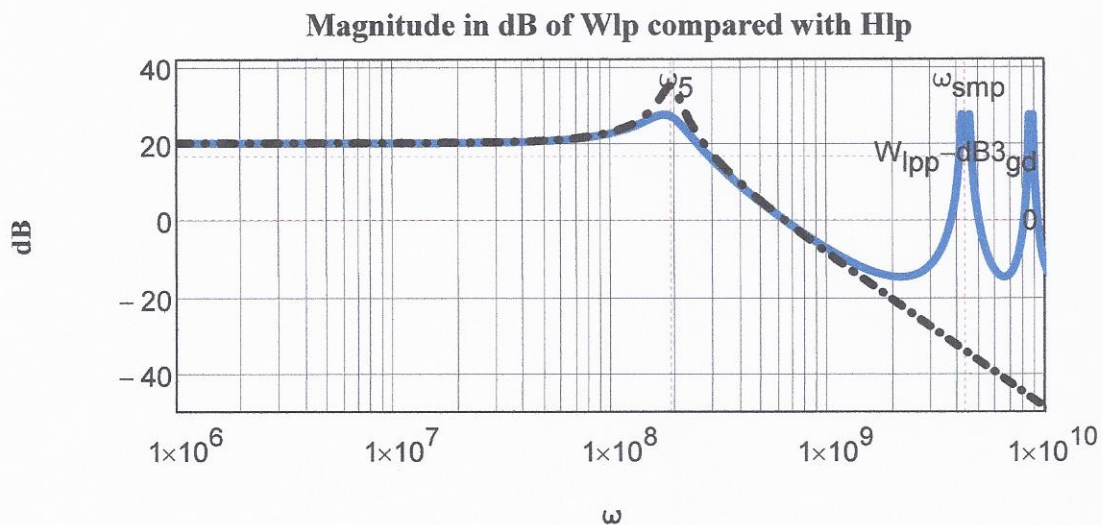


fig.:5.3.2.6.2

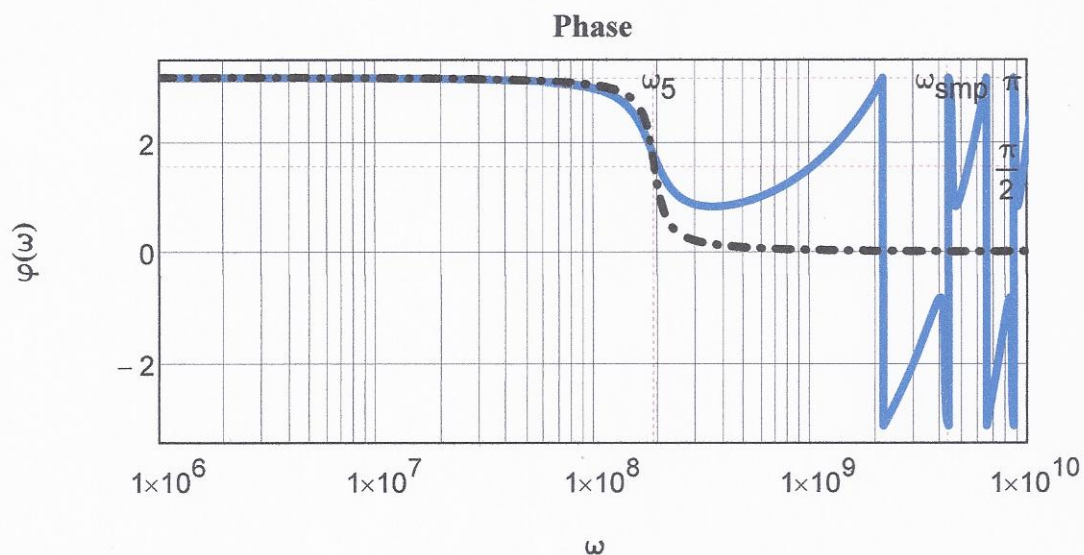


fig.:5.3.2.6.3

Block diagram of the difference equation algorithm for a second order system

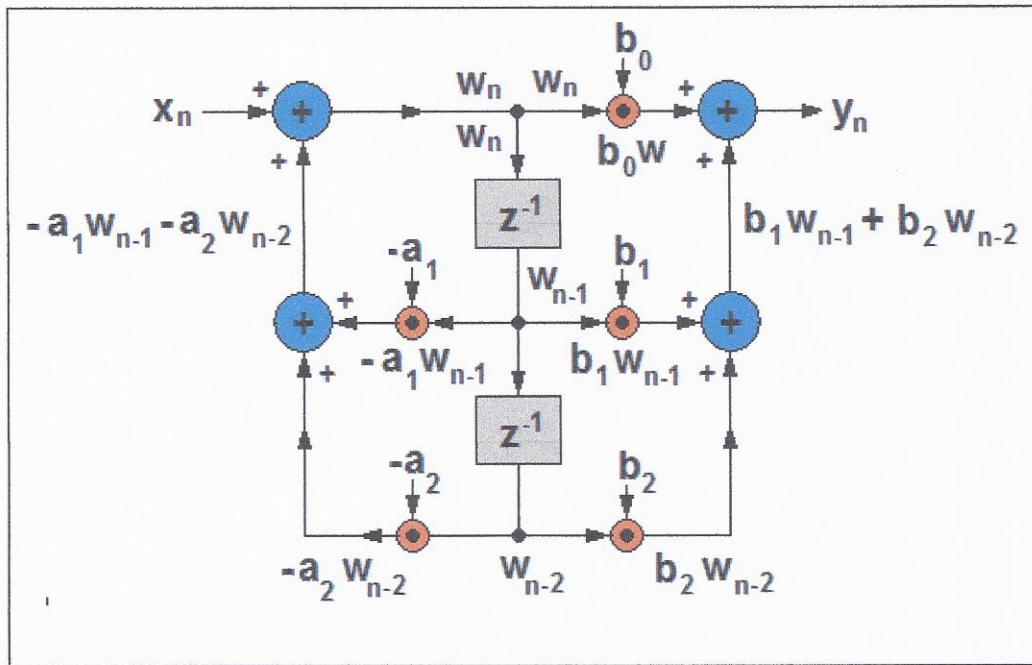


fig.:5.3.2.6.4

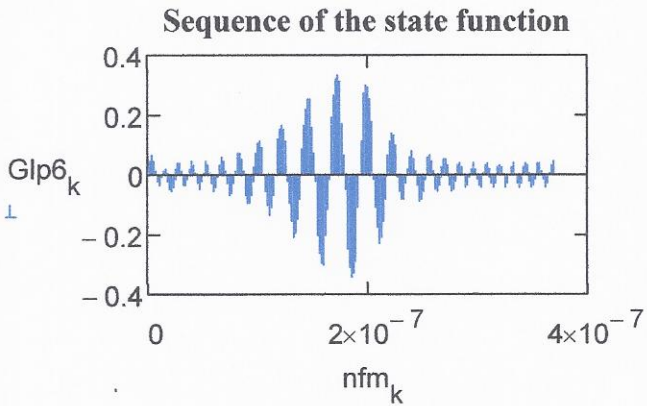


fig.:5.3.2.6.5

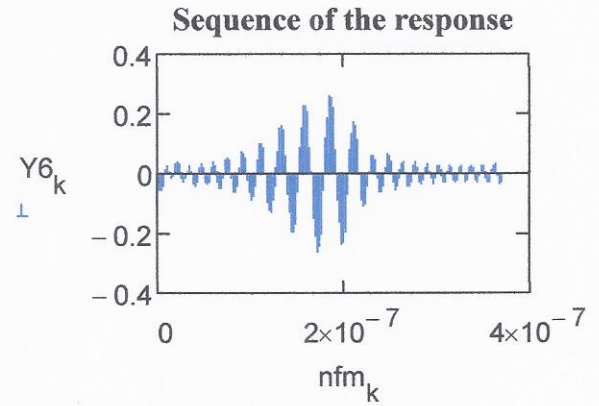


fig.:5.3.2.6.6

Spec6x := FFT(Y6)       $\max(|\text{Spec6x}|) = 0.029$       (5.3.2.6.6)

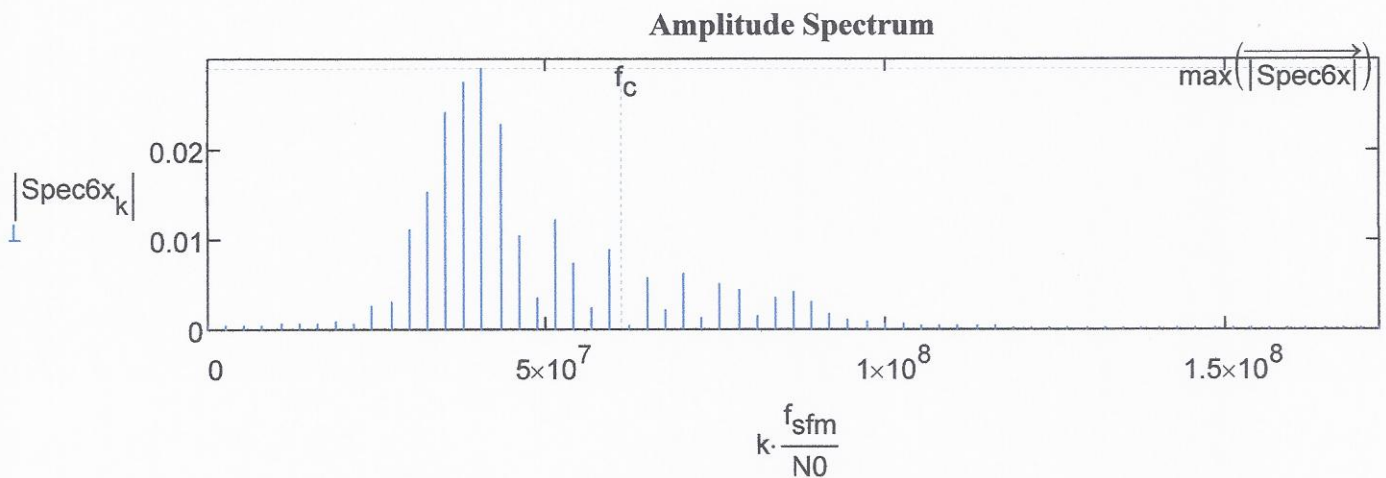


fig.:5.3.2.6.7

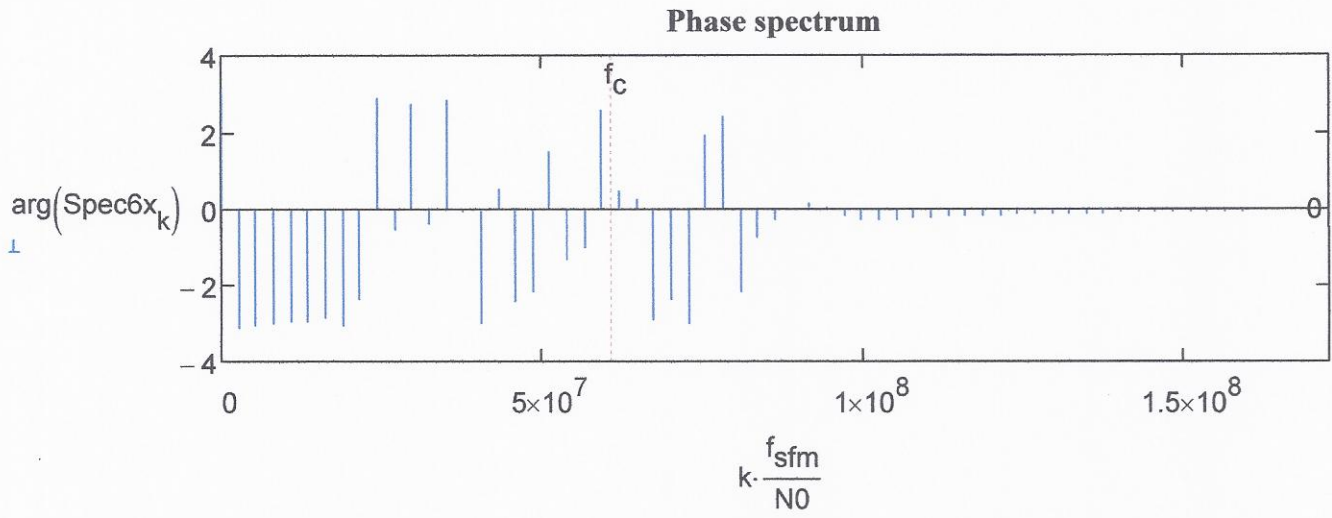


fig.:5.3.2.6.8

Frequency Responses for sampling period  $T_{\text{sfm}}$

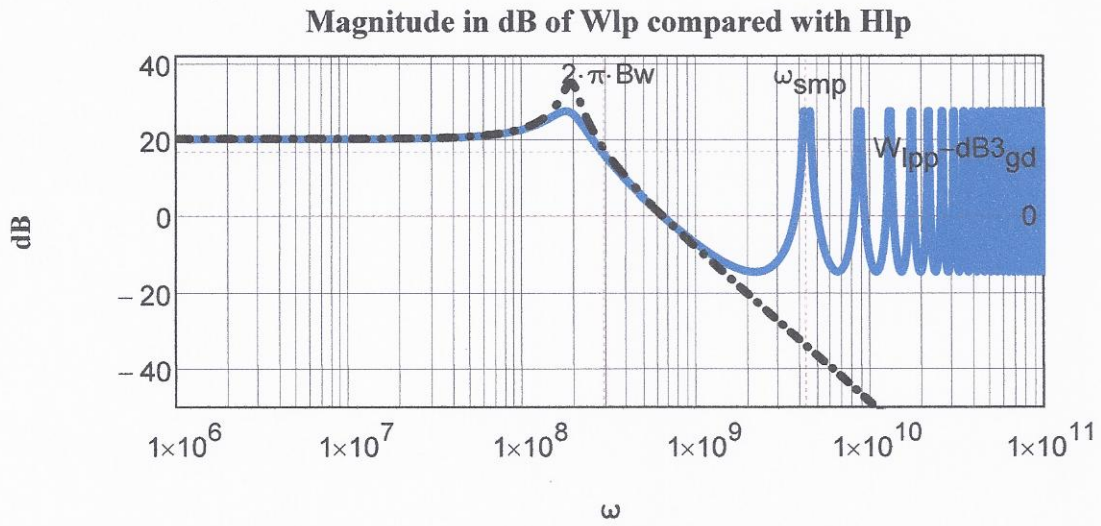


fig.:5.3.2.6.9

### 5.3 Equivalent Digital Low Pass Filter (II°order)

#### 5.3.2.7 Sequence of the (single tone) Phase Modulated carrier response.

$$\omega_{\text{mpm}} = 76.651 \cdot \frac{\text{Mrads}}{\text{sec}} \quad A_{\text{pm}} = 0.02\text{V} \quad 2 \cdot \pi \cdot f_c = 0.383 \cdot \frac{\text{Grads}}{\text{sec}} \quad \omega_c = 0.383 \cdot \frac{\text{Grads}}{\text{sec}}$$

$$m_p = 8 \quad u_{10k} := \frac{v_{\text{pm}}(n_{\text{pm}k}, \omega_c, \omega_{\text{mpm}}, A_{\text{pm}}, m_p)}{\text{volt}} \quad (5.3.2.7.1)$$

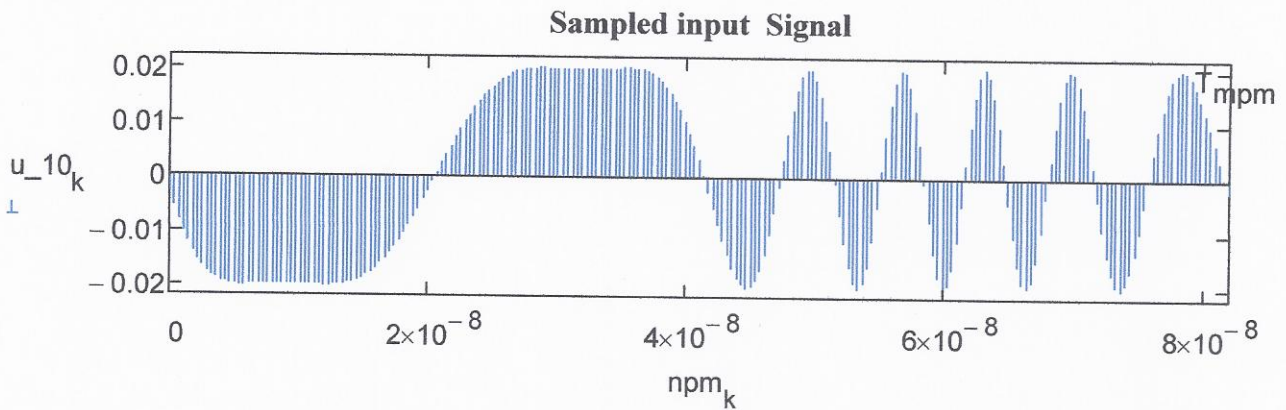


fig.:5.3.2.7.1

$$\text{Place } T_s := T_{\text{smp}} \quad T_s = 0.362 \cdot \text{ns} \quad \omega_{\text{smp}} := \frac{2 \cdot \pi}{T_s}$$

$$v7_i(k) := u_{10k} \quad (5.3.2.7.2)$$

$$\text{svsr7} := \text{CANONIC2LP}(v7_i, A_5, \zeta, \omega_5, T_s, N0) \quad (5.3.2.7.3)$$

$$\text{svsr7} = (-0.048 \quad 2.013 \quad 1.018 \quad \{256,1\} \quad \{256,1\} \quad 1.069)$$

$$a7 := \text{svsr7}_{0,0} \quad b7 := \text{svsr7}_{0,1} \quad c7 := \text{svsr7}_{0,2} \quad \text{Glp7} := \text{svsr7}_{0,3} \quad Y7 := \text{svsr7}_{0,4} \quad d7 := \text{svsr7}_{0,5}$$

$$a7 = -0.04822531 \quad b7 = 2.0128600823 \quad c7 = 1.0176826132 \quad d7 = 1.069$$

you get the following result for the t. f. as a function of z:

$$T_s = 0.362 \cdot \text{ns} \quad H_{\text{lp}}(z) := \begin{cases} \frac{a7}{z^{-2} - b7 \cdot z^{-1} + c7} & \text{if } \zeta \neq \omega_5 \\ \frac{a7}{(d7 - z^{-1})^2} & \text{otherwise} \end{cases} \quad (5.3.2.7.4)$$

$$W1_{\text{lp}} := 20 \cdot \log \left( \left| \frac{A_5 \cdot \omega_5}{2 \cdot \zeta} \right| \right)$$



# BODE PLOTS (Low Pass (II° order)):

Frequency Responses for sampling period  $T_{spm}$

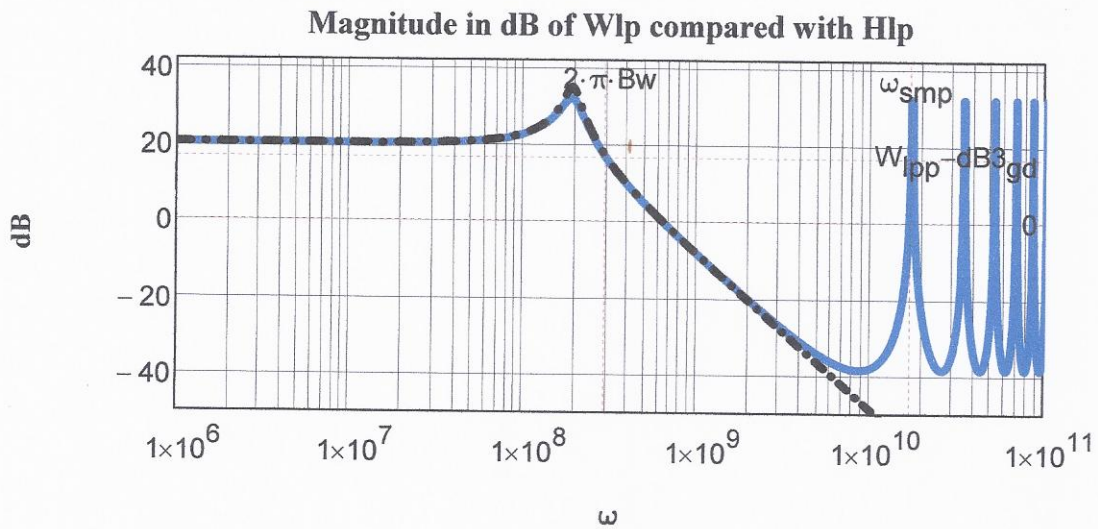


fig.:5.3.2.7.2

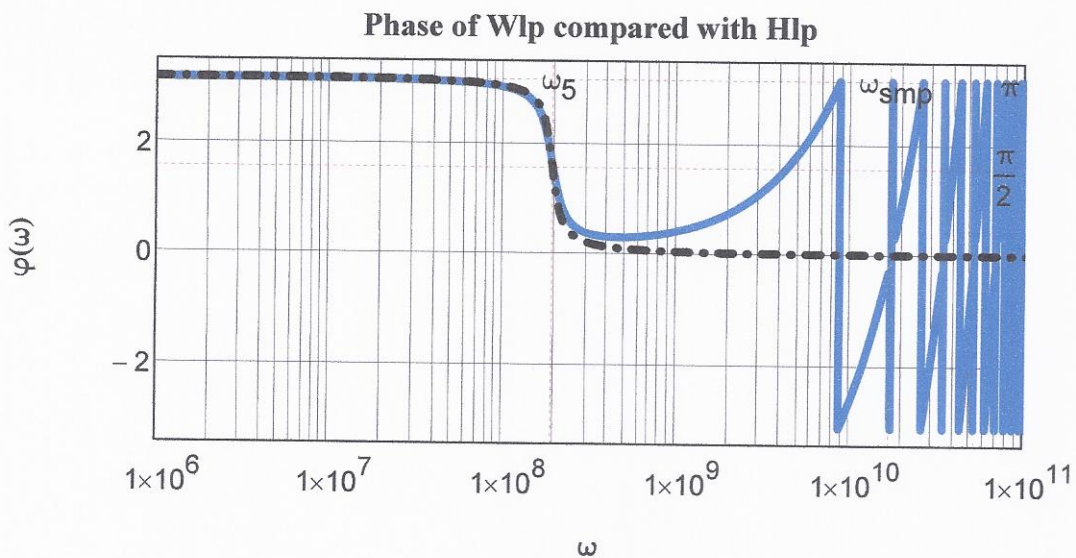


fig.:5.3.2.7.3

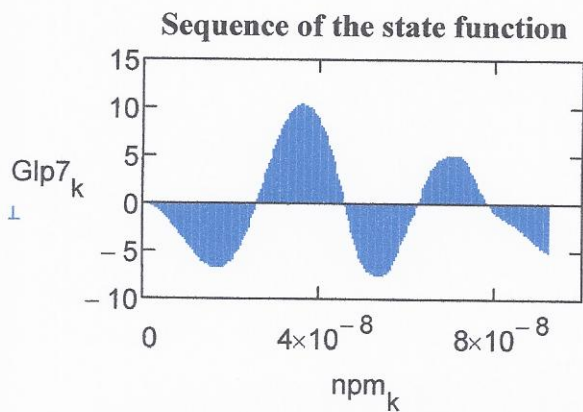


fig.:5.3.2.7.4

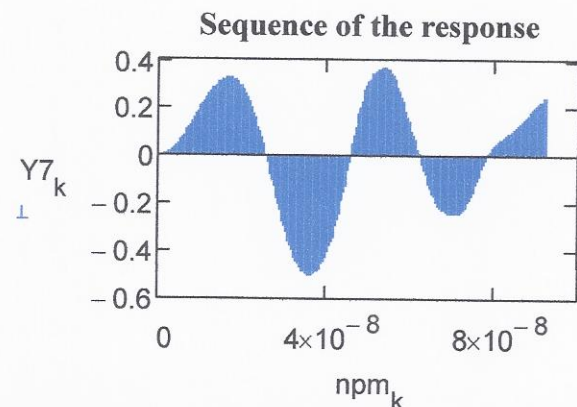


fig.:5.3.2.7.5

Block diagram of the difference equation algorithm for a second order system

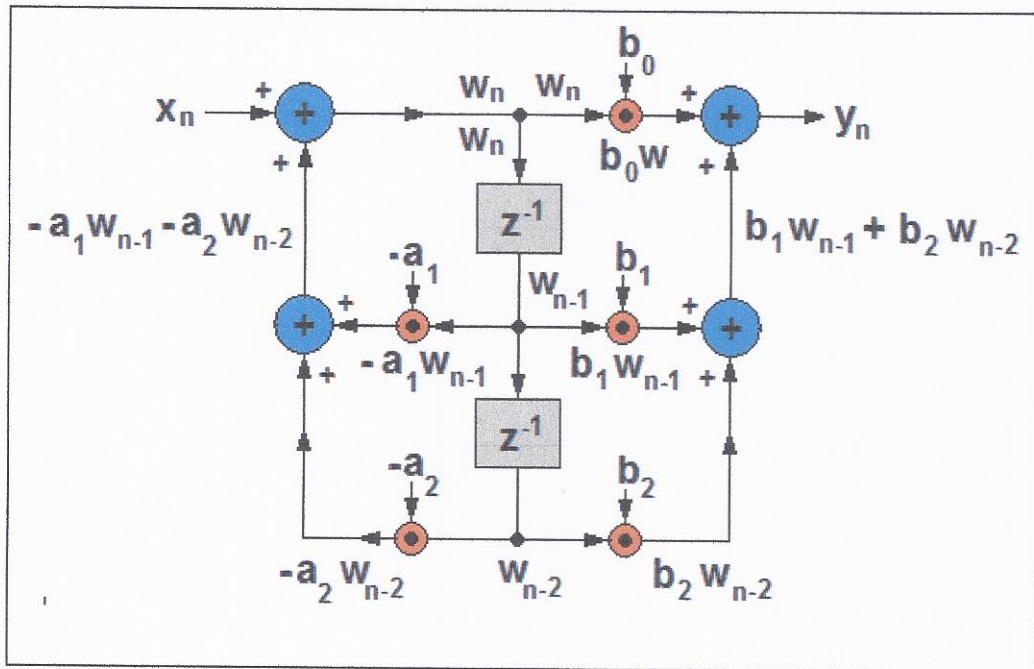


fig.:5.3.2.7.6

$$\text{Spec7x} := \text{FFT}(Y7) \quad \max(|\text{Spec7x}|) = 0.117 \quad (5.3.2.7.5)$$

Amplitude Spectrum

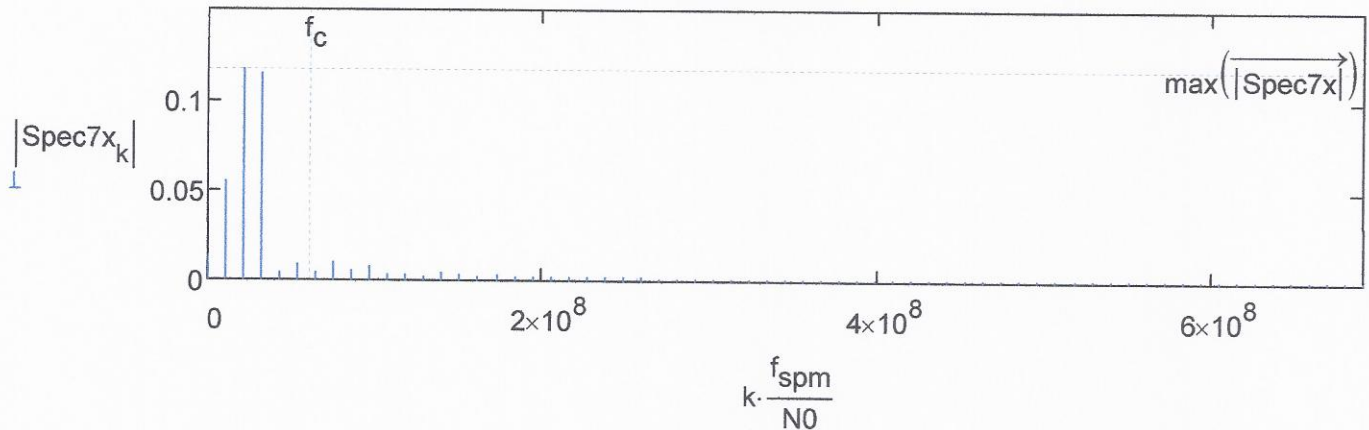


fig.:5.3.2.7.7

Phase spectrum

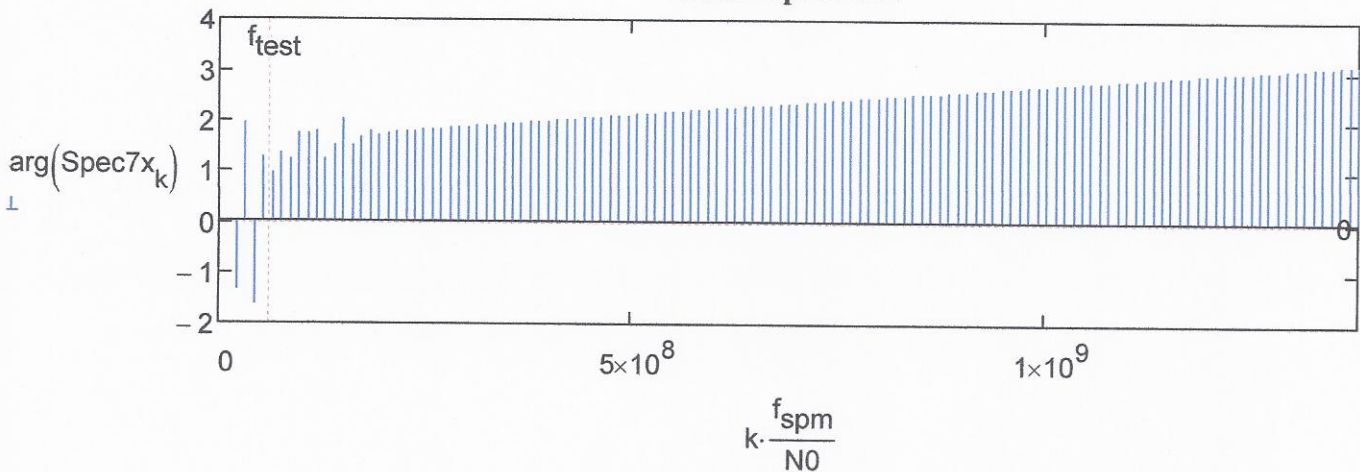


fig.:5.3.2.7.8

Frequency Responses for the sampling period  $T_{\text{smp}}$

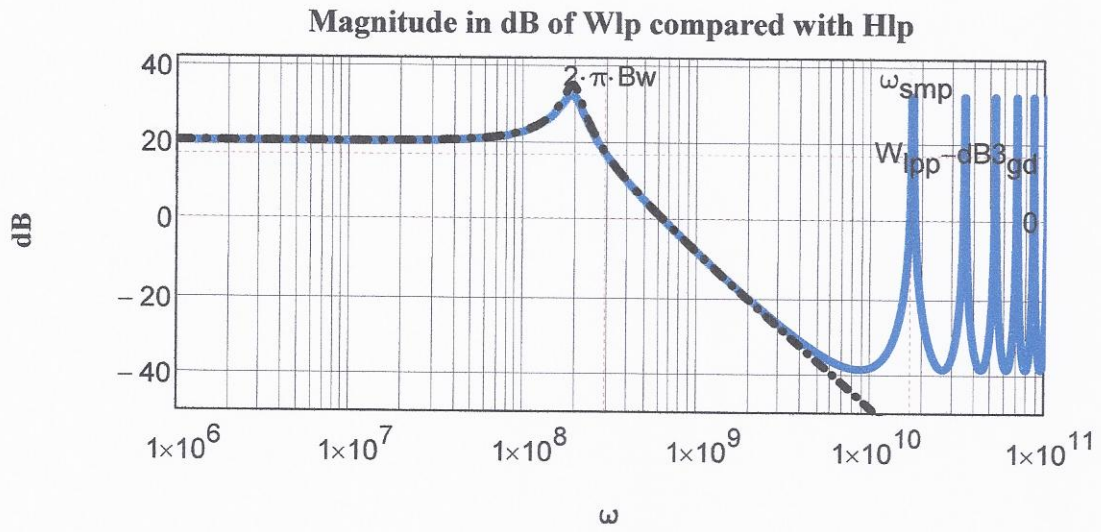


fig.:5.3.2.7.9



## 5.4

### **Synthetic Division Algorithm To Generate The Transfer Function Sequence. Output Produced By A Discrete Convolution .**

Given the Z transform of the system's transfer function

$$H_{lp}(z) = \begin{cases} \frac{A_1}{z^{-2} - B_1 \cdot z^{-1} + C_1} & \text{if } \zeta \neq \omega_5 \\ \frac{A_1}{(D_1 - z^{-1})^2} & \text{otherwise} \end{cases} \quad (5.4.1)$$

it is possible to obtain the sequence of the pulse response by a synthetic division of the z transfer function. Once defined the orders of the Numerator  $N_{num} := 0$  and of the Denominator of the t. f.:  $M_{den} := 2$ , then proceed with the calculation of the coefficients corresponding to the chosen sampling frequency:

$$\begin{aligned} A_1 &:= A_5 \cdot \omega_5^2 \cdot T_s^2 & B_1 &:= 2 \cdot (1 + \zeta \cdot T_s) & C_1 &:= T_s \cdot (\omega_5^2 \cdot T_s + 2 \cdot \zeta) + 1 & D_1 &:= T_s \cdot \omega_5 + 1 \\ \underline{N1} &:= N_{num} + M_{den} & \underline{\nu} &:= 1 .. N0 - 1 \end{aligned} \quad (5.4.2)$$

Define two vectors with N0 rows, namely: "b" for the coefficients of the numerator and "a" for the coefficient of the denominator:

Numerator Coeffs.	Denominator Co eff.	
$\underline{b}_\nu := 0.0$	$\underline{a}_\nu := 0.0$	
$\underline{b}_0 := A_1$	$\underline{a}_0 := \begin{cases} C_1 & \text{if } \zeta \neq \omega_5 \\ D_1^2 & \text{otherwise} \end{cases}$	$a_0 = 1.018$

(5.4.3)

$\underline{b}_1 := 0$	$\underline{a}_1 := \begin{cases} -B_1 & \text{if } \zeta \neq \omega_5 \\ -2 \cdot D_1 & \text{otherwise} \end{cases}$	$a_1 = -2.013$
------------------------	---	----------------

(5.4.4)

$\underline{b}_2 := 0$	$\underline{a}_2 := 1$	$a_2 = 1$
------------------------	------------------------	-----------

(5.4.5)

Then write the algorithm for the synthetic division:

$$N1 = 2 \quad h_k := 0 \quad h_0 := \frac{b_0}{a_0} \quad h_\nu := \frac{1}{a_0} \cdot \left[ b_\nu - \sum_{i=1}^{\nu} (h_{\nu-i} \cdot a_i) \right] \quad (5.4.6)$$

The output as a response to an input signal can be obtained by a convolution integral as follows:

$$y_{30\nu} := \sum_{k=0}^{\nu} (if(\nu - k \geq 0, h_k \cdot u_{1\nu-k}, 0)) \quad (5.4.7)$$

Now, to save space when applying the previous algorithm, it is convenient to call the following program ("SYNDIVC" acronym of: Synthetic Division and Convolution), it calculates the coefficient of the Z t. f. with correct sampling period linked to the signal bandwidth:

► SYNDIVC



## 5.4 Transfer Function Sequence Obtained by The Synthetic Division.

### 5.4.1 Sequence of the voltage Step response.

$$u_{10} = 5V$$

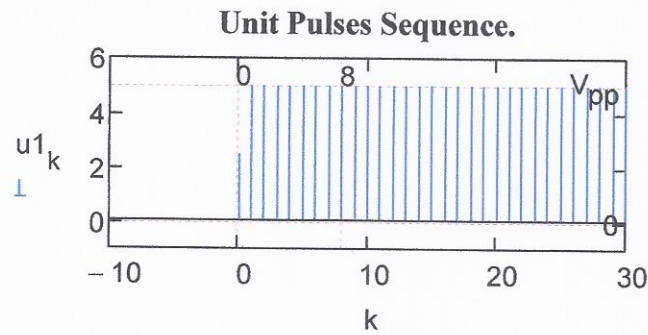


fig.:5.4.1.1

$$T_{sstp} = 1.639 \cdot ns \quad \text{conv1} := \text{SYNDIVC}\left(\frac{u_1}{V}, A_5, \zeta, \omega_5, T_{sstp}, N0\right) \quad (5.4.1.1)$$

$$\text{conv1} = (-0.987 \quad 2.058 \quad 1.157 \quad \{256,1\} \quad \{256,1\} \quad \{256,1\} \quad \{256,1\} \quad 27.225 \quad 32.194)$$

$$a0 := \text{conv1}_{0,3} \quad b0 := \text{conv1}_{0,4} \quad h := \text{conv1}_{0,5} \quad y_{10} := \text{conv1}_{0,6} \quad S0 := \text{conv1}_{0,7} \quad E0 := \text{conv1}_{0,8}$$

T. F. Numerator coefficients:

$$a_0^T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \\ \hline 0 & 1.157 & -2.058 & 1 & 0 & 0 & 0 & 0 & & \dots \\ \hline \end{array}$$

T. F. Denominator coefficients:

$$b_0^T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \\ \hline 0 & -0.987 & 0 & 0 & 0 & 0 & 0 & 0 & & \dots \\ \hline \end{array}$$

Impulse response sequence

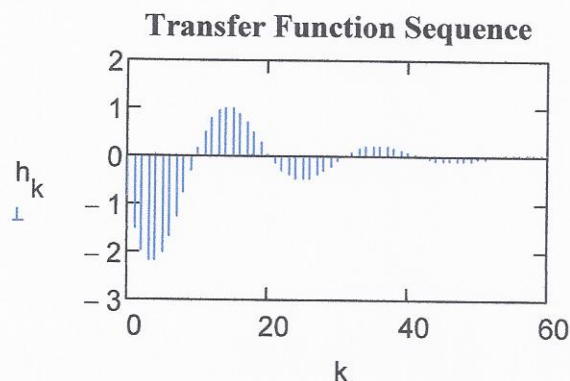
$$h^T = \begin{array}{|c|c|c|c|c|c|c|} \hline & 0 & 1 & 2 & 3 & 4 & \\ \hline 0 & -0.853 & -1.518 & -1.963 & -2.18 & & \dots \\ \hline \end{array}$$


fig.:5.4.1.2

Stability ( $S3 < \infty$ ):  $S0 = 27.225$

Energy of the sequence h3:  $E0 = 32.194$

$$t := 0 \cdot T_{\text{test}}, 0 \cdot T_{\text{test}} + \frac{20 \cdot T_5 - 0 \cdot T_{\text{test}}}{1000} .. 20 \cdot T_5$$

Step response of the II<sup>o</sup> order Low Pass Filter

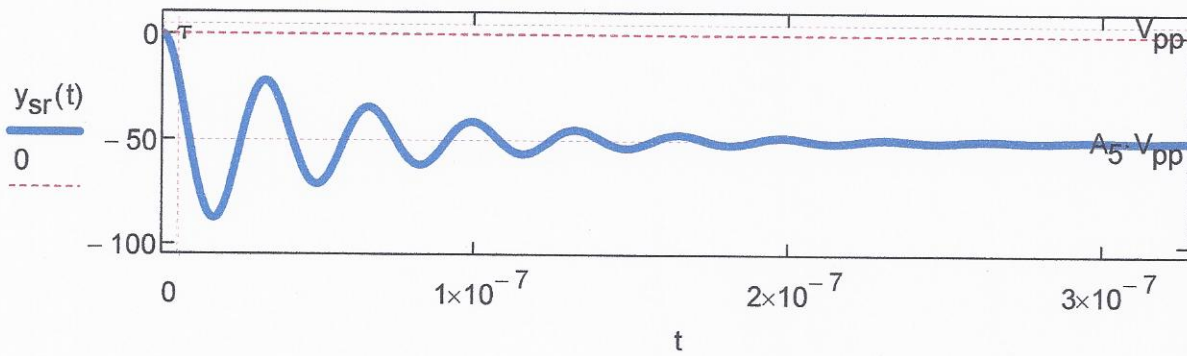


fig.:5.4.1.3

$$V_{pp} = 5V$$

Input Sequence

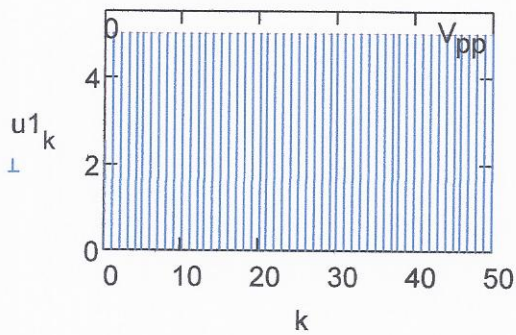


fig.:5.4.1.4

Sequence of the convolution

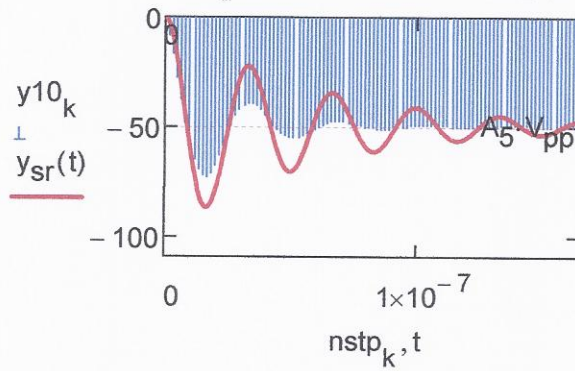


fig.:5.4.1.5

## 5.4 Transfer Function Sequence Obtained by The Synthetic Division.

### 5.4.2 Sequence of the Short Voltage Pulse response.

$$\text{conv2} := \text{SYNDIVC}(u44, A_5, \zeta, \omega_5, T_{\text{svp}}, N0) \quad (5.4.2.1)$$

$$\text{conv2} = (-43.865 \quad 2.388 \quad 5.774 \quad \{256,1\} \quad \{256,1\} \quad \{256,1\} \quad \{256,1\} \quad 11.593 \quad 67.932)$$

$$a2 := \text{conv2}_{0,3} \quad b2 := \text{conv2}_{0,4} \quad h2 := \text{conv2}_{0,5} \quad y11 := \text{conv2}_{0,6}$$

T. F. Numerator coefficients:

$$a2^T = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 0 & 5.774 & -2.388 & 1 & 0 & 0 & 0 & 0 & \dots \\ \hline \end{array}$$

T. F. Denominator coefficients:

$$b2^T = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 0 & -43.865 & 0 & 0 & 0 & 0 & 0 & \dots \\ \hline \end{array}$$

Impulse response sequence

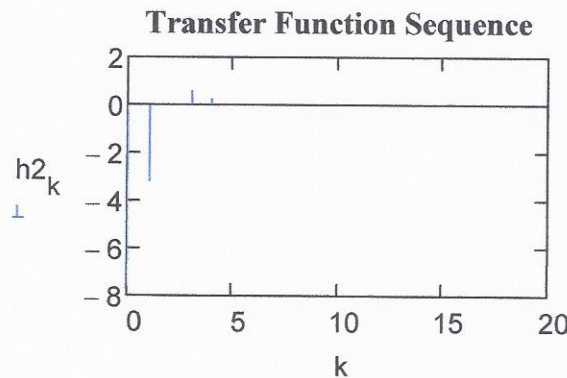
$$h2^T = \begin{array}{|c|c|c|c|c|c|} \hline & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & -7.597 & -3.141 & 0.017 & 0.551 & \dots \\ \hline \end{array}$$


fig.:5.4.2.1

$$t_w := -2 \cdot \tau_{pw}, -2 \cdot \tau_{pw} + \frac{4 \cdot \tau_{pw}}{5000} \dots 2 \cdot \tau_{pw}$$

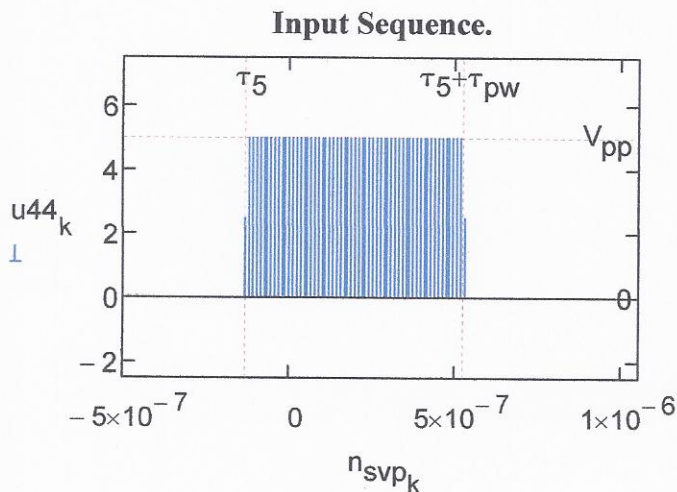


fig.:5.4.2.2

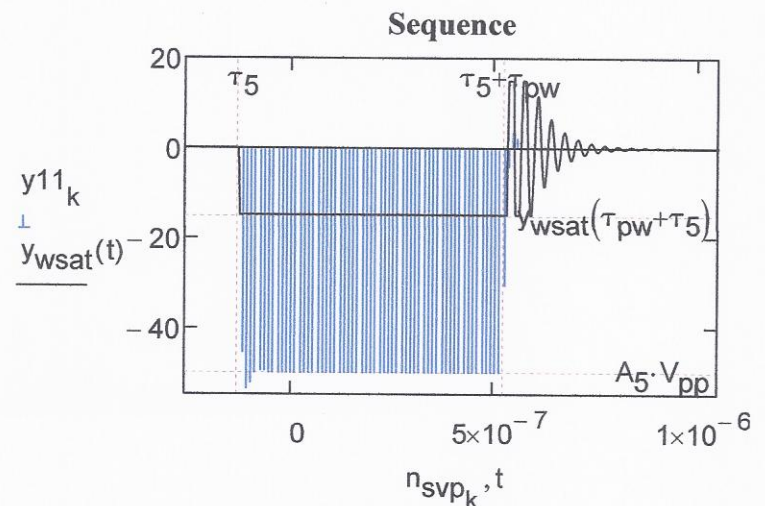


fig.:5.4.2.3

Spec11x := fft(y11)

Amplitude Spectrum

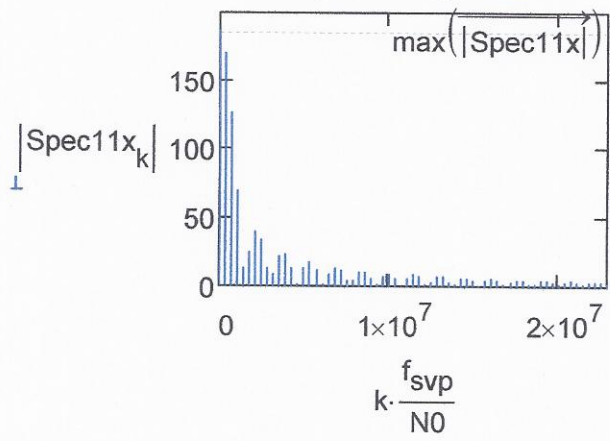


fig.:5.4.2.4

Phase spectrum

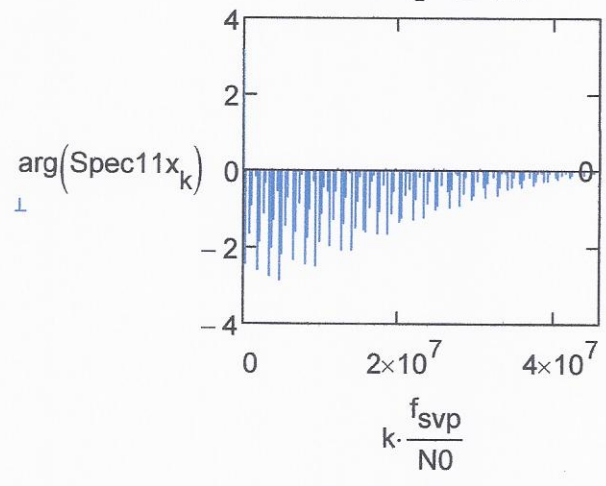


fig.:5.4.2.5



## 5.4 Transfer Function Sequence Obtained by The Synthetic Division.

### 5.4.3 Sequence of the Sawtooth response:

$$\text{conv3} := \text{SYNDIVC}\left(\frac{u55}{V}, A_5, \zeta, \omega_5, T_{\text{ssw}}, N0\right) \quad (5.4.3.1)$$

$$\text{conv3} = (-0.011 \quad 2.006 \quad 1.007 \quad \{256,1\} \quad \{256,1\} \quad \{256,1\} \quad \{256,1\} \quad 35.693 \quad 6.382)$$

$$a03 := \text{conv1}_{0,3} \quad b03 := \text{conv1}_{0,4} \quad h3 := \text{conv1}_{0,5} \quad y12 := \text{conv3}_{0,6}$$

T. F. Numerator coefficients:

$$a03^T = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 0 & 1.157 & -2.058 & 1 & 0 & 0 & 0 & 0 & \dots \\ \hline \end{array}$$

T. F. Denominator coefficients:

$$b03^T = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 0 & -0.987 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \hline \end{array}$$

Impulse response sequence

$$h3^T = \begin{array}{|c|c|c|c|c|c|} \hline & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & -0.853 & -1.518 & -1.963 & -2.18 & \dots \\ \hline \end{array}$$

Transfer Function Sequence

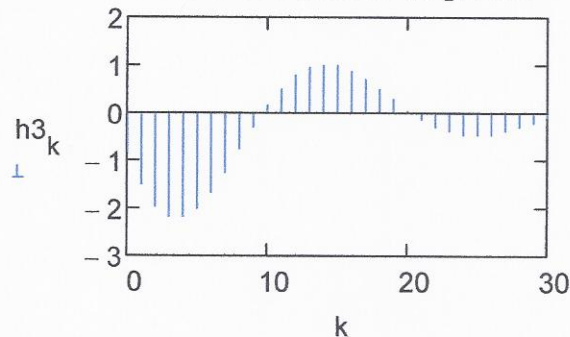


fig.:5.4.3.1

Input Sequence.

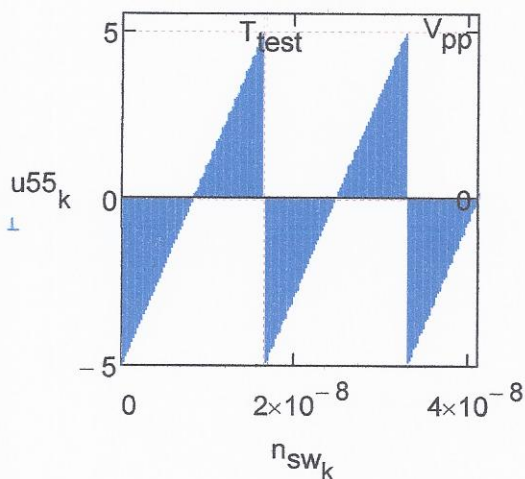


fig.:5.4.3.2

Output Sequence

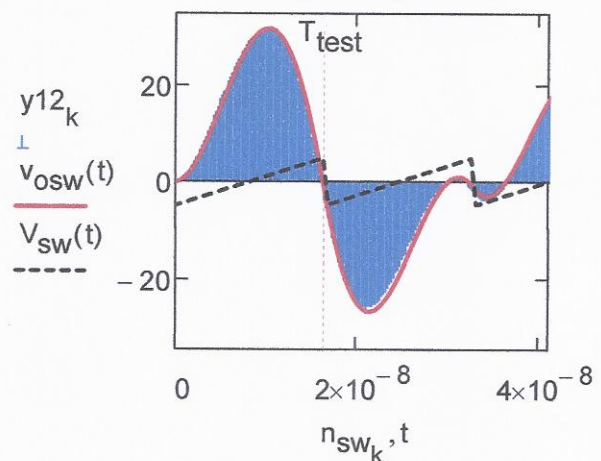


fig.:5.4.3.3

Spec12x := fft(y12)

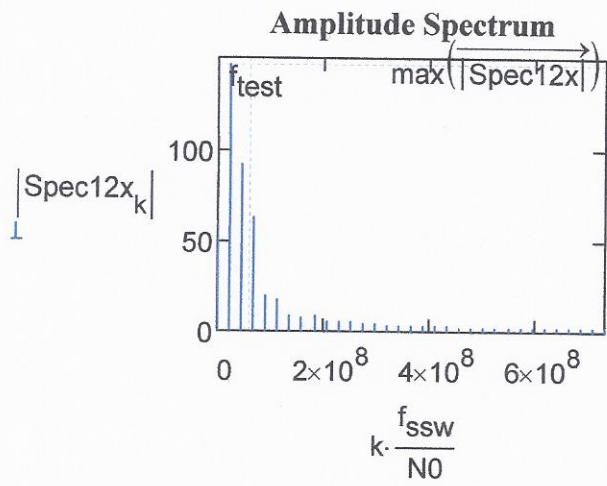


fig.:5.4.3.4

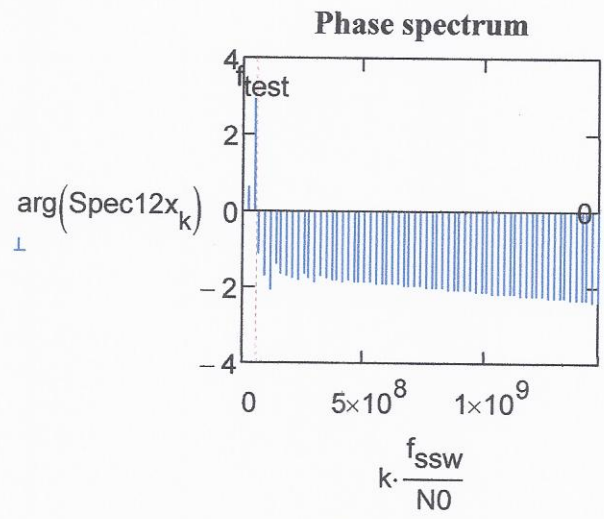


fig.:5.4.3.5

## 5.4 Transfer Function Sequence Obtained by The Synthetic Division.

### 5.4.4 Sequence of the Bipolar Square Wave response.

$$v11_i(k) := sqwk \quad (5.4.4.1)$$

$$\text{conv4} := \text{SYNDIVC}(sqw, A_5, \zeta, \omega_5, T_{ssqw}, N0) \quad (5.4.4.2)$$

$$\text{conv4} = (-0.011 \quad 2.006 \quad 1.007 \quad \{256,1\} \quad \{256,1\} \quad \{256,1\} \quad \{256,1\} \quad 35.693 \quad 6.382)$$

$$a04 := \text{conv4}_{0,3} \quad b04 := \text{conv4}_{0,4} \quad h4 := \text{conv4}_{0,5} \quad y13 := \text{conv4}_{0,6}$$

T. F. Numerator coefficients:

$$a04^T = \begin{array}{c|cccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 0 & 1.007 & -2.006 & 1 & 0 & 0 & 0 & 0 & \dots \end{array}$$

T. F. Denominator coefficients:

$$b04^T = \begin{array}{c|cccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 0 & -0.011 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \end{array}$$

Impulse response sequence

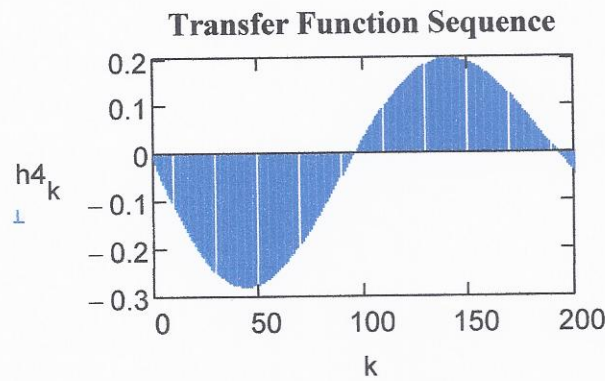
$$h4^T = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & -0.011 & -0.021 & -0.032 & -0.042 & \dots \end{array}$$


fig.:5.4.4.1

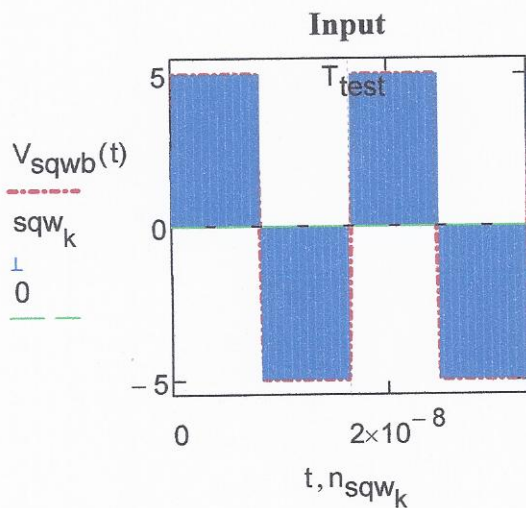


fig.:5.4.4.2

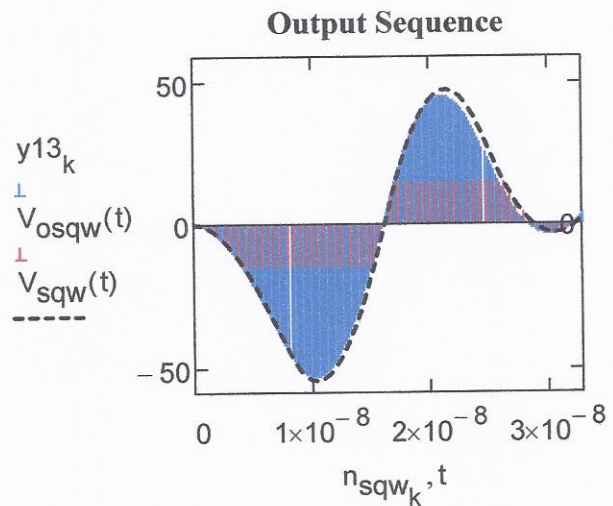


fig.:5.4.4.3

Spec13x := fft(y13)

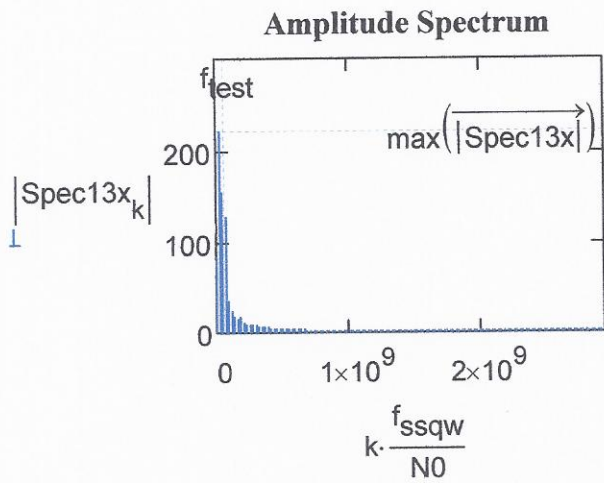


fig.:5.4.4.4

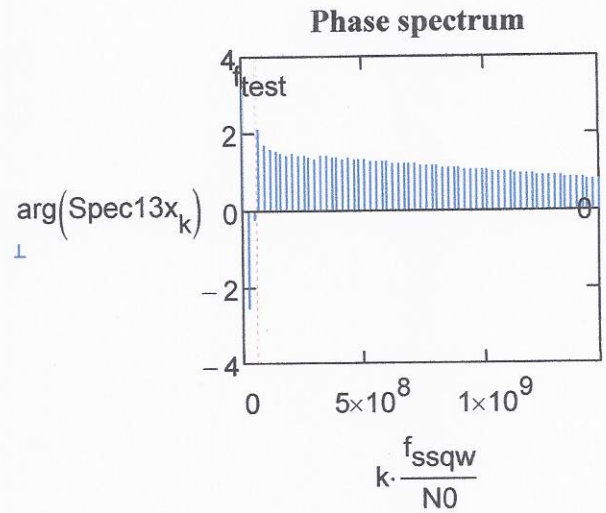


fig.:5.4.4.5



## 5.4 Transfer Function Sequence Obtained by The Synthetic Division.

### 5.4.5 Sequence of the AM (single tone) Signal response.

$$\text{conv5} := \text{SYNDIVC}(u_7, A_5, \zeta, \omega_5, T_{\text{sam}}, N0) \quad (5.4.5.1)$$

$$\text{conv5} = (-17.135 \quad 2.242 \quad 2.956 \quad \{256,1\} \quad \{256,1\} \quad \{256,1\} \quad \{256,1\} \quad 13.627 \quad 55.912)$$

$$a05 := \text{conv5}_{0,3} \quad b05 := \text{conv5}_{0,4} \quad h5 := \text{conv5}_{0,5} \quad y14 := \text{conv5}_{0,6}$$

T. F. Numerator coefficients:

$$a05^T = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 0 & 2.956 & -2.242 & 1 & 0 & 0 & 0 & 0 & \dots \\ \hline \end{array}$$

T. F. Denominator coefficients:

$$b05^T = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 0 & -17.135 & 0 & 0 & 0 & 0 & 0 & \dots \\ \hline \end{array}$$

Impulse response sequence

$$h5^T = \begin{array}{|c|c|c|c|c|c|} \hline & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & -5.797 & -4.398 & -1.375 & 0.445 & \dots \\ \hline \end{array}$$

$$m_{\text{am}} = 55.0\%$$

Transfer Function Sequence

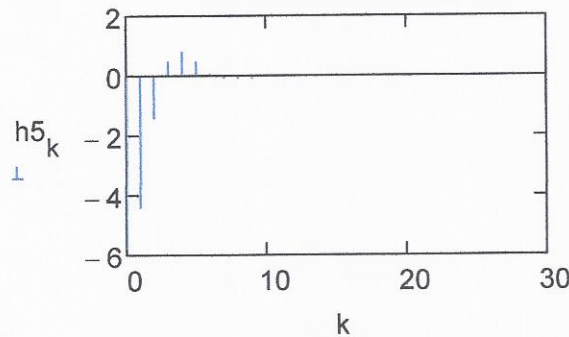


fig.:5.4.5.1

Input Sequence.

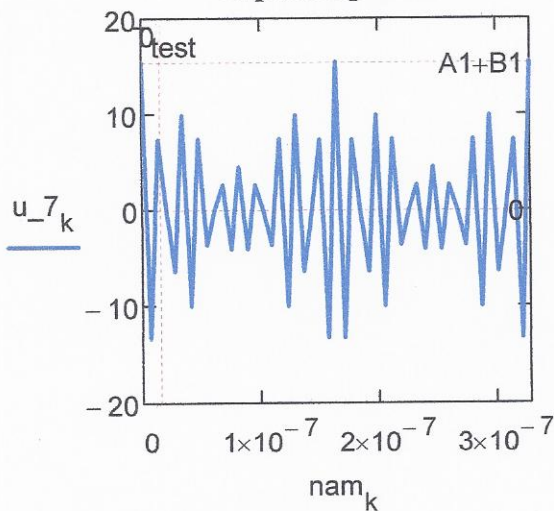


fig.:5.4.5.2

Output Sequence

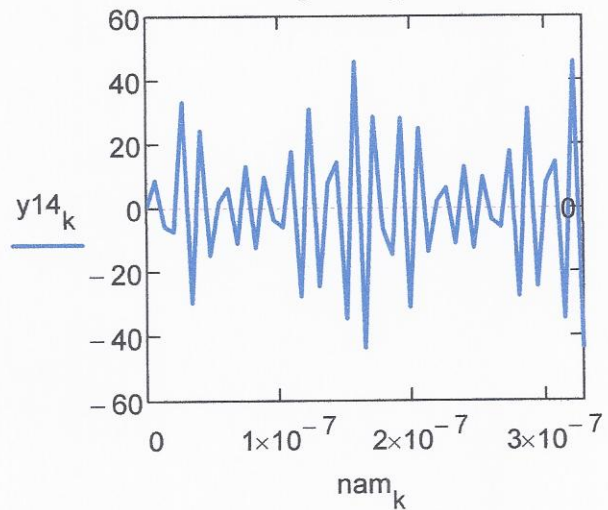


fig.:5.4.5.3

Sampled signal:

$$\text{Spec14x} := \text{fft}(y14)$$

(5.4.5.2)

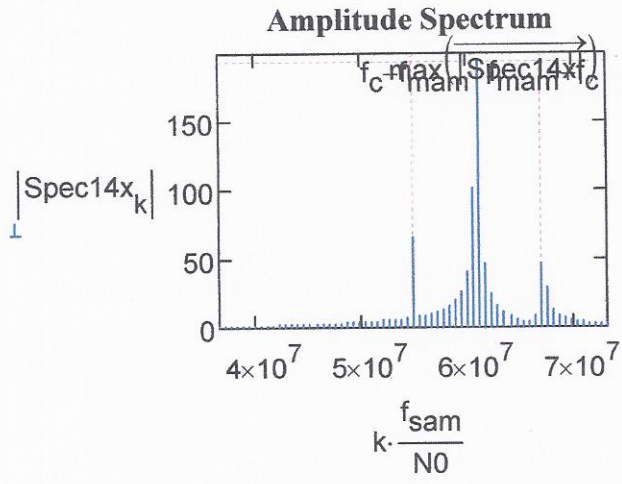


fig.:5.4.5.4

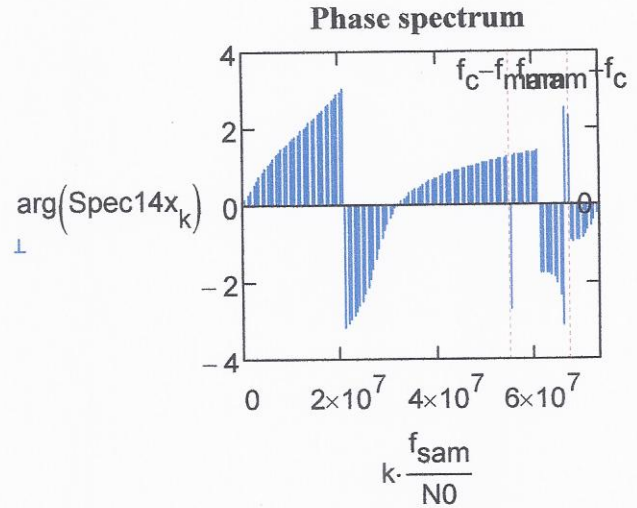


fig.:5.4.5.5

## 5.4 Transfer Function Sequence Obtained by The Synthetic Division.

### 5.4.6 Sequence of the (single tone) Frequency Modulated carrier response.

$$v13_i(k) := u8_k \quad (5.4.6.1)$$

$$\text{conv6} := \text{SYNDIVC}(u8, A_5, \zeta, \omega_5, T_{\text{sfm}}, N0) \quad (5.4.6.2)$$

$$\text{conv6} = (-0.772 \quad 2.051 \quad 1.129 \quad \{256,1\} \quad \{256,1\} \quad \{256,1\} \quad \{256,1\} \quad 28.99 \quad 30.554)$$

$$a06 := \text{conv6}_{0,3} \quad b06 := \text{conv6}_{0,4} \quad h6 := \text{conv6}_{0,5} \quad y15 := \text{conv6}_{0,6}$$

T. F. Numerator coefficients:

$$a06^T = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 0 & 1.129 & -2.051 & 1 & 0 & 0 & 0 & 0 & \dots \\ \hline \end{array}$$

T. F. Denominator coefficients:

$$b06^T = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 0 & -0.772 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \hline \end{array}$$

Impulse response sequence

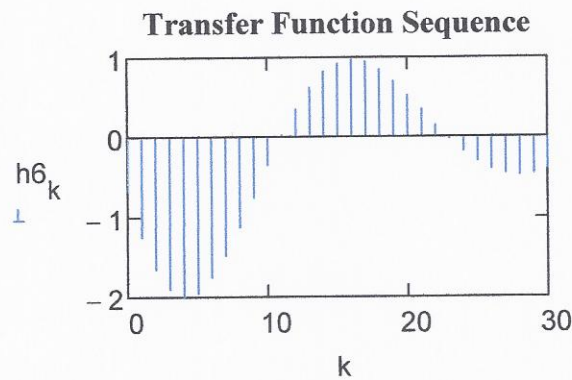
$$h6^T = \begin{array}{|c|c|c|c|c|c|} \hline & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & -0.684 & -1.243 & -1.653 & -1.904 & \dots \\ \hline \end{array}$$


fig.:5.4.6.1

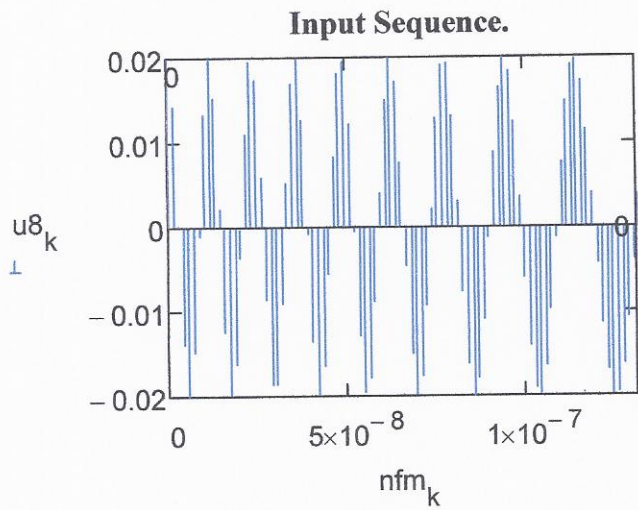


fig.:5.4.6.2

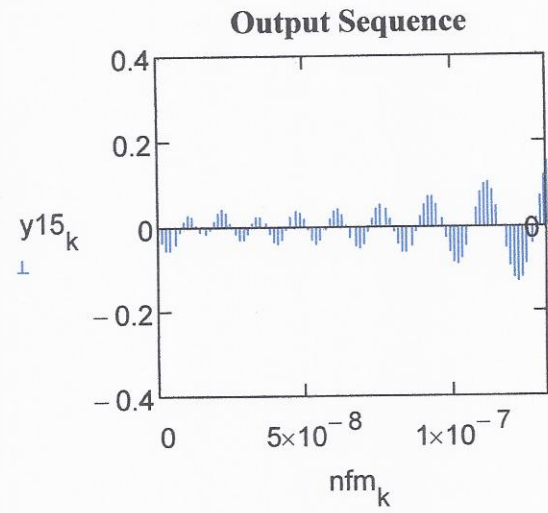


fig.:5.4.6.3

$m_f = 8$

$Spec15x := \text{fft}(y15)$

(5.4.6.3)

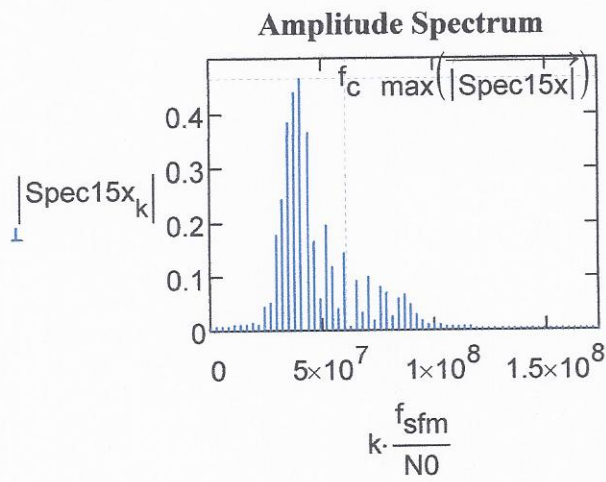


fig.:5.4.6.4

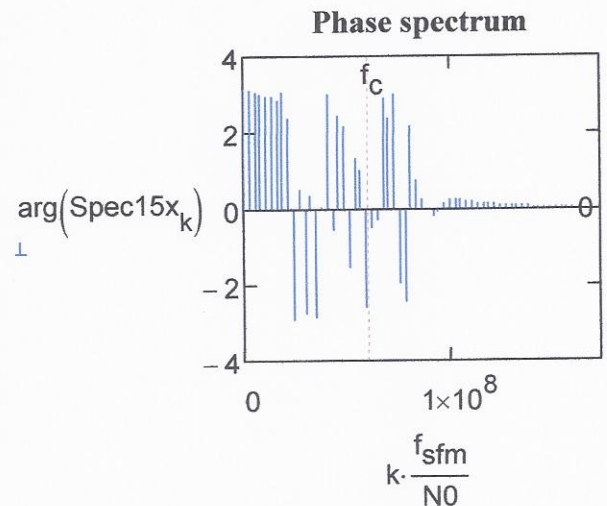


fig.:5.4.6.5

$A_{fm} \cdot A_5 = -0.2V$        $\omega_{mfm} = 0.019 \cdot \frac{\text{Grads}}{\text{sec}}$

$A_5 = -10$        $m_f = 8$



## 5.4 Transfer Function Sequence Obtained by The Synthetic Division.

### 5.4.7 Sequence of the (single tone) Phase Modulated carrier response.

$$v14_i(k) := u9_k \quad (5.4.7.1)$$

$$\text{conv7} := \text{SYNDIVC}(u9, A_5, \zeta, \omega_5, T_{\text{spm}}, N0) \quad (5.4.7.2)$$

$$\text{conv7} = (-0.048 \quad 2.013 \quad 1.018 \quad \{256,1\} \quad \{256,1\} \quad \{256,1\} \quad \{256,1\} \quad 45.268 \quad 13.507)$$

$$a07 := \text{conv7}_{0,3} \quad b07 := \text{conv7}_{0,4} \quad h7 := \text{conv7}_{0,5} \quad y16 := \text{conv7}_{0,6}$$

T. F. Numerator coefficients:

$$a07^T = \begin{array}{c|cccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 0 & 1.018 & -2.013 & 1 & 0 & 0 & 0 & 0 & \dots \end{array}$$

T. F. Denominator coefficients:

$$b07^T = \begin{array}{c|cccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 0 & -0.048 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \end{array}$$

Impulse response sequence

$$h7^T = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & -0.047 & -0.094 & -0.139 & -0.182 & \dots \end{array}$$

$$m_p = 8$$

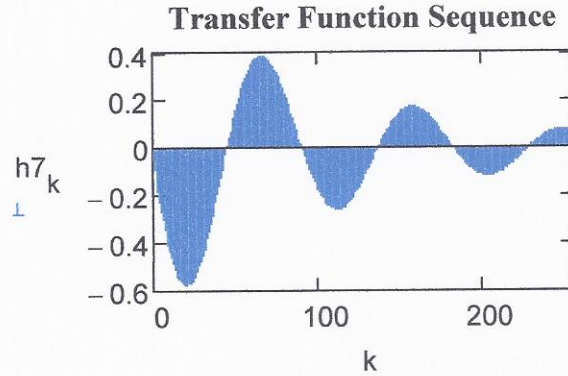


fig.:5.4.7.1

Sampled signal:  $\text{Spec16x} := \text{fft}(y16)$

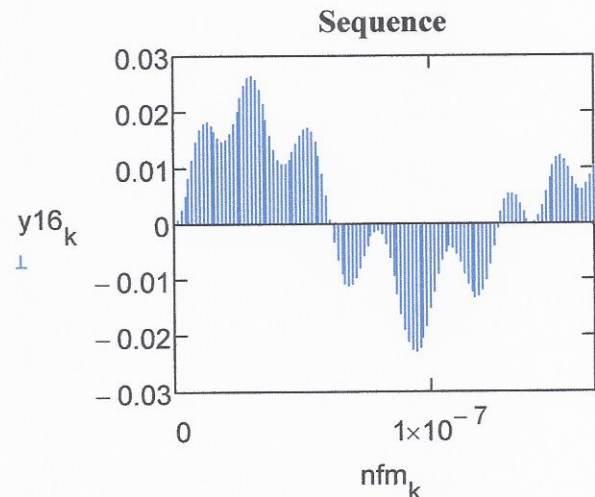
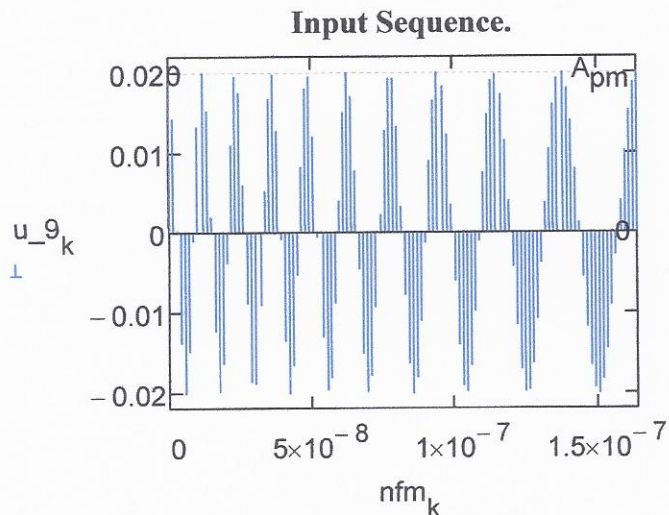


fig.:5.4.7.2

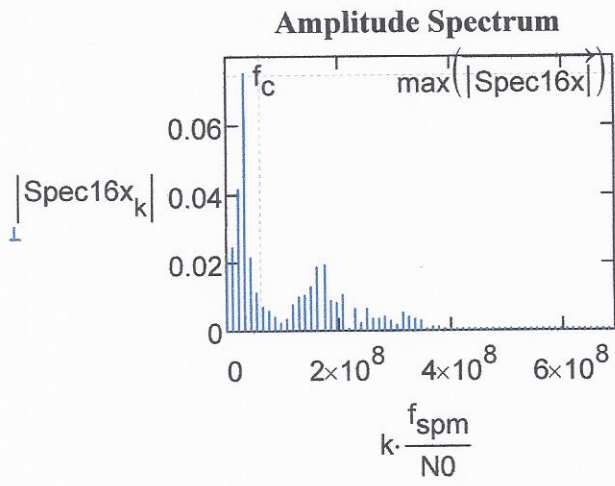


fig.:5.4.7.4

$$\omega_c = 0.383 \cdot \frac{\text{Grads}}{\text{sec}}$$

$$\omega_{mpm} = 0.077 \cdot \frac{\text{Grads}}{\text{sec}}$$

fig.:5.4.7.3

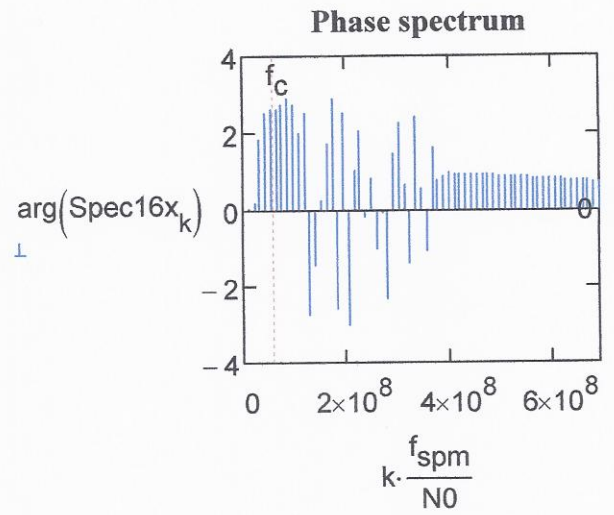


fig.:5.4.7.5

$$A_5 = -10 \quad m_p = 8$$

## 5.5

### Search of the discrete time sequence of the output by a discrete convolution

$$T_s := T_{ssw}$$

$$A_0 := A_5 \cdot \omega_5^2 \cdot T_s^2 \quad B_0 := 2 \cdot (1 + \zeta \cdot T_s) \quad C_0 := T_s \cdot (\omega_5^2 \cdot T_s + 2 \cdot \zeta) + 1 \quad D_0 := T_s \cdot \omega_5 + 1$$

The sequence corresponding to the transfer function

$$H_{lp}(z) := \begin{cases} \frac{A_0}{z^{-2} - B_0 \cdot z^{-1} + C_0} & \text{if } \zeta \neq \omega_5 \\ \frac{A_0}{(D_0 - z^{-1})^2} & \text{otherwise} \end{cases} \quad (5.5.1)$$

can be found using the "invztrans" MATHCAD's operator as follows:

$$A_0 := A_0 \quad B_0 := B_0 \quad k := k \quad C_0 := C_0$$

First case:  $\zeta \neq \omega_5$

$$h1_k := \frac{A_0}{z^{-2} - B_0 \cdot z^{-1} + C_0} \text{ invztrans, } z, k \rightarrow$$

(5.5.2)

$$h1_k := \left[ \begin{aligned} & \left( \frac{B_0 - \sqrt{B_0^2 - 4 \cdot C_0}}{C_0} \right)^{k-1} \cdot (2 \cdot C_0 - B_0^2 + B_0 \cdot \sqrt{B_0^2 - 4 \cdot C_0}) \dots \left[ \left( \frac{1}{2} \right)^k \cdot \frac{A_0}{(C_0^2 \cdot \sqrt{B_0^2 - 4 \cdot C_0})} \right] \\ & + \left( \frac{B_0 + \sqrt{B_0^2 - 4 \cdot C_0}}{C_0} \right)^{k-1} \cdot (B_0^2 - 2 \cdot C_0 + B_0 \cdot \sqrt{B_0^2 - 4 \cdot C_0}) \end{aligned} \right]$$

Second case:  $\zeta = \omega_5$

$$A_0 := A_0 \quad D_0 := D_0 \quad k := k$$

$$h1_k := \frac{A_0}{(D_0 - z^{-1})^2} \text{ invztrans, } z, k \rightarrow$$

$$h1_k := \frac{A_0 \cdot \left( \frac{1}{D_0} \right)^k \cdot (k+1)}{D_0^2}$$

$$K1 := \frac{B_0 - \sqrt{B_0^2 - 4 \cdot C_0}}{C_0} \quad K2 := \left( 2 \cdot C_0 - B_0^2 + B_0 \cdot \sqrt{B_0^2 - 4 \cdot C_0} \right)$$

$$K3 := \left( \frac{B_0 + \sqrt{B_0^2 - 4 \cdot C_0}}{C_0} \right) \quad K4 := \left( B_0^2 - 2 \cdot C_0 + B_0 \cdot \sqrt{B_0^2 - 4 \cdot C_0} \right)$$

$$K5 := \text{if} \left[ \zeta \neq \omega_5, \frac{A_0}{\left( C_0^2 \cdot \sqrt{B_0^2 - 4 \cdot C_0} \right)}, 0 \right]$$

$$h1_k := \begin{cases} \left( K1^{k-1} \cdot K2 + K3^{k-1} \cdot K4 \right) \cdot \left[ \left( \frac{1}{2} \right)^k \cdot K5 \right] & \text{if } \zeta \neq \omega_5 \\ A_0 \cdot D_0^{-(k+2)} \cdot (k+1) & \text{otherwise} \end{cases} \quad (5.5.3)$$

The result is the sequence of the impulse response, here depicted:

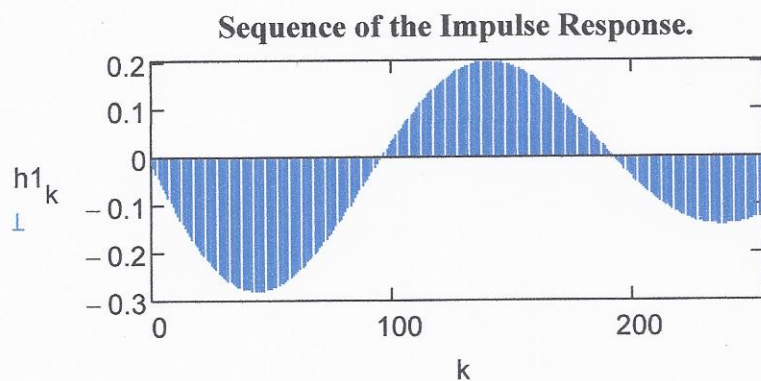


fig.:5.5.1

The Output of the Digital System is given by the **discrete convolution** between the discrete time input signal (the discrete time sequence of the sawtooth wave for this example) and the discrete impulse response of the System:

$$\underline{\nu} := 1 \dots N_0 - 1 \quad \underline{y15}_{\nu} := \sum_{k=0}^{\nu} \left( \text{if} \left( \nu - k \geq 0, h1_k \cdot \frac{u55_{\nu-k}}{V}, 0 \right) \right) \quad (5.5.4)$$



**Input Sequence.**

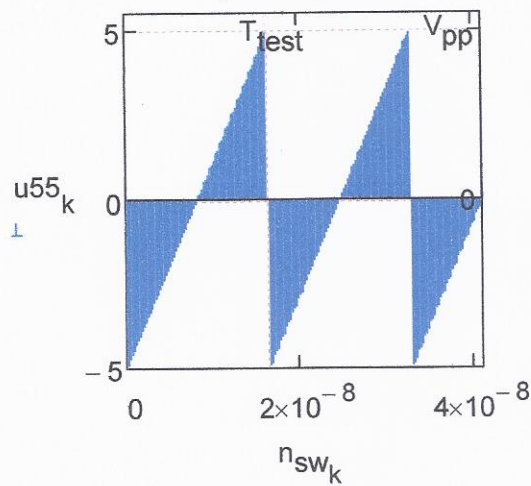
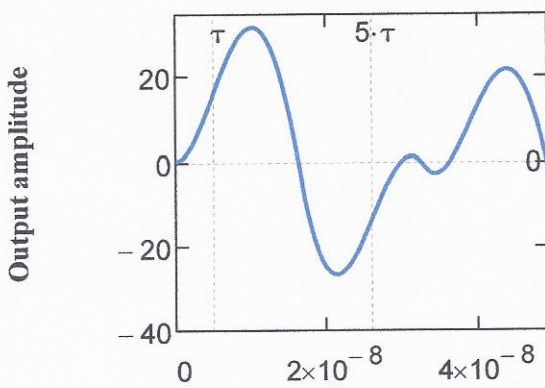


fig.:5.5.2

**Response**



time as multiple of  $\tau$

fig.:5.5.3

**Output Sequence**

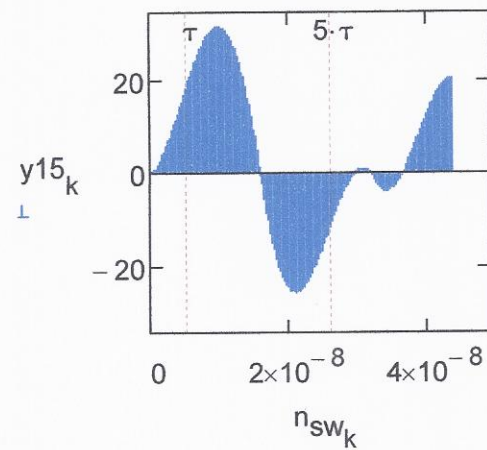


fig.:5.5.4

Knowing the sequences of any input  $u_{1\nu}$  and that of the impulse response  $h_{1\nu}$ , one can obtain

the relative Z transforms namely :  $X_{lp}(z) := \sum_{\nu=0}^{N1-1} (u_{1\nu} \cdot z^{-\nu})$  and  $H_{lps}(z) := \sum_{\nu=0}^{N0-1} (h_{1\nu} \cdot z^{-\nu})$ . The

**z-inverse transform of the product**  $Y_{lp}(z) := H_{lps}(z) \cdot X_{lp}(z)$ , which corresponds to the convolution of the two given sequences  $u_{1\nu}$ ,  $h_{1\nu}$ , provides the sought output.

**Example 1: voltage ramp as input::**

Since the t. f. is already known, you just have to calculate the input's Z transform as follows:

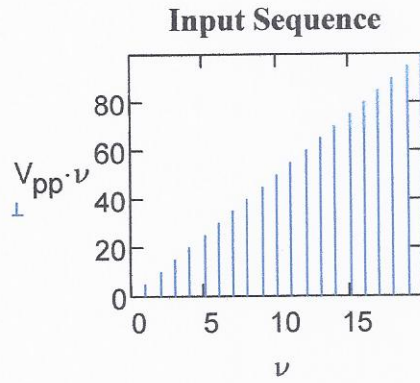


fig.:5.5.5

$$V_{pp} := V_{pp}, \nu := \nu, \quad V_{pp} \cdot \nu \text{ ztrans, } \nu \rightarrow \frac{V_{pp} \cdot z}{(z-1)^2}$$

$$\text{results: } \mathcal{Z}(V_{pp} \cdot \nu) = \frac{V_{pp} \cdot z}{(z-1)^2} \quad (5.5.5)$$

Output sequence corresponding to the z-inverse transform of the product  $H(z)V_i(z)$ :

$$V_{pp} := V_{pp} \quad A_0 := A_0 \quad B_0 := B_0 \quad C_0 := C_0 \quad z := z$$

**First case:**  $\zeta \neq \omega_5$

$$y_{16k} := \frac{A_0}{z^{-2} - B_0 \cdot z^{-1} + C_0} \cdot \frac{V_{pp} \cdot z}{(z-1)^2} \quad \left. \begin{array}{l} \text{invztrans, z, k} \\ \text{simplify, max} \rightarrow \\ \text{combine} \end{array} \right\} \blacksquare$$

To simplify place:

$$K6 := \frac{B_0 - 2}{(C_0 - B_0 + 1)^2} \quad K7 := \frac{B_0 + \sqrt{B_0^2 - 4 \cdot C_0}}{C_0}$$

$$K88 := B_0^3 - 2 \cdot B_0^2 + (1 - 3 \cdot C_0) \cdot B_0 + 4 \cdot C_0$$

$$K8 := \left[ (B_0^2 - 2 \cdot B_0 - C_0 + 1) \cdot \sqrt{B_0^2 - 4 \cdot C_0} + K88 \right]$$

$$K9 := \frac{B_0 - \sqrt{B_0^2 - 4 \cdot C_0}}{C_0}$$

$$K10 := \left[ (2 \cdot B_0 - B_0^2 + C_0 - 1) \cdot \sqrt{B_0^2 - 4 \cdot C_0} + K88 \right]$$

$$K11 := C_0 \cdot \sqrt{B_0^2 - 4 \cdot C_0} \cdot (C_0 - B_0 + 1)^2$$

Resulting sequence:  $y_{16k} := (A_0 \cdot V_{pp}) \cdot \left[ \begin{array}{l} \frac{k}{C_0 - B_0 + 1} - K6 \dots \\ + \frac{\left(\frac{1}{2}\right)^k \cdot K7^{k-1} \cdot K8}{K11} \dots \\ + (-1) \cdot \frac{\left(\frac{1}{2}\right)^k \cdot K9^{k-1} \cdot K10}{K11} \end{array} \right]$  (5.5.6)

Second case:  $\zeta = \omega_5$

$V_{pp} := V_{pp} \quad A_0 := A_0 \quad D_0 := D_0 \quad C_0 := C_0 \quad z := z$

$y_{16k} := \frac{A_0}{(D_0 - z^{-1})^2} \cdot \frac{V_{pp} \cdot z}{(z - 1)^2}$   $\left\{ \begin{array}{l} \text{invztrans, z, k} \\ \text{simplify, max} \\ \text{collect, } \left(\frac{1}{D_0}\right)^k \rightarrow \\ \text{collect, } A_0 \cdot V_{pp} \end{array} \right.$

$y_{16k} := \left[ \frac{\left(\frac{1}{D_0}\right)^k \cdot (D_0 - 1) \cdot k + 2 \cdot D_0}{D_0} - [-k \cdot (D_0 - 1) + 2] \right] \cdot \frac{(A_0 \cdot V_{pp})}{(D_0 - 1)^3 \cdot V}$  (5.5.7)

General result:

$y_{o_k} :=$	$(A_0 \cdot V_{pp}) \cdot \frac{k}{C_0 - B_0 + 1} + \frac{\left(\frac{1}{2}\right)^k \cdot (K7^{k-1} \cdot K8 - K9^{k-1} \cdot K10)}{K11} - K6$ if $\zeta \neq \omega_5$	(5.5.8)
	$\left[ \frac{\left(\frac{1}{D_0}\right)^k \cdot (D_0 - 1) \cdot k + 2 \cdot D_0}{D_0} - [-k \cdot (D_0 - 1) + 2] \right] \cdot \frac{A_0 \cdot V_{pp}}{(D_0 - 1)^3}$ otherwise	

$\zeta = 0.018 \cdot \frac{\text{Grads}}{\text{sec}}$

$\omega_5 = 0.192 \cdot \frac{\text{Grads}}{\text{sec}}$

$Q_5 = 5.4$

$Y_{o_k} := \text{if}(-V_{\text{sat}} \leq y_{o_k} \leq V_{\text{sat}}, y_{o_k}, \text{if}(y_{o_k} \leq 0.0 \cdot V, -V_{\text{sat}}, V_{\text{sat}}))$  (5.5.9)

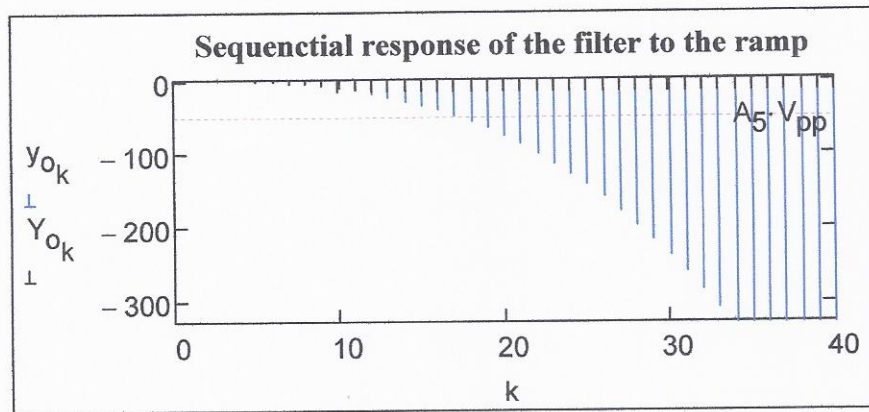


fig.:5.5.6

**Example 2:** Sinusoidal input:

$$\Delta T := \frac{2 \cdot T_{\text{test}}}{N_0 - 1} \quad \Delta T = 0.129 \cdot \text{ns}$$

Z transform of the input signal:  $\omega := \omega_{\text{test}} \quad \omega = 0.383 \cdot \frac{\text{Grads}}{\text{sec}}$

$$\Delta T := \Delta T \quad \omega := \omega \quad V_{\text{pp}} = 5V$$

$$v := v \quad V_{\text{pp}} := V_{\text{pp}} \quad V_{\text{pp}} \cdot \sin(\omega \cdot v \cdot \Delta T) \text{ ztrans, } v \rightarrow \frac{V_{\text{pp}} \cdot z \cdot \sin(\omega \cdot \Delta T)}{z^2 - 2 \cdot \cos(\omega \cdot \Delta T) \cdot z + 1}$$

Place:  $K_2 := \sin(\Delta T \cdot \omega) \quad \cos(\Delta T \cdot \omega) = \sqrt{1 - K_2^2} \quad \sqrt{1 - K_2^2} = 0.999$

$$K_2 := K_2 \quad \text{poles1} := 1 - 2 \cdot \sqrt{1 - K_2^2} \cdot z^{-1} + z^{-2} \text{ solve, } z \rightarrow \begin{pmatrix} \sqrt{1 - K_2^2} + K_2 \cdot j \\ \sqrt{1 - K_2^2} - K_2 \cdot j \end{pmatrix}$$

$$p1 := \text{poles1}_0 \quad p1 = 0.999 + 0.049j$$

$$p2 := \text{poles1}_1 \quad p2 = 0.999 - 0.049j$$

$$\frac{V_{\text{pp}} \cdot z \cdot K_2}{z^2 - 2 \cdot \sqrt{1 - K_2^2} \cdot z + 1} = \frac{K_2 \cdot V_{\text{pp}} \cdot z}{(p1 - z) \cdot (p1 - \overline{p1})} - \frac{K_2 \cdot V_{\text{pp}} \cdot z}{(p1 - \overline{p1}) \cdot (z - p1)}$$

$$\frac{K_2 \cdot V_{\text{pp}} \cdot z}{(p1 - z) \cdot (p1 - \overline{p1})} = \left[ 1 + \frac{p1}{(z - p1)} \right] \cdot \frac{(K_2 \cdot V_{\text{pp}})}{(p1 - \overline{p1})}$$

$$\frac{K_2 \cdot V_{\text{pp}} \cdot z}{(p1 - \overline{p1}) \cdot (z - p1)} = - \left[ 1 + \frac{\overline{p1}}{(z - p1)} \right] \cdot \frac{K_2 \cdot V_{\text{pp}}}{p1 - p1}$$

$$\frac{A_0}{z^{-2} - B_0 \cdot z^{-1} + C_0} = \frac{A_0}{\left( z - \frac{B_0 + \sqrt{B_0^2 - 4 \cdot C_0}}{2 \cdot C_0} \right) \cdot \left( z - \frac{B_0 - \sqrt{B_0^2 - 4 \cdot C_0}}{2 \cdot C_0} \right)}$$



$$q1 := \frac{B_0 + \sqrt{B_0^2 - 4 \cdot C_0}}{2 \cdot C_0} \quad q2 := \frac{B_0 - \sqrt{B_0^2 - 4 \cdot C_0}}{2 \cdot C_0}$$

$$q1 = 0.996 + 0.032j \quad q2 = 0.996 - 0.032j$$

$$\frac{A_0}{(z - q1) \cdot (z - q2)} = \left[ \frac{1}{(z - q1)} - \frac{1}{(z - q2)} \right] \cdot \frac{A_0}{(q1 - q2)}$$

**First case:**  $\zeta \neq \omega_5$

$$y17_k := \frac{A_0 \cdot K2 \cdot V_{pp}}{(q1 - q2) \cdot (p1 - \overline{p1})} \cdot \left[ \frac{1}{(z - q1)} - \frac{1}{(z - q2)} \right] \cdot \left[ \begin{array}{l} \frac{z}{(p1 - z)} \dots \\ + \frac{-z}{(z0 - p1)} \end{array} \right] \left| \begin{array}{l} \text{invztrans, z, using, } \nu = k \\ \text{simplify} \end{array} \right. \rightarrow$$

Computing the corresponding sequence the result returned for the symbolic operation is too large to be displayed.

$$q1 := q1 \quad p1 := p1 \quad z := z$$

$$\left[ \frac{1}{(z - q1)} \cdot \frac{z}{(p1 - z)} \right] \text{invztrans} \rightarrow -\frac{p1^n - q1^n}{p1 - q1}$$

$$q1 := q1 \quad p1 := p1 \quad z := z$$

$$\left[ \frac{1}{(z - q1)} \cdot \frac{-z}{(z - \overline{p1})} \right] \text{invztrans} \rightarrow \frac{(\overline{p1})^n - q1^n}{q1 - \overline{p1}}$$

$$q2 := q2 \quad p1 := p1 \quad z := z \quad n := n \quad k := k$$

$$\left[ \frac{1}{(z - q2)} \cdot \frac{z}{(p1 - z)} \right] \text{invztrans} \rightarrow -\frac{p1^n - q2^n}{p1 - q2}$$

$$q2 := q2 \quad p1 := p1 \quad z := z$$

$$\left[ \frac{1}{(z - q2)} \cdot \frac{z}{(z - \overline{p1})} \right] \text{invztrans} \rightarrow -\frac{(\overline{p1})^n - q2^n}{q2 - \overline{p1}}$$

$$y17_k := \frac{A_0 \cdot K2 \cdot V_{pp}}{(q1 - q2) \cdot (p1 - \overline{p1})} \cdot \left[ \frac{p1^k - q1^k}{q1 - p1} + \frac{(\overline{p1})^k - q1^k}{q1 - \overline{p1}} - \frac{p1^k - q2^k}{q2 - p1} - \frac{(\overline{p1})^k - q2^k}{q2 - \overline{p1}} \right]$$

Proof

**Proof:** First case:  $\zeta \neq \omega_5$

$$q1 := q1 \quad q2 := q2 \quad p1 := p1 \quad \overline{p1} := \overline{p1}$$

$$\frac{p1^k - q1^k}{q1 - p1} + \frac{(\overline{p1})^k - q1^k}{q1 - \overline{p1}} - \frac{p1^k - q2^k}{q2 - p1} - \frac{(\overline{p1})^k - q2^k}{q2 - \overline{p1}} \text{ ztrans, k} \rightarrow \frac{z \cdot (q1 - q2) \cdot (p1 - 2 \cdot z + \overline{p1})}{(p1 - z) \cdot (q1 - z) \cdot (q2 - z) \cdot (z - \overline{p1})}$$

$$z \left[ \frac{p1^k - q1^k}{q1 - p1} + \frac{(\overline{p1})^k - q1^k}{q1 - \overline{p1}} - \frac{p1^k - q2^k}{q2 - p1} - \frac{(\overline{p1})^k - q2^k}{q2 - \overline{p1}} \right] = \frac{z \cdot (q1 - q2) \cdot (p1 - 2 \cdot z + \overline{p1})}{(p1 - z) \cdot (q1 - z) \cdot (q2 - z) \cdot (z - \overline{p1})}$$

$$\left[ \frac{z \cdot (q1 - q2) \cdot (p1 - 2 \cdot z + \overline{p1})}{(p1 - z) \cdot (q1 - z) \cdot (q2 - z) \cdot (z - \overline{p1})} \right] \left| \begin{array}{l} \text{parfrac, z} \\ \text{simplify} \end{array} \right. \rightarrow \frac{z \cdot (q1 - q2) \cdot (p1 - 2 \cdot z + \overline{p1})}{(p1 - z) \cdot (q1 - z) \cdot (q2 - z) \cdot (z - \overline{p1})}$$

$$z \left[ \frac{p1^k - q1^k}{q1 - p1} + \frac{(\overline{p1})^k - q1^k}{q1 - \overline{p1}} - \frac{p1^k - q2^k}{q2 - p1} - \frac{(\overline{p1})^k - q2^k}{q2 - \overline{p1}} \right] = \frac{\overline{p1} \cdot (q1 - q2)}{-(q1 - \overline{p1}) \cdot (q2 - \overline{p1}) \cdot (z - \overline{p1})} \dots$$

$$+ \frac{p1 \cdot (q1 - q2) \cdot (p1 - \overline{p1})}{(p1 - q1) \cdot (p1 - q2) \cdot (p1 - z) \cdot (p1 - \overline{p1})} \dots$$

$$+ (-1) \cdot \frac{q1 \cdot (p1 - 2 \cdot q1 + p1)}{-(p1 - q1) \cdot (q1 - z) \cdot (q1 - \overline{p1})} \dots$$

$$+ (-1) \cdot \frac{q2 \cdot (q1 - q2) \cdot (p1 - 2 \cdot q2 + p1)}{(p1 - q2) \cdot (q1 - q2) \cdot (q2 - z) \cdot (q2 - \overline{p1})}$$

▣ Proof

**Second case case:**  $\zeta = \omega_5$

$$y17_k := \frac{A_0 \cdot K2 \cdot V_{pp}}{(p1_0 - \overline{p1_0})} \cdot \frac{1}{(D_0 - z^{-1})^2} \left[ \begin{array}{l} \frac{z}{(p1_0 - z)} \dots \\ + \frac{-z}{(z - \overline{p1_0})} \end{array} \right] \left| \begin{array}{l} \text{invztrans, z, using, n = k} \\ \text{simplify} \end{array} \right. \rightarrow$$

Computing the corresponding sequence the result returned for the symbolic operation is too large to be displayed.

$$\frac{1}{(D_0 - z^{-1})^2} \cdot \frac{z}{(p1_0 - z)} \text{ invztrans, using, n = k} \rightarrow$$

$$\frac{1}{(D_0 - z^{-1})^2} \cdot \frac{-z}{(z - \overline{p1_0})} \text{ invztrans, using, n = k} \rightarrow$$

$$y_{17k} = \frac{A_0 \cdot K_2 \cdot V_{pp}}{(p_1 - \overline{p_1})} \cdot \left[ \begin{aligned} & \frac{D_0 \cdot (k+2) \cdot p_1 - k - 1}{D_0^2 \cdot (D_0 \cdot p_1 - 1)^2} \cdot \left(\frac{1}{D_0}\right)^k - \frac{(p_1)^{k+2}}{(D_0 \cdot p_1 - 1)^2} \dots \\ & + \frac{\overline{p_1} \cdot (k+2) \cdot D_0 - (k+1)}{D_0^2 \cdot (D_0 \cdot \overline{p_1} - 1)^2} \cdot \left(\frac{1}{D_0}\right)^k - \frac{(\overline{p_1})^{k+2}}{(D_0 \cdot \overline{p_1} - 1)^2} \end{aligned} \right]$$

▣ Proof

**Proof: Second case:**  $\zeta = \omega_5$

$$\frac{A_0 \cdot K_2 \cdot V_{pp}}{(p_1 - \overline{p_1})} \cdot \left[ \begin{aligned} & \frac{D_0 \cdot (k+2) \cdot p_1 - k - 1}{D_0^2 \cdot (D_0 \cdot p_1 - 1)^2} \cdot \left(\frac{1}{D_0}\right)^k - \frac{(p_1)^{k+2}}{(D_0 \cdot p_1 - 1)^2} \dots \\ & + \frac{\overline{p_1} \cdot (k+2) \cdot D_0 - (k+1)}{D_0^2 \cdot (D_0 \cdot \overline{p_1} - 1)^2} \cdot \left(\frac{1}{D_0}\right)^k - \frac{(\overline{p_1})^{k+2}}{(D_0 \cdot \overline{p_1} - 1)^2} \end{aligned} \right] \xrightarrow{\text{ztrans, } k} - \frac{A_0 \cdot K_2 \cdot V_{pp} \cdot z^3 \cdot (p_1 - \overline{p_1})}{(D_0 \cdot z - 1)^2 \cdot (p_1 - z) \cdot (p_1 - \overline{p_1})}$$

$$z \left[ \frac{A_0 \cdot K_2 \cdot V_{pp}}{(p_{10} - \overline{p_{10}})} \cdot \left[ \begin{aligned} & \frac{D_0 \cdot (k+2) \cdot p_{10} - k - 1}{D_0^2 \cdot (D_0 \cdot p_{10} - 1)^2} \cdot \left(\frac{1}{D_0}\right)^k - \frac{(p_{10})^{k+2}}{(D_0 \cdot p_{10} - 1)^2} \dots \\ & + \frac{\overline{p_{10}} \cdot (k+2) \cdot D_0 - (k+1)}{D_0^2 \cdot (D_0 \cdot \overline{p_{10}} - 1)^2} \cdot \left(\frac{1}{D_0}\right)^k - \frac{(\overline{p_{10}})^{k+2}}{(D_0 \cdot \overline{p_{10}} - 1)^2} \end{aligned} \right] \right] = \frac{A_0 \cdot K_2 \cdot V_{pp} \cdot z^3 \cdot (p_1 - 2 \cdot z)}{(D_0 \cdot z - 1)^2 \cdot (p_1 - z) \cdot (p_1 - \overline{p_1})}$$

▣ Proof

**General result:**

$$y_{17k} := \frac{A_0 \cdot K_2 \cdot V_{pp}}{(p_1 - \overline{p_1})} \cdot \left[ \begin{aligned} & \frac{1}{(q_1 - q_2)} \cdot \left[ \begin{aligned} & \frac{(p_1)^k - q_1^k}{q_1 - p_1} \dots \\ & + \frac{(\overline{p_1})^k - q_1^k}{q_1 - \overline{p_1}} - \frac{(p_1)^k - q_2^k}{q_2 - p_1} - \frac{(\overline{p_1})^k - q_2^k}{q_2 - \overline{p_1}} \end{aligned} \right] \quad \text{if } \zeta \neq \omega_5 \\ & \left[ \begin{aligned} & \frac{D_0 \cdot (k+2) \cdot p_1 - k - 1}{D_0^2 \cdot (D_0 \cdot p_1 - 1)^2} \cdot \left(\frac{1}{D_0}\right)^k - \frac{(p_1)^{k+2}}{(D_0 \cdot p_1 - 1)^2} \dots \\ & + \frac{\overline{p_1} \cdot (k+2) \cdot D_0 - (k+1)}{D_0^2 \cdot (D_0 \cdot \overline{p_1} - 1)^2} \cdot \left(\frac{1}{D_0}\right)^k - \frac{(\overline{p_1})^{k+2}}{(D_0 \cdot \overline{p_1} - 1)^2} \end{aligned} \right] \quad \text{otherwise} \end{aligned} \right]$$

(5.5.10)



$$y_{17}^T = \begin{array}{c|cccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline 0 & 0 & 0 & -0.054j & -0.16j & -0.319j & -0.53j & -0.79j & -1.101j & \dots \end{array}$$

Sequence of the sinusoidal response compared with the input

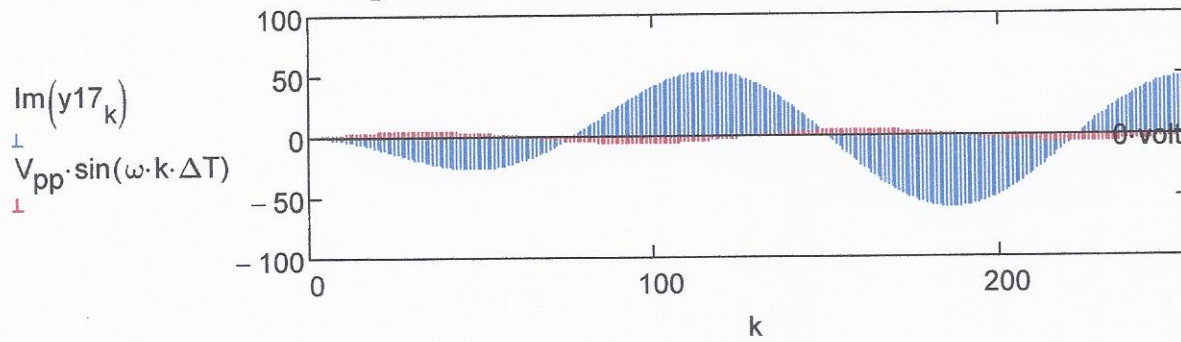


fig.:5.5.7

## Bode plots:

$$H_{lp}(z) := \begin{cases} \frac{A_0}{z^{-2} - B_0 \cdot z^{-1} + C_0} & \text{if } \zeta \neq \omega_5 \\ \frac{A_0}{(D_0 - z^{-1})^2} & \text{otherwise} \end{cases} \quad (5.5.11)$$

$$1.1 \cdot \frac{A_0 \cdot K_2 \cdot V_{pp}}{(p_1 - \bar{p}_1)} = 0.029jV$$

$$A_5 = -10 \quad 20 \cdot \log(|W_{lp}(j \cdot \sqrt{\omega_5 \cdot \omega_{smp}})|) = -19.036$$

$$H_{lpdB}(\omega) := 20 \cdot \log(|H_{lp}(e^{j \cdot \omega \cdot T_s})|)$$

$$\omega := \frac{\omega_5}{U \cdot 10^{10}}, \frac{\omega_5}{U \cdot 10^{10}} + \frac{\omega_5 \cdot 10 \cdot U - \frac{\omega_5}{U \cdot 10^{10}}}{4 \cdot U^2} \dots 10 \cdot U \cdot \omega_5$$



BODE Plots of  $H(z)$  compared with that of  $W(j\omega)$

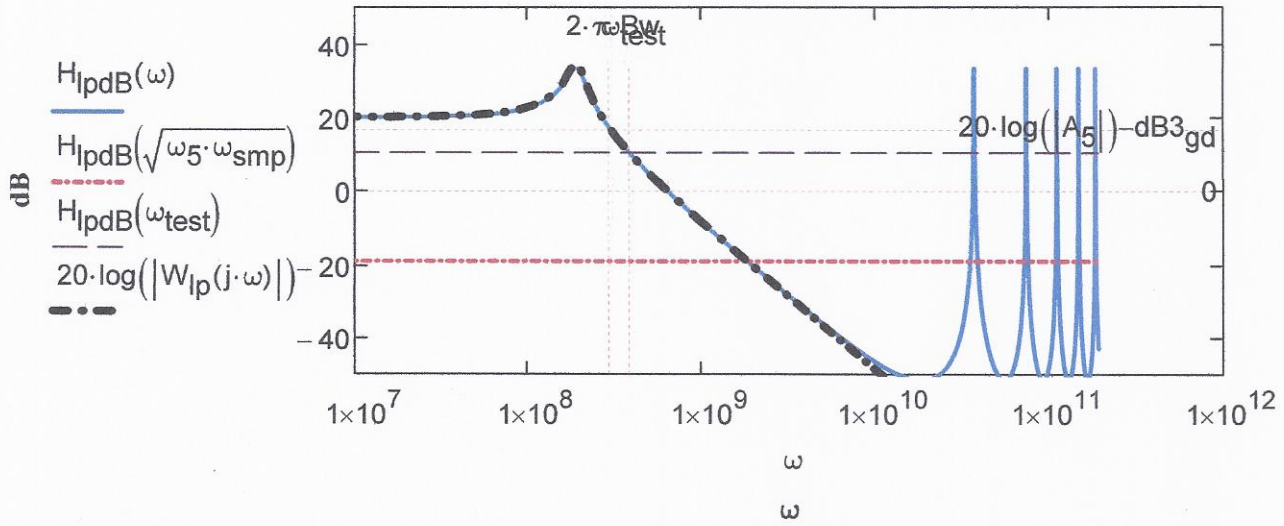


fig.:5.5.8

## 5.6

### The bilinear transformation

#### 5.6.1 Z-transfer function of the II° Order Low Pass Digital Filter.

On the other hand, using the bilinear transformation:  $s = \frac{2}{T_s} \cdot \left( \frac{1-z^{-1}}{1+z^{-1}} \right)$ , (5.6.1)

we get a new t. f. for the given system, in the z domain.

$$A_5 := A_5 \quad \omega_5 := \omega_5 \quad \zeta := \zeta \quad T_s := T_s \quad \omega_5 = 0.192 \cdot \frac{\text{Grads}}{\text{sec}}$$

$$H_4(z) := \frac{A_5 \cdot \omega_5^2}{s^2 + 2 \cdot \zeta \cdot s + \omega_5^2} \left| \begin{array}{l} \text{substitute, } s = \frac{2}{T_s} \cdot \left( \frac{1-z^{-1}}{1+z^{-1}} \right) \\ \text{collect, } z \\ \text{collect, } A_5 \cdot (T_s \cdot \omega_5)^2 \end{array} \right. \rightarrow \quad (5.6.2)$$

$$H_4(z) = \frac{(z^{-1} + 1)^2}{z^{-2} + z^{-1} \cdot \frac{2 \cdot T_s^2 \cdot \omega_5^2 - 8}{T_s^2 \cdot \omega_5^2 - 4 \cdot \zeta \cdot T_s + 4} + \frac{4 \cdot T_s \cdot \zeta + T_s^2 \cdot \omega_5^2 + 4}{T_s^2 \cdot \omega_5^2 - 4 \cdot \zeta \cdot T_s + 4}} \cdot \frac{A_5 \cdot T_s^2 \cdot \omega_5^2}{T_s^2 \cdot \omega_5^2 - 4 \cdot \zeta \cdot T_s + 4}$$

$$H_4(z) := \frac{(z^{-1} + 1)^2}{z^{-2} + z^{-1} \cdot \frac{2 \cdot T_s^2 \cdot \omega_5^2 - 8}{T_s^2 \cdot \omega_5^2 - 4 \cdot \zeta \cdot T_s + 4} \dots} \cdot \frac{A_5 \cdot T_s^2 \cdot \omega_5^2}{T_s^2 \cdot \omega_5^2 - 4 \cdot \zeta \cdot T_s + 4} \quad (5.6.3)$$

$$+ \frac{4 \cdot T_s \cdot \zeta + T_s^2 \cdot \omega_5^2 + 4}{T_s^2 \cdot \omega_5^2 - 4 \cdot \zeta \cdot T_s + 4}$$

Parameters necessary for the design of the digital filter:

Consider the sampling time  $T_s = 1.708 \times 10^{-4} \cdot \mu\text{s}$

$$\alpha_1 := \frac{A_5 \cdot \omega_5^2 \cdot T_s^2}{T_s^2 \cdot \omega_5^2 - 4 \cdot \zeta \cdot T_s + 4} \quad \beta_1 := \frac{2 \cdot T_s^2 \cdot \omega_5^2 - 8}{T_s^2 \cdot \omega_5^2 - 4 \cdot \zeta \cdot T_s + 4} \quad \gamma_1 := \frac{T_s^2 \cdot \omega_5^2 + 4 \cdot \zeta \cdot T_s + 4}{T_s^2 \cdot \omega_5^2 - 4 \cdot \zeta \cdot T_s + 4}$$

$$\alpha_1 = -0.0026847177$$

$$\beta_1 = -2.005003071$$

$$\gamma_1 = 1.0060769577$$

result for the t. f. as a function of  $z^{-1}$ :

$$\omega_{\text{smp}} = 17.338 \cdot \frac{\text{Grads}}{\text{sec}}$$

$$\omega_5 = 0.192 \cdot \frac{\text{Grads}}{\text{sec}}$$

$$H_4(z) := \alpha_1 \cdot \frac{(z^{-1} + 1)^2}{z^{-2} + \beta_1 \cdot z^{-1} + \gamma_1} \quad (5.6.4)$$

If  $\zeta = \omega_5$

$$x = z^{-1} \quad H_{lp}(x) := A_5 \cdot \frac{\omega_5^2}{(s + \omega_5)^2} \quad \left\{ \begin{array}{l} \text{substitute, } s = \frac{2}{T_s} \cdot \left( \frac{1-x}{1+x} \right) \\ \text{collect, } x \\ \text{factor} \end{array} \right. \rightarrow$$

$$\text{z transfer function: } H_{lp}(z) = \frac{(A_5 \cdot T_s^2 \cdot \omega_5^2)}{(T_s \cdot \omega_5 - 2)^2} \cdot \frac{(z^{-1} + 1)^2}{\left[ z^{-1} + \frac{T_s \cdot \omega_5 + 2}{(T_s \cdot \omega_5 - 2)} \right]^2} \quad (5.6.5)$$

$$\alpha_{11} := \frac{(A_5 \cdot T_s^2 \cdot \omega_5^2)}{(T_s \cdot \omega_5 - 2)^2} \quad \beta_{22} := \frac{T_s \cdot \omega_5 + 2}{(T_s \cdot \omega_5 - 2)}$$

$$H_4(z) := \left[ \begin{array}{l} \alpha_1 \cdot \frac{(z^{-1} + 1)^2}{z^{-2} + \beta_1 \cdot z^{-1} + \gamma_1} \quad \text{if } \zeta \neq \omega_5 \\ \alpha_{11} \cdot \frac{(z^{-1} + 1)^2}{(z^{-1} + \beta_{22})^2} \quad \text{otherwise} \end{array} \right] \quad (5.6.6)$$

Now compare the Bode diagrams of  $H_4(z)$  with those of  $H_{lp}(z)$ . To do that, the coefficients of  $H_{lp}(z)$  must be redefined with the given sampling time  $T_s$ . Therefore they are rewritten here below:

$$\underline{A0} := A_5 \cdot \omega_5^2 \cdot T_s^2 \quad \underline{B0} := 2 \cdot (1 + \zeta \cdot T_s) \quad \underline{C0} := T_s \cdot (\omega_5^2 \cdot T_s + 2 \cdot \zeta) + 1 \quad \underline{D0} := T_s \cdot \omega_5 + 1$$

$$A0 = -0.01070921$$

$$B0 = 2.006060171$$

$$C0 = 1.0071310916$$

$$D0 = 1.033$$

$$H_{lp}(z) := \left[ \begin{array}{l} \frac{A0}{z^{-2} - B0 \cdot z^{-1} + C0} \quad \text{if } \zeta \neq \omega_5 \\ \frac{A0}{(D0 - z^{-1})^2} \quad \text{otherwise} \end{array} \right] \quad (5.6.7)$$

# BODE PLOTS (Low Pass (II° order)):

$$\omega := \frac{\omega_5}{20 \cdot U}, \frac{\omega_5}{20 \cdot U} + \frac{\omega_5 \cdot U - \frac{\omega_5}{20 \cdot U}}{U^2} \dots 10 \cdot U \cdot \omega_5$$

Frequency response of the II° order Low Pass Filter

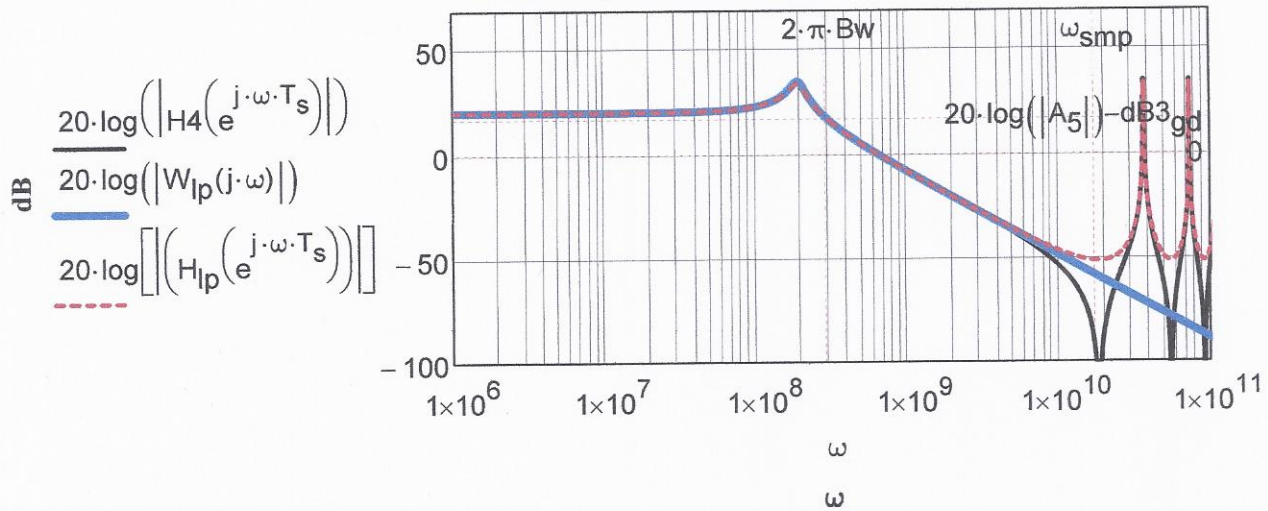


fig.:5.6.1

Phase Response of the II° order Low Pass Filter

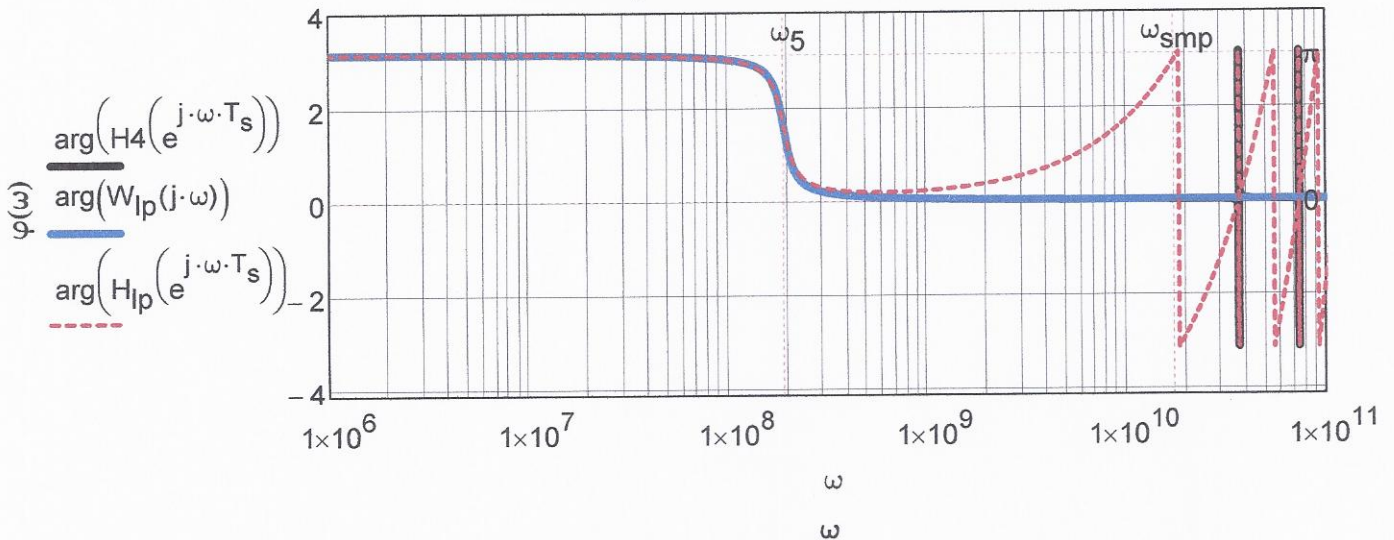


fig.:5.6.2



## 5.6 The bilinear transformation

### 5.6.2 Difference equations (II<sup>o</sup> order Low Pass filter). Canonical form.

$$H_4(z) = \alpha_1 \cdot \frac{(z^{-1} + 1)^2}{z^{-2} + \beta_1 \cdot z^{-1} + \gamma_1} \quad (5.6.2.1)$$

$$H_4(z) = \frac{Y_2(z)}{X(z)} = \frac{Y_2(z)}{W_1(z)} \cdot \frac{W_1(z)}{X(z)} \quad (5.6.2.2)$$

$$\frac{Y_2(z)}{W_1(z)} = \alpha_1 \cdot (1 + 2 \cdot z^{-1} + z^{-2}) \quad (5.6.2.3)$$

$$Y_2(z) = \alpha_1 \cdot (1 + 2 \cdot z^{-1} + z^{-2}) \cdot W_1(z) \quad (5.6.2.4)$$

$$\boxed{y_2(\nu) = \alpha_1 \cdot ((w_1(\nu) + 2 \cdot w_1(\nu - 1) + w_1(\nu - 2)))} \quad (5.6.2.5)$$

$$\frac{W_1(z)}{X(z)} = \frac{1}{z^{-2} + \beta_1 \cdot z^{-1} + \gamma_1} \quad (5.6.2.6)$$

$$X(z) = (z^{-2} + \beta_1 \cdot z^{-1} + \gamma_1) \cdot W_1(z) \quad (5.6.2.7)$$

$$X(z) = \gamma_1 \cdot W_1(z) + \beta_1 \cdot z^{-1} \cdot W_1(z) + z^{-2} \cdot W_1(z) \quad (5.6.2.8)$$

$$\boxed{x(\nu) = \gamma_1 \cdot w_1(\nu) + \beta_1 \cdot w_1(\nu - 1) + w_1(\nu - 2)} \quad (5.6.2.9)$$

The corresponding set of difference equations is:

$$w_1(\nu) = \frac{x(\nu)}{\gamma_1} - \frac{\beta_1}{\gamma_1} \cdot w_1(\nu - 1) - \frac{1}{\gamma_1} \cdot w_1(\nu - 2) \quad (5.6.2.10)$$

$$y_2(\nu) = \alpha_1 \cdot (w_1(\nu) + 2 \cdot w_1(\nu - 1) + w_1(\nu - 2)) \quad (5.6.2.11)$$

$$\alpha_1 := \alpha_1 \quad \beta_1 := \beta_1 \quad \gamma_1 := \gamma_1$$

Z T. Initial value theorem:  $\lim_{z \rightarrow \infty} \left( \alpha_1 \cdot \frac{1 + 2 \cdot z^{-1} + z^{-2}}{z^{-2} + \beta_1 \cdot z^{-1} + \gamma_1} \right) \rightarrow \frac{\alpha_1}{\gamma_1}$

Z T. Final value theorem:  $\lim_{z \rightarrow 0} \left( \alpha_1 \cdot \frac{1 + 2 \cdot z^{-1} + z^{-2}}{z^{-2} + \beta_1 \cdot z^{-1} + \gamma_1} \right) \rightarrow \alpha_1$

## 5.6 The bilinear transformation

### 5.6.2.1 Sequence of the voltage Step response.

Recurrence relations:

$$v_i(\nu) := u1_\nu \quad (5.6.2.1.1)$$

$$1) \quad w1(\nu) := \begin{cases} \frac{v_i(\nu) - \beta_1 \cdot w1(\nu - 1) - w1(\nu - 2)}{\gamma_1} & \text{if } \nu > 1 \\ \frac{\alpha_1}{\gamma_1} \cdot v_i(\nu) & \text{if } \nu = 0 \\ \left( \frac{v_i(1) - \beta_1 \cdot w1(0)}{\gamma_1} \right) & \text{if } \nu = 1 \end{cases} \quad (5.6.2.1.2)$$

$$2) \quad y2(\nu) := \begin{cases} \alpha_1 \cdot ((w1(\nu) + 2 \cdot w1(\nu - 1) + w1(\nu - 2))) & \text{if } \nu > 1 \\ \alpha_1 \cdot ((w1(1) + 2 \cdot w1(0))) & \text{if } \nu = 1 \\ \alpha_1 \cdot w1(0) & \text{otherwise} \end{cases} \quad (5.6.2.1.3)$$

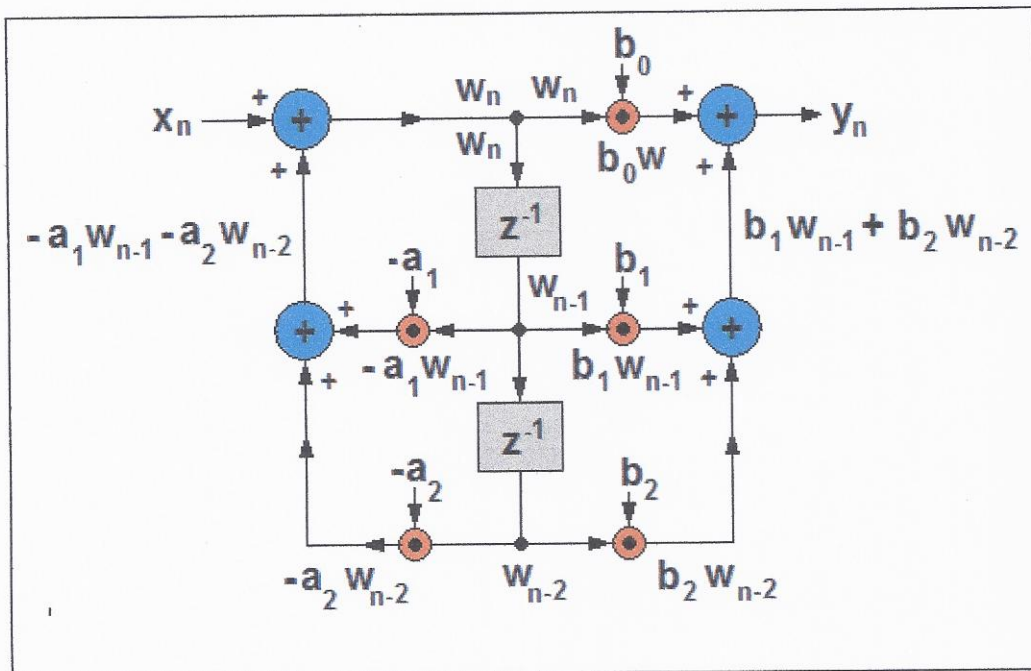
Same relations but vectorized:  $W1_\nu := 0.0$

$$1) \quad W1_\nu := \begin{cases} \frac{\frac{u1_\nu}{\text{volt}} - \beta_1 \cdot W1_{\nu-1} - W1_{\nu-2}}{\gamma_1} & \text{if } \nu > 1 \\ \frac{1}{\gamma_1} \cdot \frac{u1_\nu}{\text{volt}} & \text{if } \nu = 0 \\ \frac{\frac{u1_1}{\text{volt}} - \beta_1 \cdot W1_0}{\gamma_1} & \text{otherwise} \end{cases} \quad (5.6.2.1.4)$$

$$2) \quad Y2_\nu := \begin{cases} \alpha_1 \cdot (W1_\nu + 2 \cdot W1_{\nu-1} + W1_{\nu-2}) & \text{if } \nu > 1 \\ \alpha_1 \cdot W1_0 & \text{if } \nu = 0 \\ \alpha_1 \cdot (W1_1 + 2 \cdot W1_0) & \text{otherwise} \end{cases} \quad (5.6.2.1.5)$$

► BILINEAR

Block diagram of the difference equation algorithm for a second order system



Input signal sequence

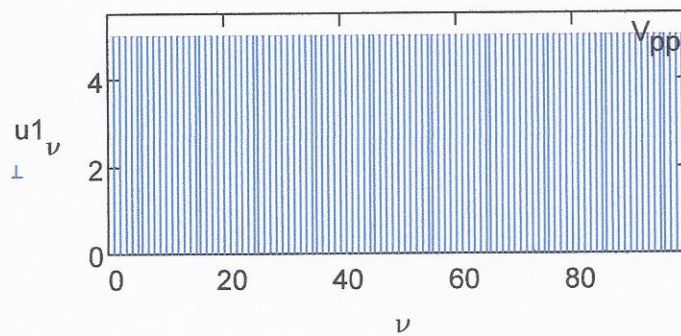


fig.:5.6.2.1.1

$$A_5 = -10 \quad b11 := \text{BILINEAR}\left(\frac{u1}{V}, A_5, \zeta, \omega_5, T_{\text{sstp}}, N0\right) \quad (5.6.2.1.6)$$

$$b11 = (-0.248 \quad -1.959 \quad 1.058 \quad \{256,1\} \quad \{256,1\})$$

$$a00 := b11_{0,0} \quad b00 := b11_{0,1} \quad c00 := b11_{0,2} \quad W00 := b11_{0,3} \quad Y11 := b11_{0,4}$$

$$a5 = -17.13472986$$

$$b5 = 2.2424068406$$

$$c5 = 2.9558798269$$

Sequence of the state function

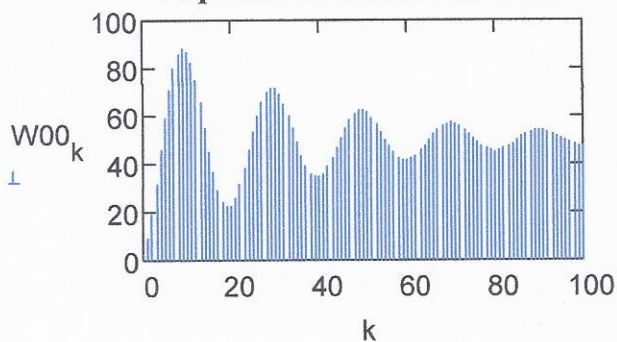


fig.:5.6.2.1.2

Output Sequence

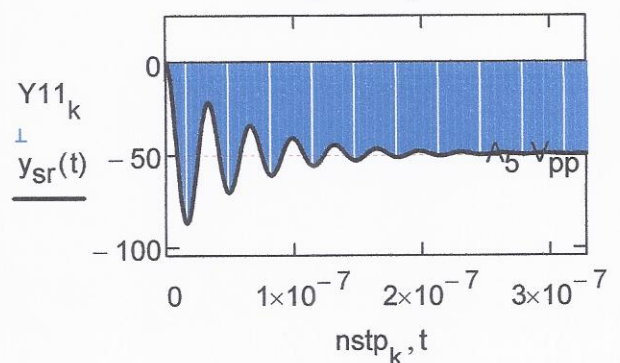


fig.:5.6.2.1.3

## 5.6 The bilinear transformation

### 5.6.2.2 Sequence of the Short Voltage Pulse response.

$$T_{\text{test}} = 16.394 \cdot \text{ns} \quad T_{\text{svp}} = 10.93 \cdot \text{ns} \quad \tau = 5.218 \times 10^{-3} \cdot \mu\text{s}$$

$$\text{Chosen Test signal period, } T_{\text{test}} = 16.394 \cdot \text{ns} \quad \frac{1}{T_{\text{test}}} = 0.061 \cdot \text{GHz}$$

Short pulse sequence of amplitude  $V_i$ :

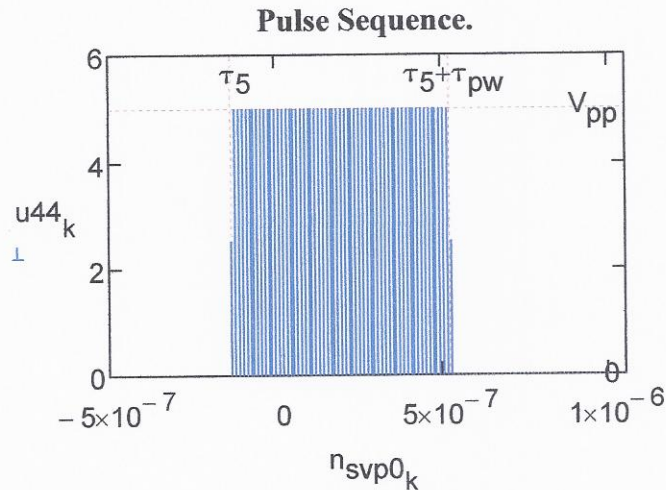


fig.:5.6.2.2.1

$$b12 := \text{BILINEAR}(u44, A_5, \zeta, \omega_5, T_{\text{svp}}, N0) \quad (5.6.2.2.1)$$

$$b12 = (-5.764 \quad 0.102 \quad 1.204 \quad \{256, 1\} \quad \{256, 1\})$$

$$a11 := b12_{0,0} \quad b11 := b12_{0,1} \quad c11 := b12_{0,2} \quad W11 := b12_{0,3} \quad Y22 := b12_{0,4}$$

$$a11 = -5.7635166 \quad b11 = 0.101563937 \quad c11 = 1.2038427013$$

Block diagram of the difference equation algorithm for a second order system



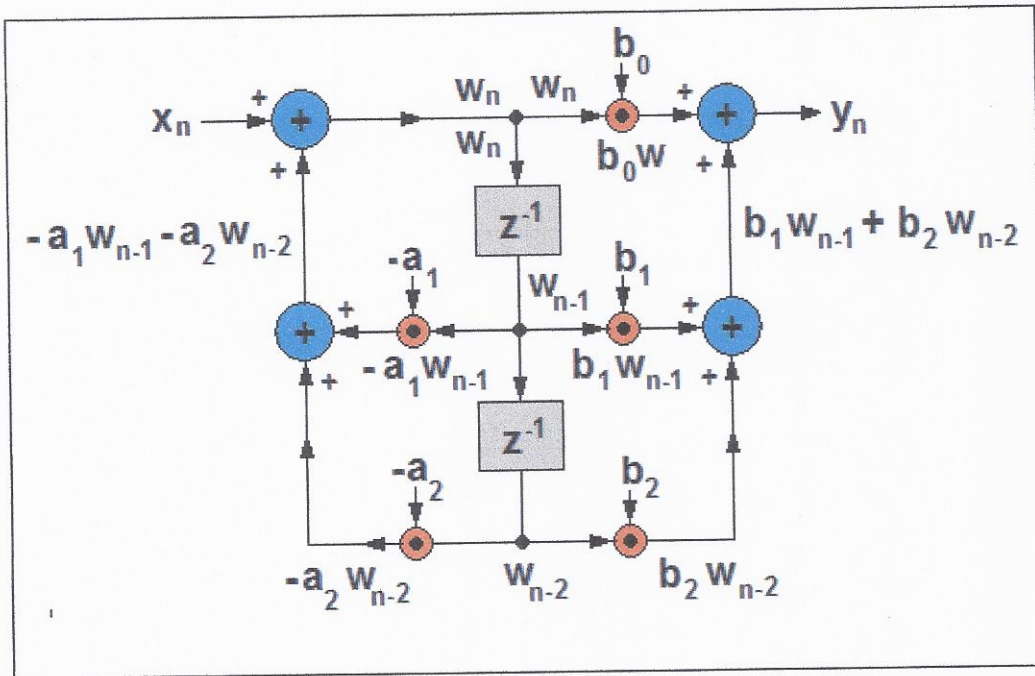


fig.:5.6.2.2.2

**Sequence of the voltage Step response.**

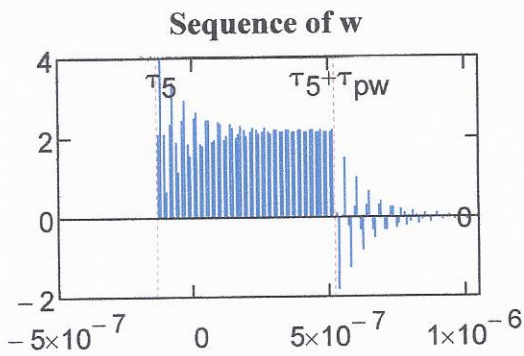


fig.:5.6.2.2.3

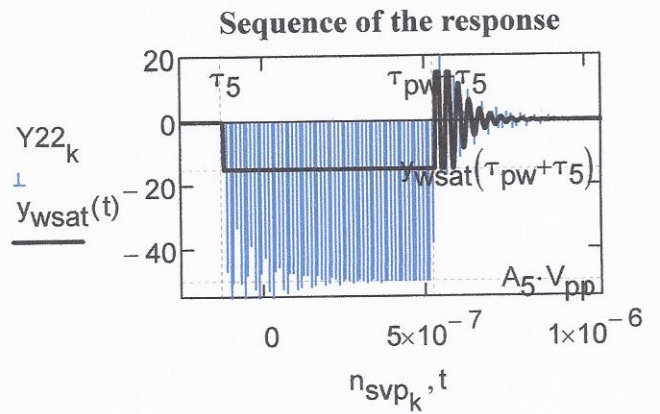


fig.:5.6.2.2.4

Spec18x := fft(Y22)

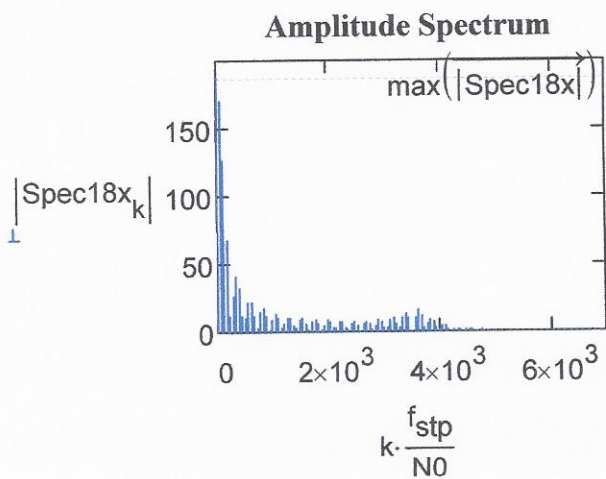


fig.:5.6.2.2.5

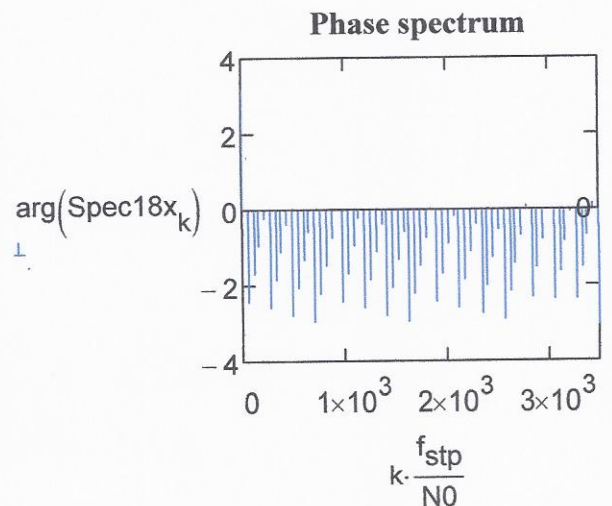
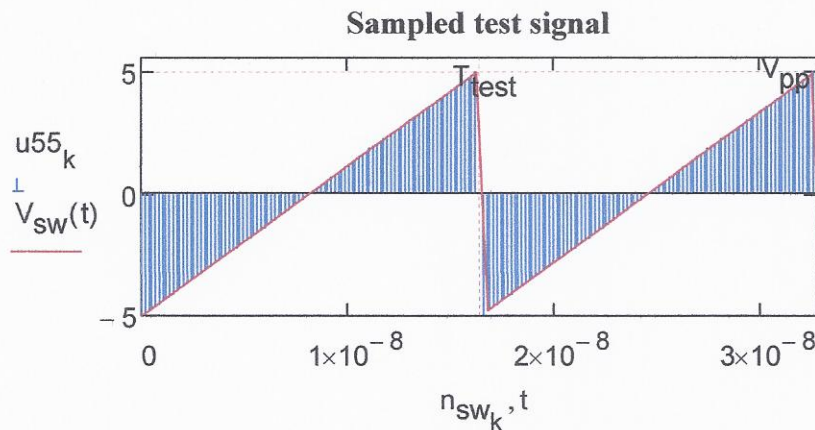


fig.:5.6.2.2.6

## 5.6 The bilinear transformation

### 5.6.2.3 Sequence of the Sawtooth Response



$$\frac{T_5}{T_{\text{test}}} = 2$$

fig.:5.6.2.3.1

$$\omega_5 = 191.627 \cdot \frac{\text{Mrads}}{\text{sec}}$$

Signal bandwidth:  $B_{\text{sw}} = 2.928 \times 10^3 \cdot \text{MHz}$

$$f_{\text{ssaw}} := 2 \cdot B_{\text{sw}}$$

$$f_{\text{ssaw}} = 5.856 \times 10^3 \cdot \text{MHz}$$

$$\omega_{\text{ssaw}} := 2 \cdot \pi \cdot f_{\text{ssaw}}$$

$$\text{Parseval}_{\text{sw}} = 16.466$$

$$\text{Average1.volt} = 0 \cdot \text{V}$$

$$\text{RMS1.volt} = 2.887 \times 10^3 \cdot \text{mV}$$

$$T_{\text{test}} = 1.639 \times 10^{-8} \text{ s}$$

$$\zeta = 0.018 \cdot \frac{\text{Grads}}{\text{sec}}$$

$$T_{\text{ssaw}} := \frac{1}{f_{\text{ssaw}}}$$

$$\omega_{\text{ssaw}} = 36.792 \cdot \frac{\text{Grads}}{\text{sec}}$$

$$\zeta \cdot T_s = 3.03 \times 10^{-3}$$

$$T_{\text{ssaw}} = 1.708 \times 10^{-10} \text{ s} \quad A_5 = -10$$

$$Q_5 = 5.4$$

$$T_5 = 0.033 \cdot \mu\text{s}$$

$$\frac{f_{\text{ssaw}}}{f_{\text{test}}} = 96$$

$$T_{\text{test}} = 0.016 \cdot \mu\text{s}$$

$$A_5 = -10$$

$$\text{bl3} := \text{BILINEAR}\left(\frac{u10}{V}, A_5, \zeta, \omega_5, T_{\text{ssaw}}, N0\right) \quad (5.6.2.3.1)$$

$$\text{bl3} = \left(-2.685 \times 10^{-3} \quad -2.005 \quad 1.006 \quad \{256, 1\} \quad \{256, 1\}\right)$$

$$\text{a22} := \text{bl3}_{0,0} \quad \text{b22} := \text{bl3}_{0,1} \quad \text{c22} := \text{bl3}_{0,2} \quad \text{W22} := \text{bl3}_{0,3} \quad \text{Y33} := \text{bl3}_{0,4}$$

$$\text{a22} = -0.00268472$$

$$\text{b22} = -2.0050030707$$

$$\text{c22} = 1.0060769577$$

Block diagram of the difference equation algorithm for a second order system

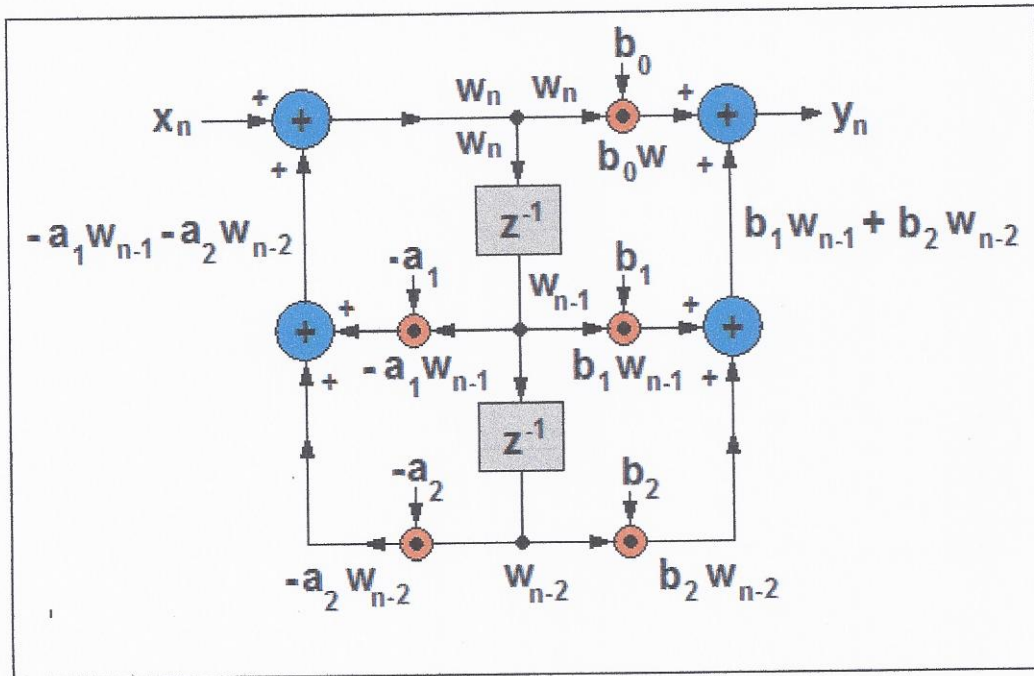


fig.:5.6.2.3.2

$A_5 = -10$

**Sequence of the Sawtooth response.**

$Q_5 = 5.4$

$T_{test} = 0.016 \cdot \mu s$

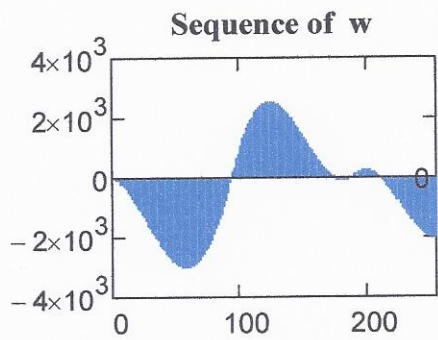


fig.:5.6.2.3.3

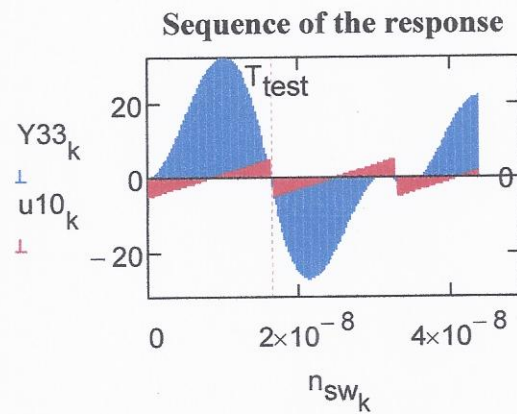


fig.:5.6.2.3.4

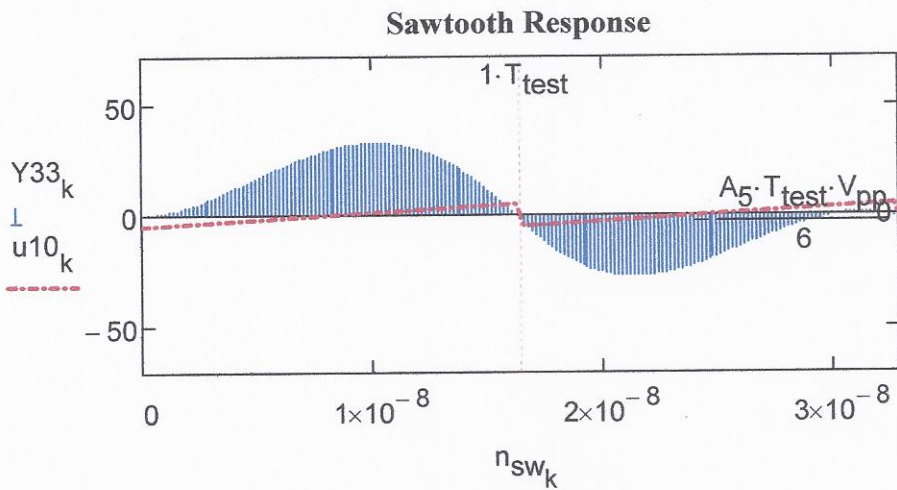


fig.:5.6.2.3.5

$$\text{SpeC}_{00x} := \text{fft}(Y33)$$

**Amplitude Spectrum**

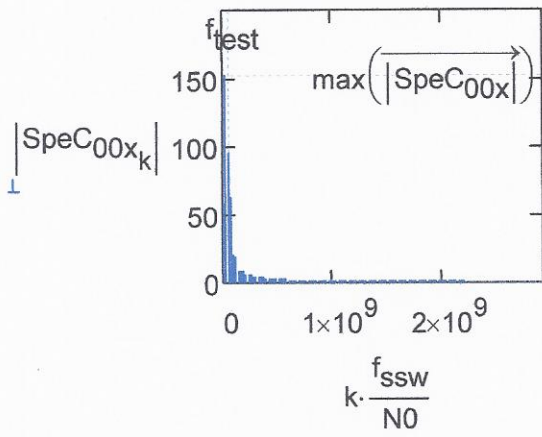


fig.:5.6.2.3.6

**Phase spectrum**

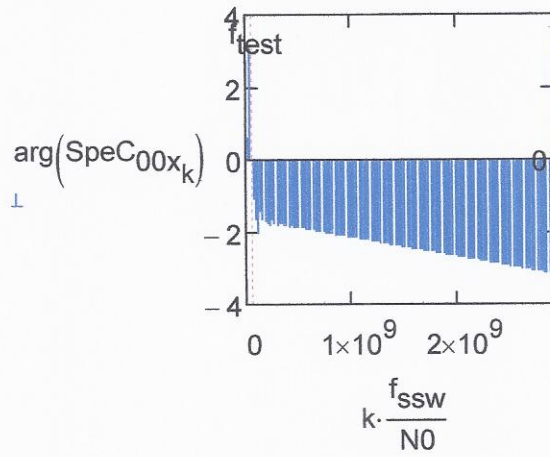


fig.:5.6.2.3.7

$$\max(|\text{SpeC}_{00x_k}|) = 153.4 \frac{1}{V} \cdot V$$



## 5.6 The bilinear transformation

### 5.6.2.4 Sequence of the Bipolar Square Wave response.

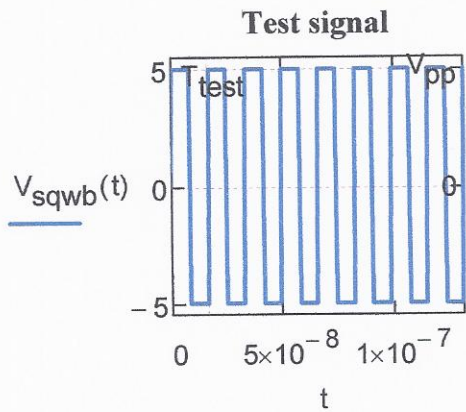


fig.:5.6.2.4.1

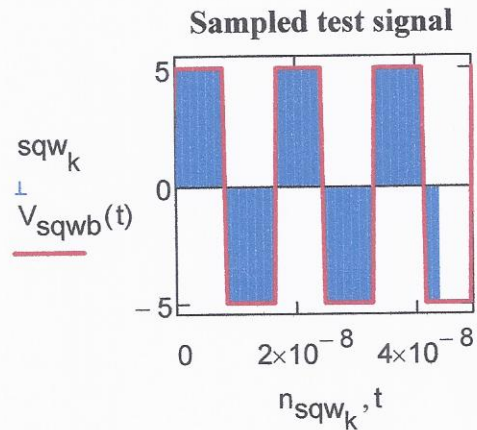


fig.:5.6.2.4.2

$$bl4 := \text{BILINEAR}(sqw, A_5, \zeta, \omega_5, T_{ssqw}, N0) \quad (5.6.2.4.1)$$

$$bl4 = \left( -2.685 \times 10^{-3} \quad -2.005 \quad 1.006 \quad \{256, 1\} \quad \{256, 1\} \right)$$

$$a33 := bl4_{0,0} \quad b33 := bl4_{0,1} \quad c33 := bl4_{0,2} \quad W33 := bl4_{0,3} \quad Y44 := bl4_{0,4}$$

$$a33 = -0.00268472 \quad b33 = -2.0050030707 \quad c33 = 1.0060769577$$

Block diagram of the difference equation algorithm for a second order system

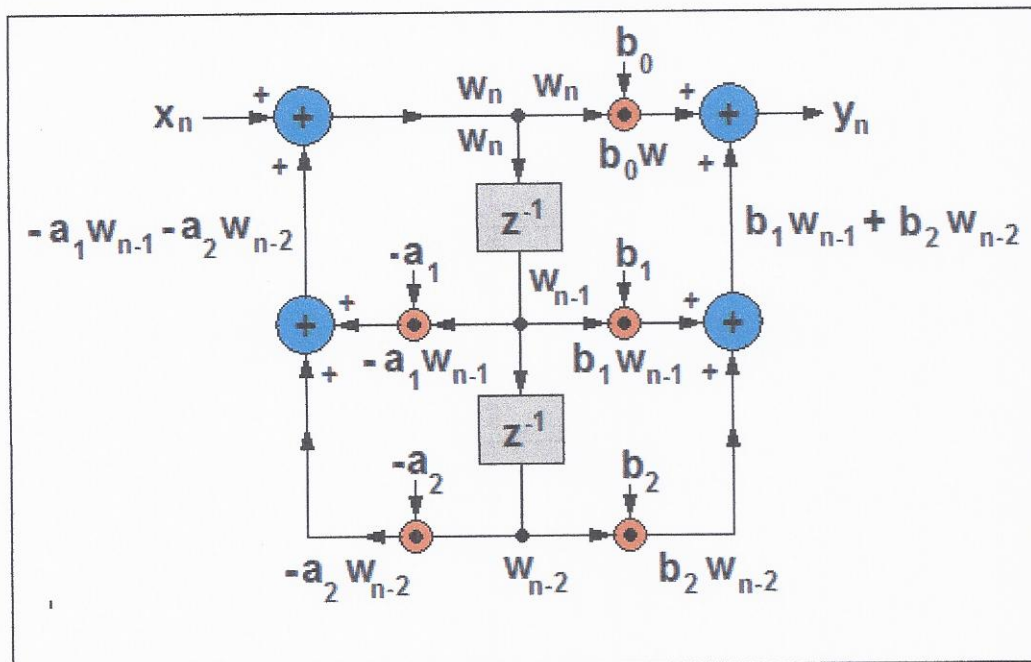


fig.:5.6.2.4.3

**Sequence of the Bipolar Square Wave response.**

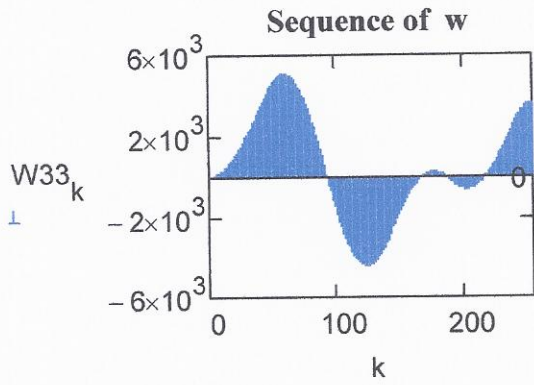


fig.:5.6.2.4.4

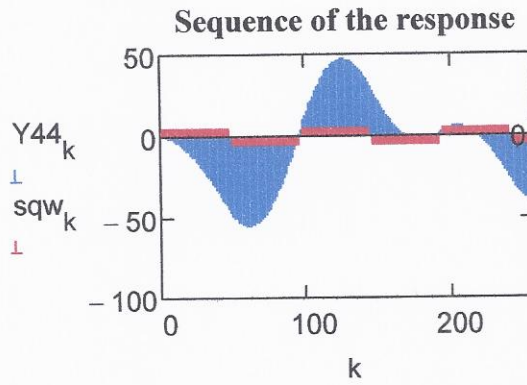


fig.:5.6.2.4.5

$$\text{SpeC}_{01x} := \text{fft}(Y44) \quad (5.6.2.4.2)$$

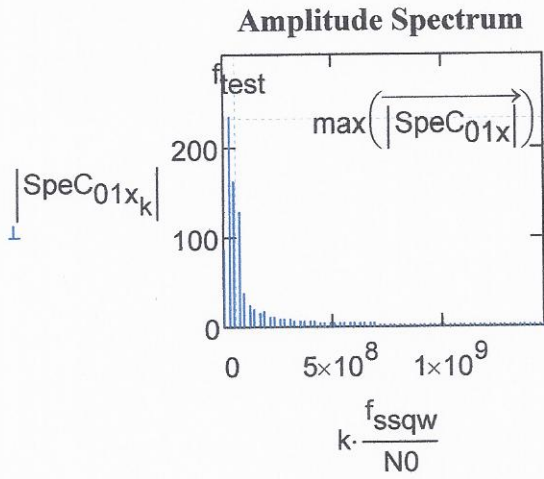


fig.:5.6.2.4.6

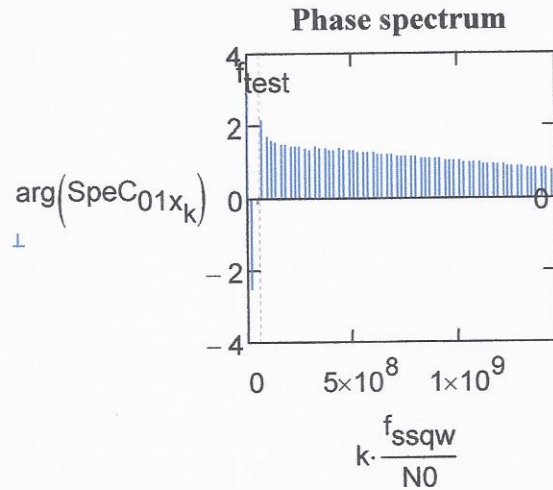


fig.:5.6.2.4.7

$$\max(|\text{SpeC}_{01x}|) = 230.965$$

## 5.6 The bilinear transformation

### 5.6.2.5 Sequence of the (single tone) AM Signal response.

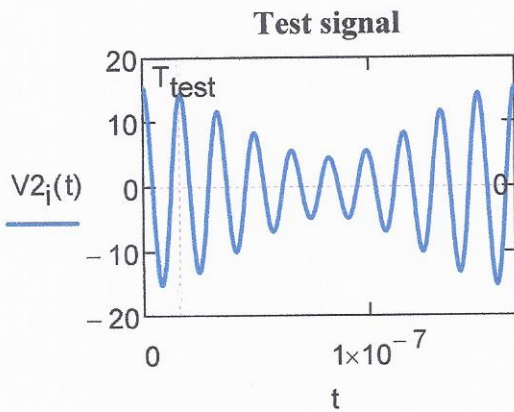


fig.:5.6.2.5.1

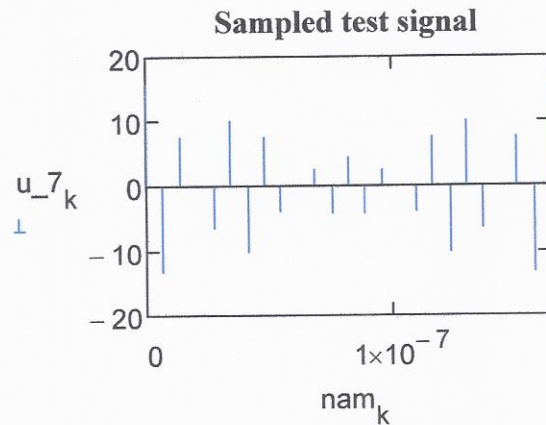


fig.:5.6.2.5.2

$$bl5 := \text{BILINEAR}(u\_7, A_5, \zeta, \omega_5, T_{\text{sam}}, N0) \quad (5.6.2.5.1)$$

$$bl5 = (-3.277 \quad -0.875 \quad 1.185 \quad \{256,1\} \quad \{256,1\})$$

$$a44 := bl5_{0,0} \quad b44 := bl5_{0,1} \quad c44 := bl5_{0,2} \quad W44 := bl5_{0,3} \quad Y55 := bl5_{0,4}$$

$$a44 = -3.27707905 \quad b44 = -0.8746131198 \quad c44 = 1.1854447394$$

Block diagram of the difference equation algorithm for a second order system

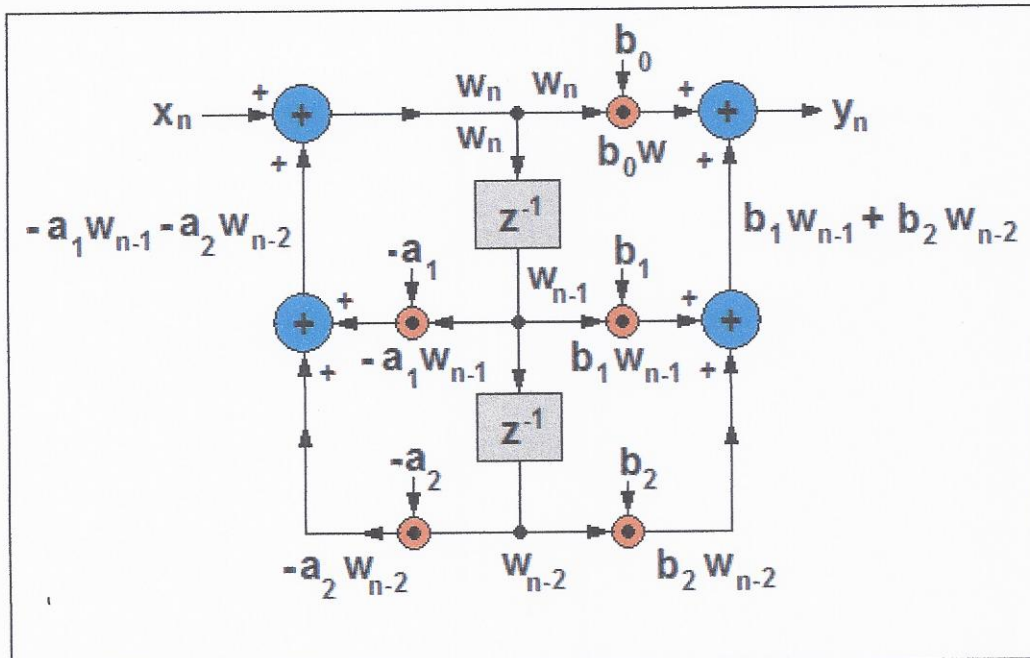


fig.:5.6.2.5.3

**Sequence of the AM (single tone) Signal response.**

**Sequence of w**

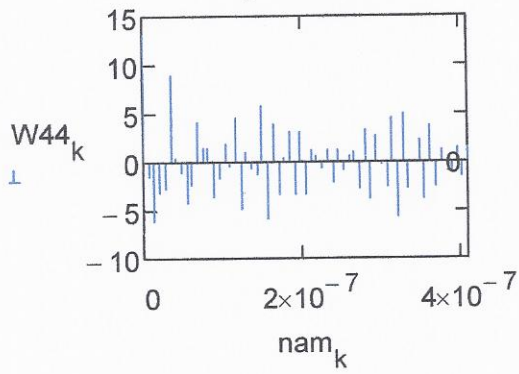


fig.:5.6.2.5.4

**Sequence of the response**

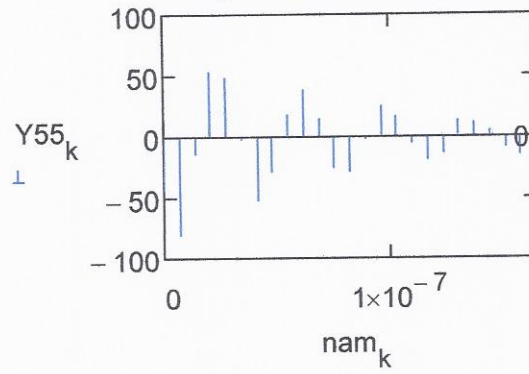


fig.:5.6.2.5.5

$$\text{SpeC01x} := \text{fft}(Y55)$$

**Amplitude Spectrum**

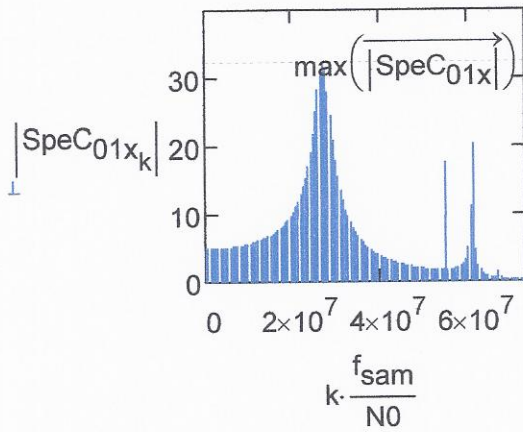


fig.:5.6.2.5.6

**Phase spectrum**

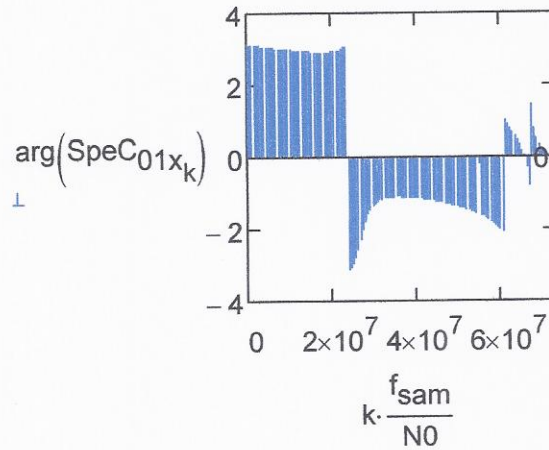


fig.:5.6.2.5.7

$$\max(|\text{SpeC01x}|) = 32.497$$



## 5.6 The bilinear transformation

### 5.6.2.6 Sequence of the (single tone) Frequency Modulated carrier response .

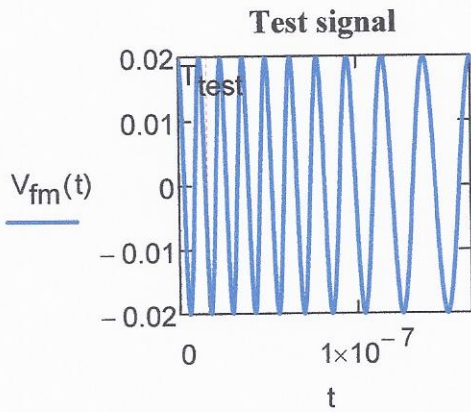


fig.:5.6.2.6.1

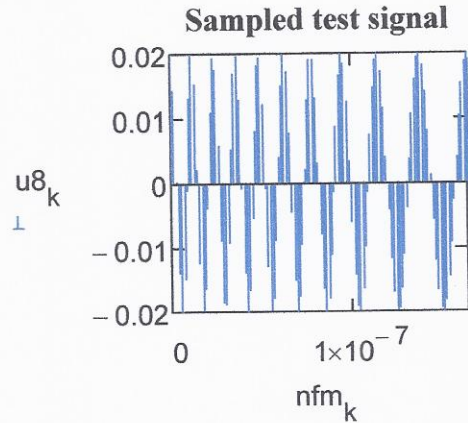


fig.:5.6.2.6.2

$$bl6 := \text{BILINEAR}(u8, A_5, \zeta, \omega_5, T_{\text{sfm}}, N0) \quad (5.6.2.6.1)$$

$$bl6 = (-0.194 \quad -1.974 \quad 1.052 \quad \{256,1\} \quad \{256,1\})$$

$$a55 := bl6_{0,0} \quad b55 := bl6_{0,1} \quad c55 := bl6_{0,2} \quad W5 := bl6_{0,3} \quad Y66 := bl6_{0,4}$$

$$a55 = -0.19414962 \quad b55 = -1.9741133834 \quad c55 = 1.0517732332$$

Block diagram of the difference equation algorithm for a second order system

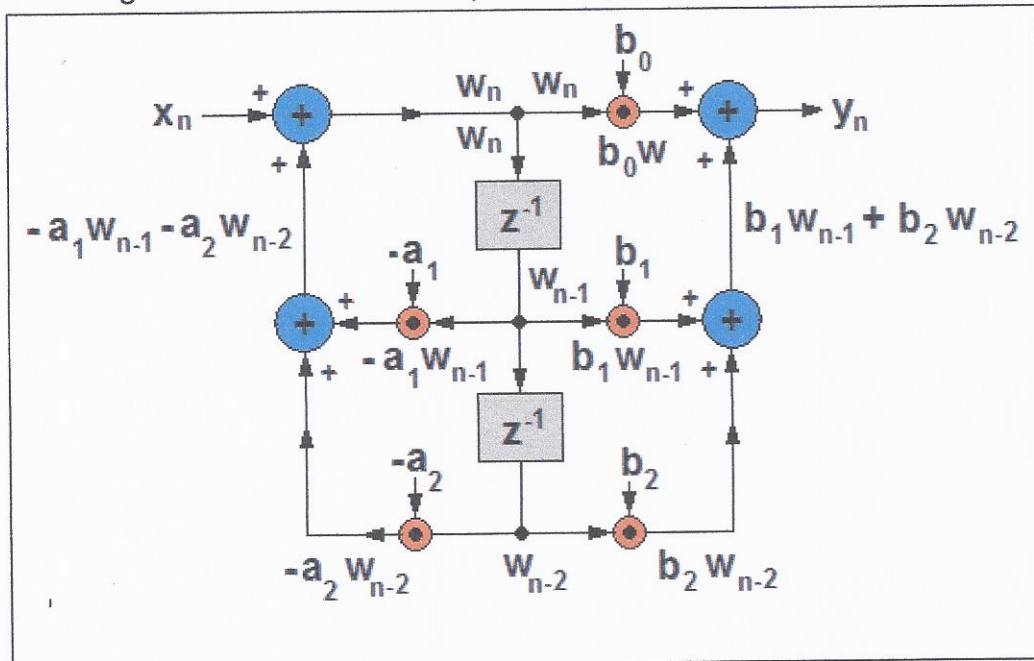


fig.:5.6.2.6.3

**Sequence of the Frequency (single tone) Modulated carrier response.**

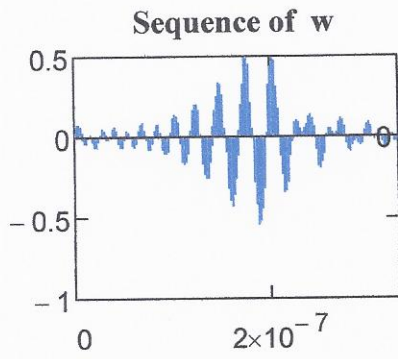


fig.:5.6.2.6.4

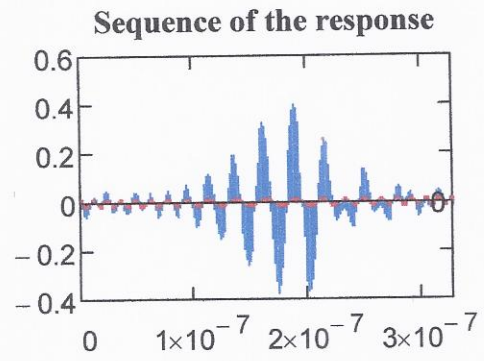


fig.:5.6.2.6.5

$$\text{SpeC}_{02x} := \text{fft}(Y66)$$

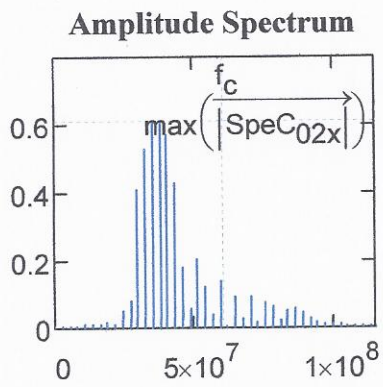


fig.:5.6.2.6.6

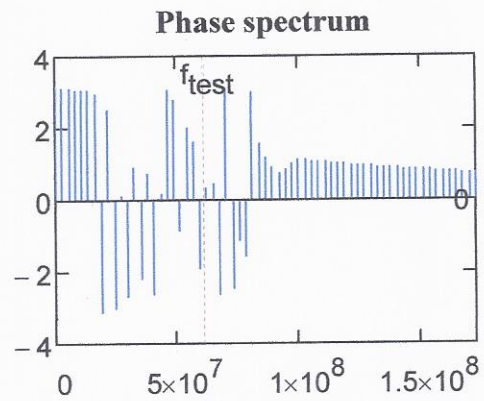


fig.:5.6.2.6.7

$$\max(|\text{SpeC}_{02x}|) = 0.614$$

## 5.6 The bilinear transformation

### 5.6.2.7 Sequence of the (single tone) Phase Modulated carrier response.

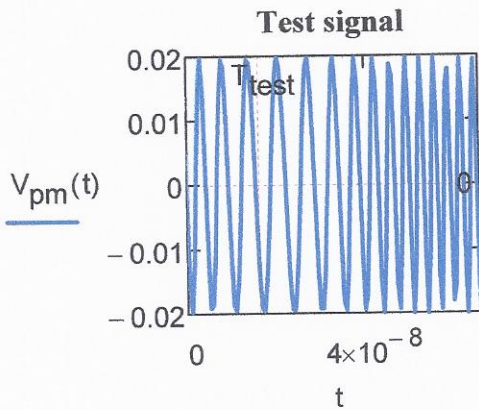


fig.:5.6.2.7.1

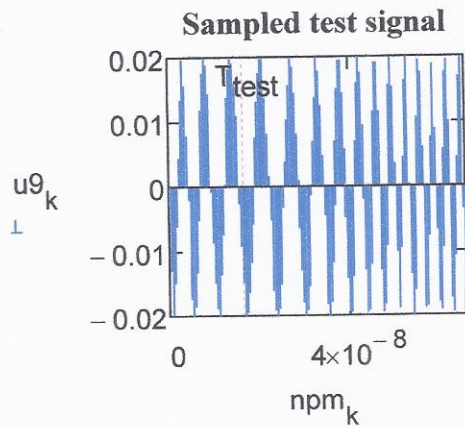


fig.:5.6.2.7.2

$$bl7 := BILINEAR(u9, A_5, \zeta, \omega_5, T_{spm}, N0) \quad (5.6.2.7.1)$$

$$bl7 = (-0.012 \quad -2.008 \quad 1.013 \quad \{256,1\} \quad \{256,1\})$$

$$a66 := bl7_{0,0} \quad b66 := bl7_{0,1} \quad c66 := bl7_{0,2} \quad W66 := bl7_{0,3} \quad Y77 := bl7_{0,4}$$

$$a66 = -0.01211965 \quad b66 = -2.0080797634 \quad c66 = 1.0129276215$$

Block diagram of the difference equation algorithm for a second order system

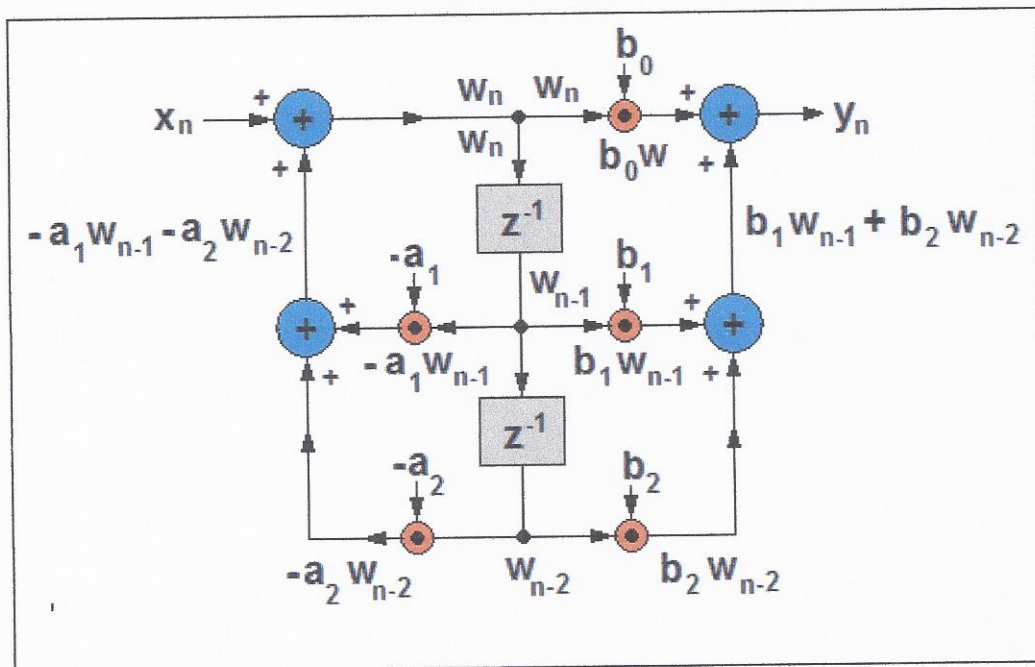


fig.:5.6.2.7.3

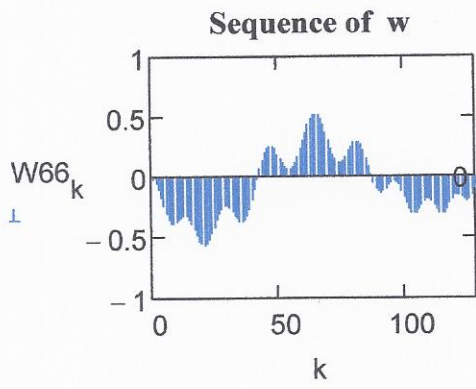


fig.:5.6.2.7.4

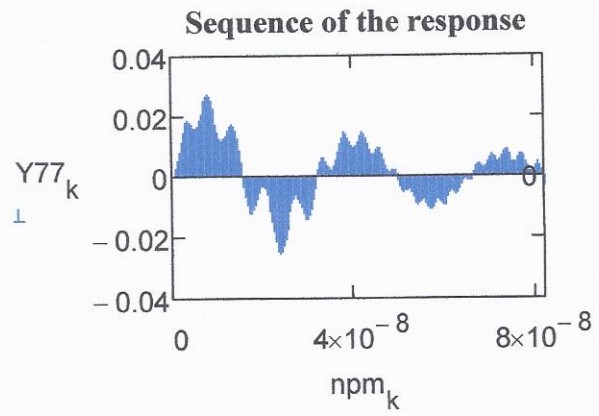


fig.:5.6.2.7.5

$$\text{SpeC}_{03x} := \text{fft}(Y77)$$

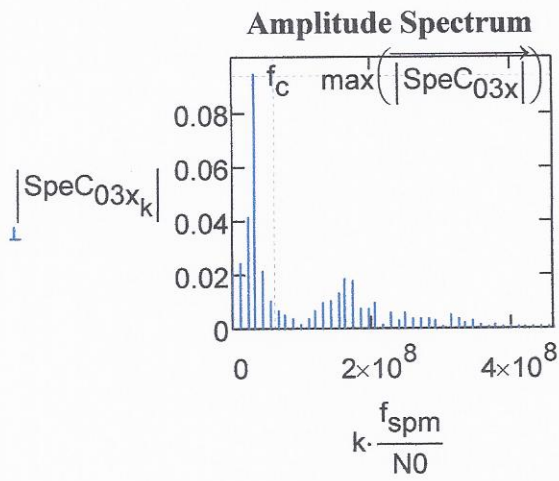


fig.:5.6.2.7.6

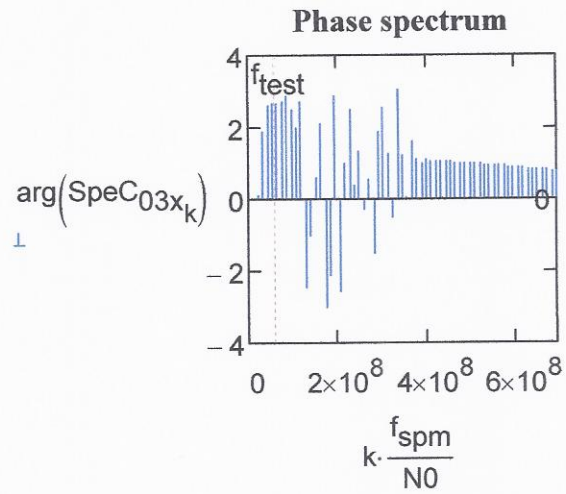


fig.:5.6.2.7.7

$$\max(|\text{SpeC}_{00x}|) = 153.4$$



## 5.7

### Synthetic Division algorithm (considering the bilinear transformation).

Search of the sequence corresponding to the z t. f. knowing its numerator and denominator coefficients:

$$\alpha_2 := \begin{cases} \frac{A_5 \cdot \omega_5^2 \cdot T_s^2}{T_s^2 \cdot \omega_5^2 - 4 \cdot \zeta \cdot T_s + 4} & \text{if } \zeta \neq \omega_5 \\ \frac{A_5 \cdot T_s^2 \cdot \omega_5^2}{(T_s \cdot \omega_5 - 2)^2} & \text{otherwise} \end{cases}, \beta_2 := \begin{cases} \frac{2 \cdot T_s^2 \cdot \omega_5^2 - 8}{T_s^2 \cdot \omega_5^2 - 4 \cdot \zeta \cdot T_s + 4} & \text{if } \zeta \neq \omega_5 \\ 2 \cdot \frac{T_s^2 \cdot \omega_5^2 - 4}{(T_s \cdot \omega_5 - 2)^2} & \text{otherwise} \end{cases}$$

$$\gamma_2 := \begin{cases} \frac{T_s^2 \cdot \omega_5^2 + 4 \cdot \zeta \cdot T_s + 4}{T_s^2 \cdot \omega_5^2 - 4 \cdot \zeta \cdot T_s + 4} & \text{if } \zeta \neq \omega_5 \\ \frac{(T_s \cdot \omega_5 + 2)^2}{(T_s \cdot \omega_5 - 2)^2} & \text{otherwise} \end{cases}$$

$$\zeta - \omega_5 = -173.884 \cdot \frac{\text{Mrads}}{\text{sec}}$$

$$\alpha_2 = -2.685 \times 10^{-3}$$

$$\beta_2 = -2.005$$

$$\gamma_2 = 1.006$$

The z t. f. is:

$$H_{lp}(z) := \alpha_2 \cdot \frac{1 + 2 \cdot z^{-1} + z^{-2}}{z^{-2} + \beta_2 \cdot z^{-1} + \gamma_2} \quad (5.7.1)$$

Numerator  $\underline{N} := 2$

Denominator  $\underline{M} := 2$

$$\underline{N1} := \underline{N} + \underline{M}$$

Numerator Coeffs.

Denominator Coeffs.

$$b_\nu := 0.0$$

$$a_\nu := 0.0$$

$$h_{3k} := 0$$

$$b_0 := \alpha_2$$

$$a_0 := \gamma_2$$

$$a_0 = \bullet$$

$$b_1 := 2 \cdot \alpha_2$$

$$a_1 := \beta_2$$

$$a_1 = \bullet$$

$$b_2 := \alpha_2$$

$$a_2 := 1$$

$$a_2 = \bullet$$

$$N1 = 4 \quad h_{30} := \frac{b_0}{a_0} \quad h_{3\nu} := \frac{1}{a_0} \cdot \left[ b_\nu - \sum_{i=1}^{\nu} (h_{3\nu-i} \cdot a_i) \right] \quad (5.7.2)$$

In this worksheet will be used the following program:

SYNDIVBL

### 5.7 Synthetic Division algorithm (considering the bilinear transformation).

#### 5.7.1 Sequence of the voltage Step response.

$$T_{s5} := T_{sstp} \quad \text{syndbl} := \text{SYNDIVBL}\left(\frac{u1}{V}, A_5, \zeta, \omega_5, T_{sstp}, N0\right) \quad (5.7.1.1)$$

$$\text{syndbl} = (-0.248 \quad -1.959 \quad 1.058 \quad \{256,1\} \quad \{256,1\})$$

$$a5 := \text{syndbl}_0,0 \quad b5 := \text{syndbl}_0,1 \quad c5 := \text{syndbl}_0,2 \quad h3 := \text{syndbl}_0,3 \quad Y24 := \text{syndbl}_0,4$$

$$a5 = -0.24783425$$

$$b5 = -1.9593019235$$

$$c5 = 1.0584356239$$

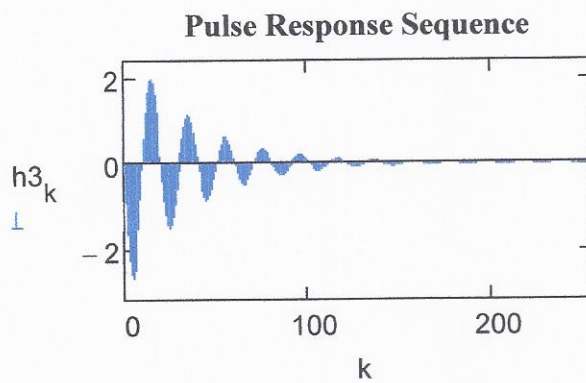


fig.:5.7.1.1

$$t := 0 \cdot \tau, \frac{40 \cdot \tau}{1000} .. 40 \cdot \tau$$

$$y24_\nu := \sum_{k=0}^{\nu} (\text{if}(\nu - k \geq 0, h3_k \cdot u1_{\nu-k}, 0)) \quad (5.7.1.2)$$

$$V_{pp} = 5V$$

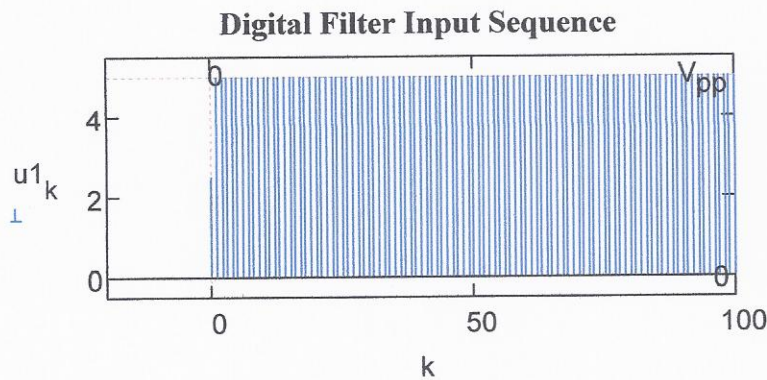


fig.:5.7.1.2

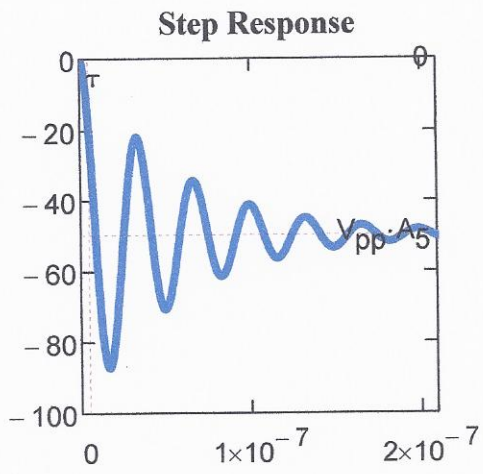


fig.:5.7.1.3

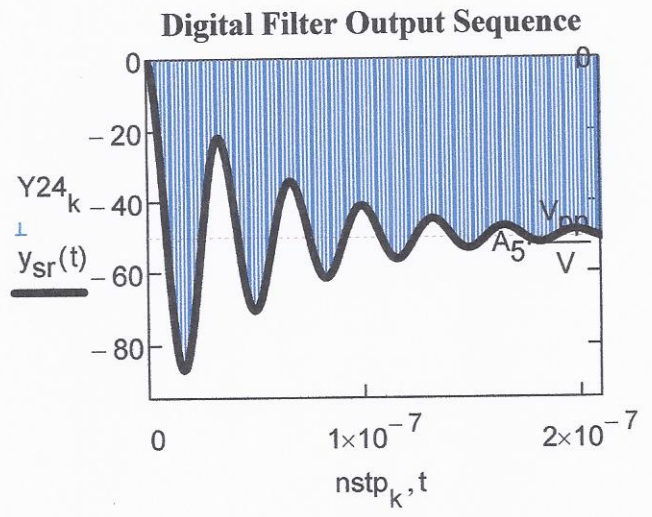


fig.:5.7.1.4

5.7 Synthetic Division algorithm (considering the bilinear transformation).

5.7.2 Sequence of the Short Voltage Pulse response.

$$\text{syndbl1} := \text{SYNDIVBL}(u44, A_5, \zeta, \omega_5, T_{\text{svp}}, N0) \quad (5.7.2.1)$$

$$\text{syndbl1} = (-5.764 \quad 0.102 \quad 1.204 \quad \{256,1\} \quad \{256,1\})$$

$$h4 := \text{syndbl1}_{0,3} \quad Y25 := \text{syndbl1}_{0,4} \quad (5.7.2.2)$$

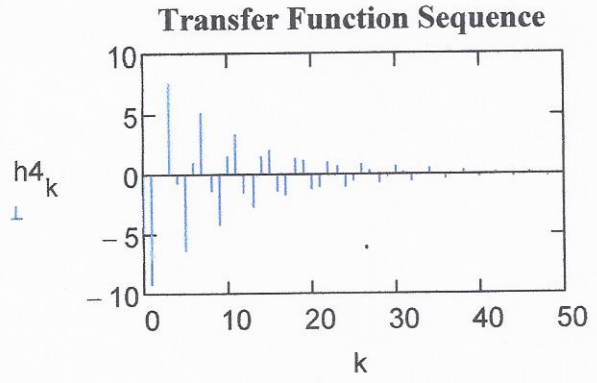


fig.:5.7.2.1

$$\tau_5 = -131.154 \cdot \text{ns} \quad t := 1 \cdot \tau_5, \frac{\tau_5 + 8 \cdot (\tau_{\text{pw}} + \tau_5) - 1 \cdot \tau_5}{10^{15}} .. \tau_5 + 8 \cdot (\tau_{\text{pw}} + \tau_5)$$

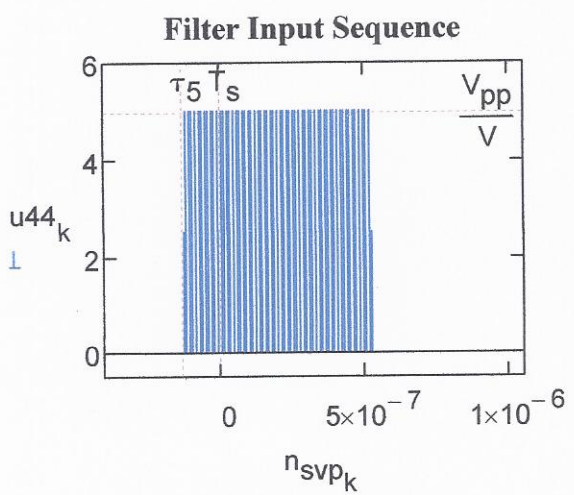


fig.:5.7.2.2

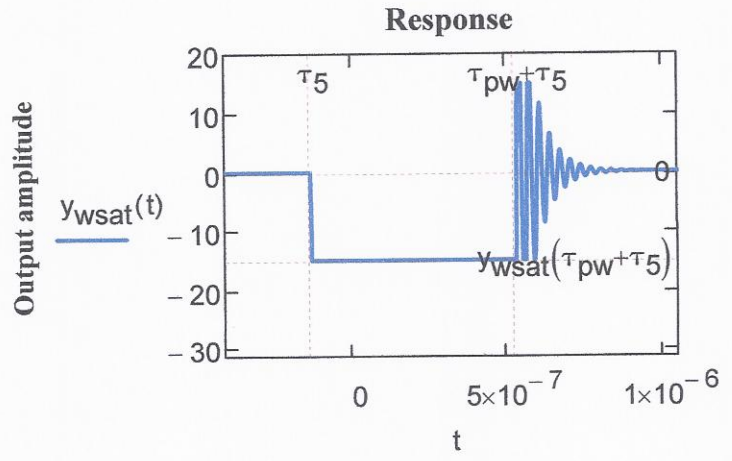


fig.:5.7.2.3



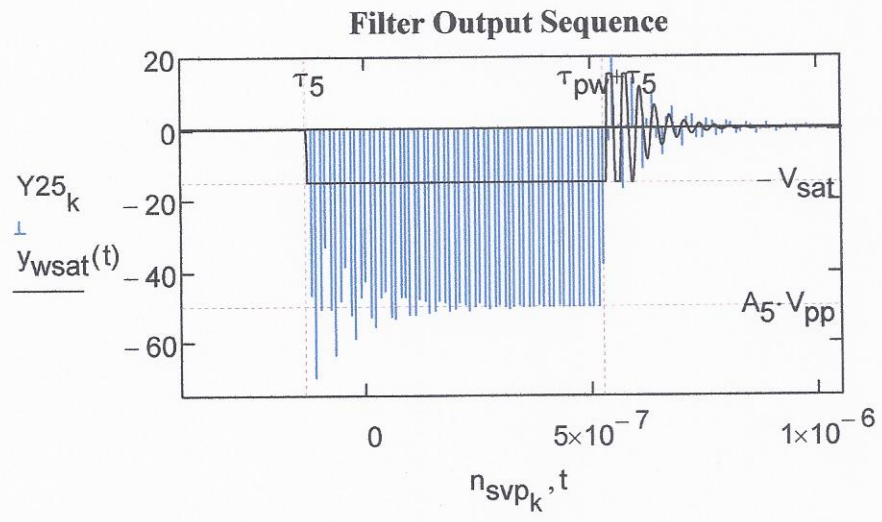


fig.:5.7.2.4

## 5.7 Synthetic Division algorithm (considering the bilinear transformation).

### 5.7.3 Sequence of the Sawtooth response:

$$\text{syndbl2} := \text{SYNDIVBL}\left(\frac{u55}{V}, A_5, \zeta, \omega_5, T_{\text{SSW}}, N0\right) \quad (5.7.3.1)$$

$$\text{syndbl2} = \left(-2.685 \times 10^{-3} \quad -2.005 \quad 1.006 \quad \{256, 1\} \quad \{256, 1\}\right)$$

$$\underline{h5}_k := \text{syndbl2}_{0,3} \quad Y27 := \text{syndbl2}_{0,4} \quad (5.7.3.2)$$

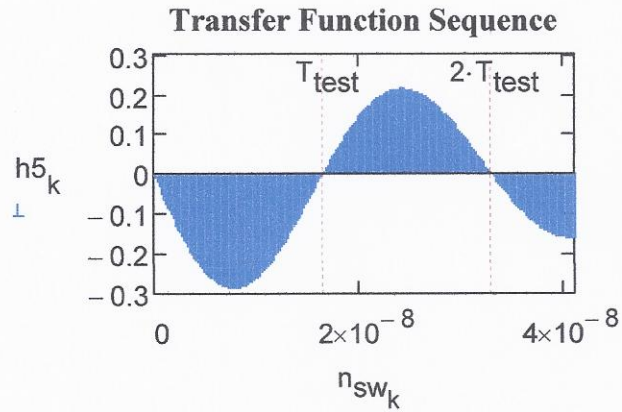


fig.:5.7.3.1

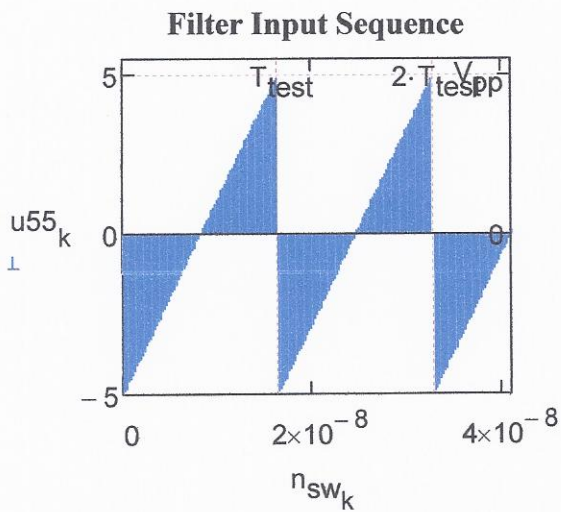


fig.:5.7.3.2

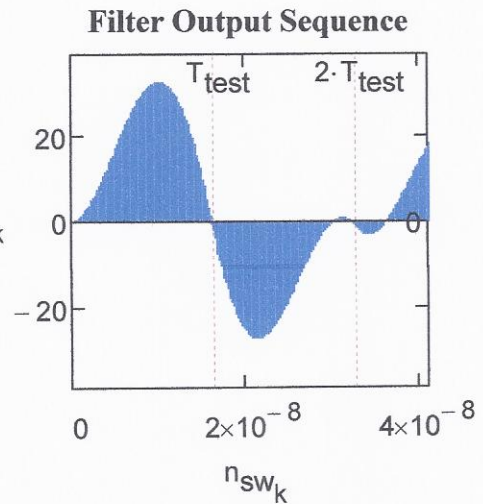


fig.:5.7.3.3

$$A_5 \cdot V_{\text{pp}} = -50V$$

## 5.7 Synthetic Division algorithm (considering the bilinear transformation).

### 5.7.4 Sequence of the Bipolar Square Wave response.

$$\text{syndbl3} := \text{SYNDIVBL}(\text{sqw}, A_5, \zeta, \omega_5, T_{\text{ssqw}}, N0) \quad (5.7.4.1)$$

$$\text{syndbl3} = \left( -2.685 \times 10^{-3} \quad -2.005 \quad 1.006 \quad \{256, 1\} \quad \{256, 1\} \right)$$

$$\text{h6} := \text{syndbl3}_{0,3} \quad Y26 := \text{syndbl3}_{0,4} \quad (5.7.4.2)$$

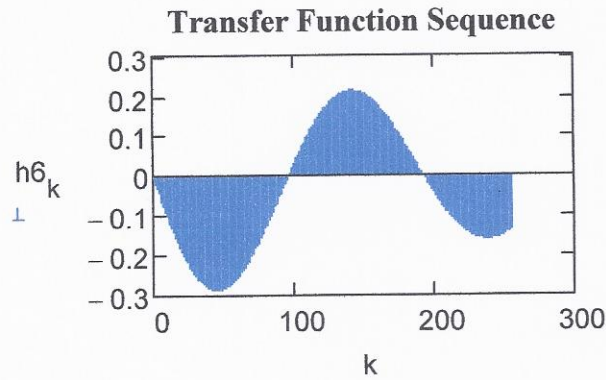


fig.:5.7.4.1

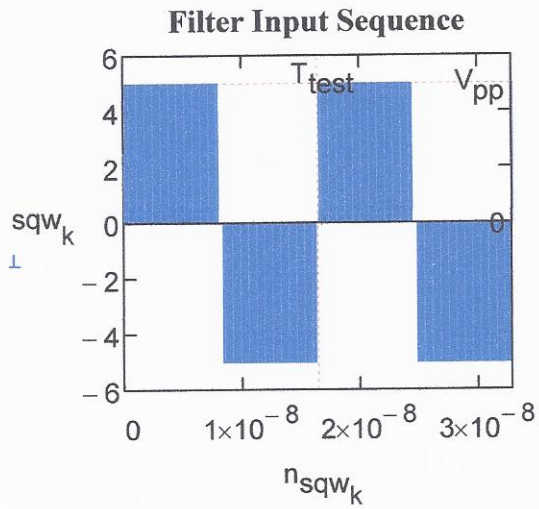


fig.:5.7.4.2

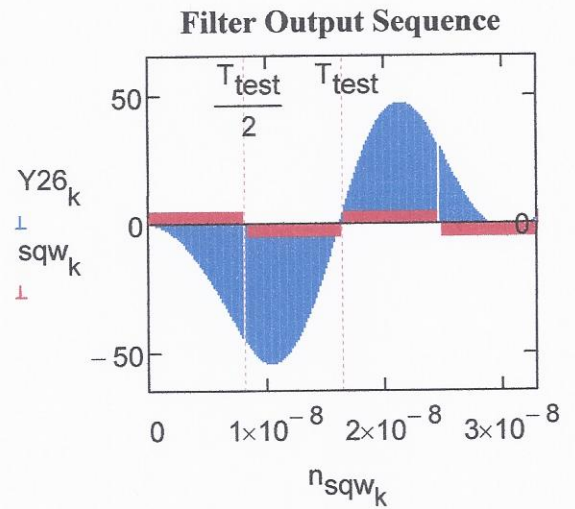


fig.:5.7.4.3

### 5.7 Synthetic Division algorithm (considering the bilinear transformation).

#### 5.7.5 Sequence of the (single tone) AM Signal response.

$$\text{syndbl4} := \text{SYNDIVBL}(u\_7, A_5, \zeta, \omega_5, T_{\text{sam}}, N_0) \quad (5.7.5.1)$$

$$\text{syndbl4} = (-3.277 \quad -0.875 \quad 1.185 \quad \{256,1\} \quad \{256,1\})$$

$$\text{h7}_k := \text{syndbl4}_{0,3} \quad Y28 := \text{syndbl4}_{0,4} \quad (5.7.5.2)$$

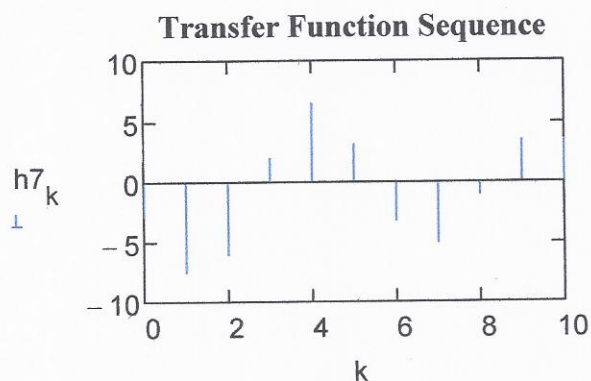


fig.:5.7.5.1

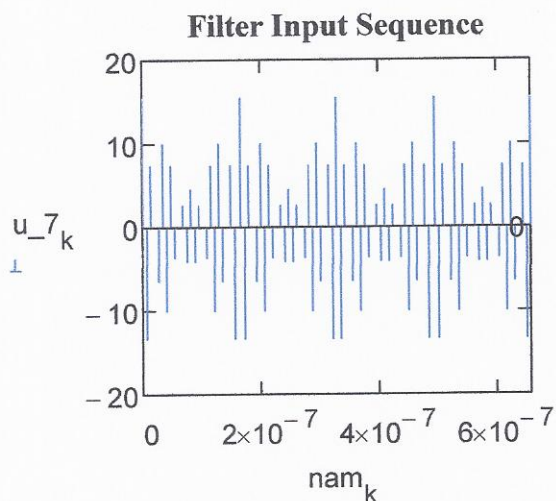


fig.:5.7.5.2

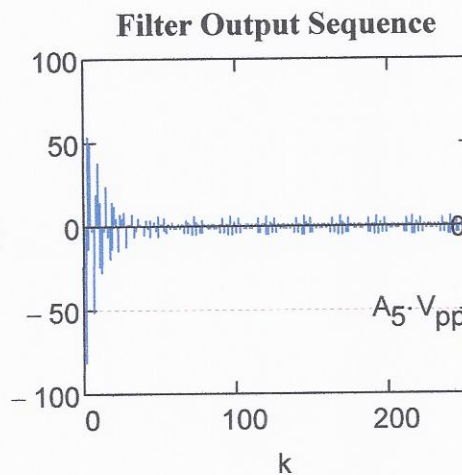


fig.:5.7.5.3



## 5.7 Synthetic Division algorithm (considering the bilinear transformation).

### 5.7.6 Sequence of the (single tone) Frequency Modulated carrier response.

$$\text{syndbl5} := \text{SYNDIVBL}(u8, A_5, \zeta, \omega_5, T_{\text{sfm}}, N0) \quad (5.7.6.1)$$

$$\text{syndbl5} = (-0.194 \quad -1.974 \quad 1.052 \quad \{256,1\} \quad \{256,1\})$$

$$h8 := \text{syndbl5}_{0,3} \quad Y29 := \text{syndbl5}_{0,4} \quad (5.7.6.2)$$

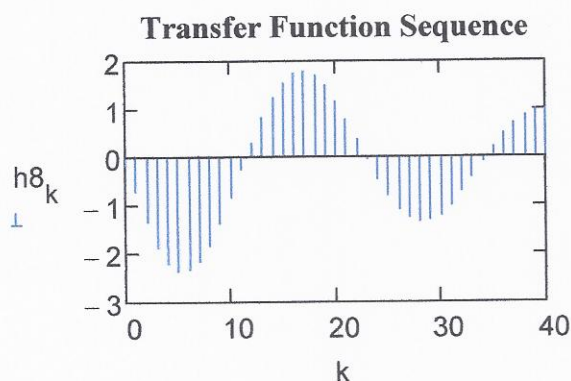


fig.:5.7.6.1

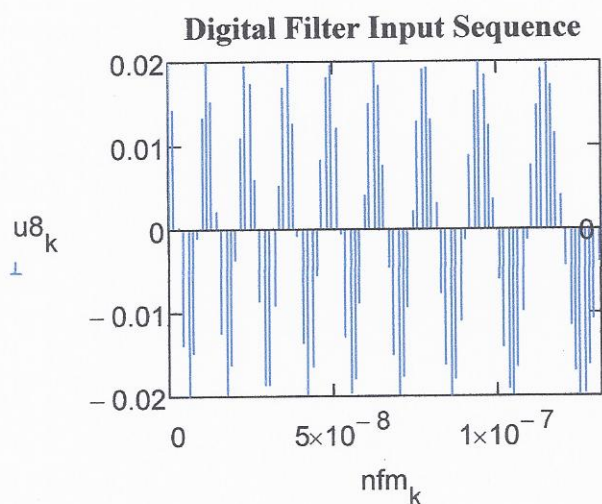


fig.:5.7.6.2

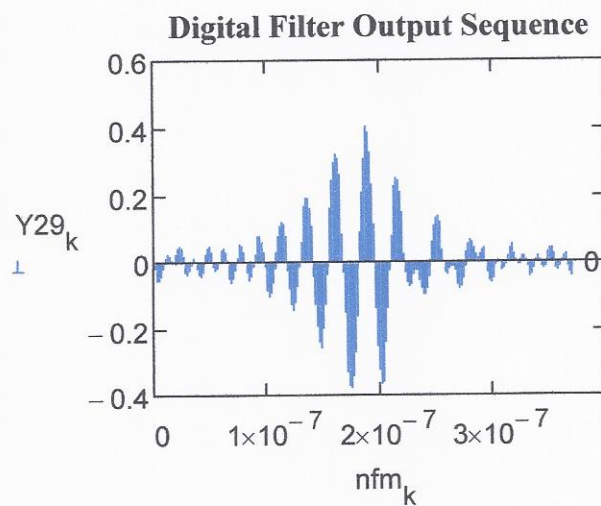


fig.:5.7.6.3

**5.7 Synthetic Division algorithm (considering the bilinear transformation).**

**5.7.7 Sequence of the (single tone) Phase Modulated carrier response.**

$$\text{syndbl6} := \text{SYNDIVBL}(u_9, A_5, \zeta, \omega_5, T_{\text{spm}}, N0) \quad (5.7.7.1)$$

$$m_p = 8$$

$$\text{syndbl6} = (-0.012 \quad -2.008 \quad 1.013 \quad \{256,1\} \quad \{256,1\})$$

$$A_{\text{fm}} = 0.02V$$

$$h9 := \text{syndbl6}_{0,3} \quad Y30 := \text{syndbl6}_{0,4} \quad (5.7.7.2)$$

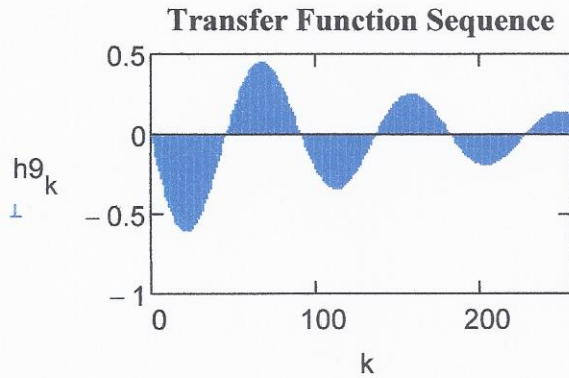


fig.:5.7.7.1

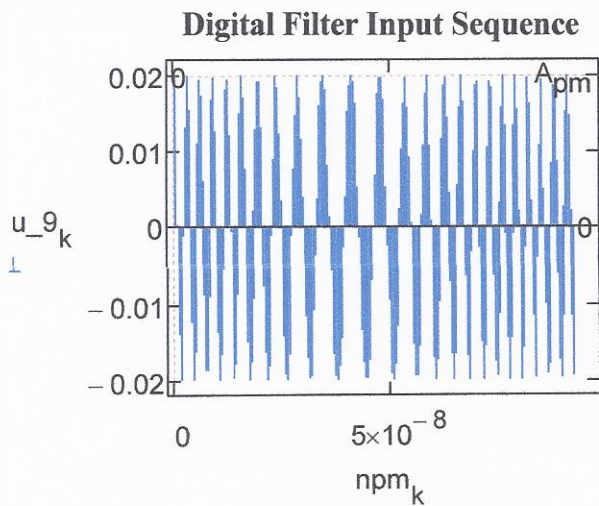


fig.:5.7.7.2

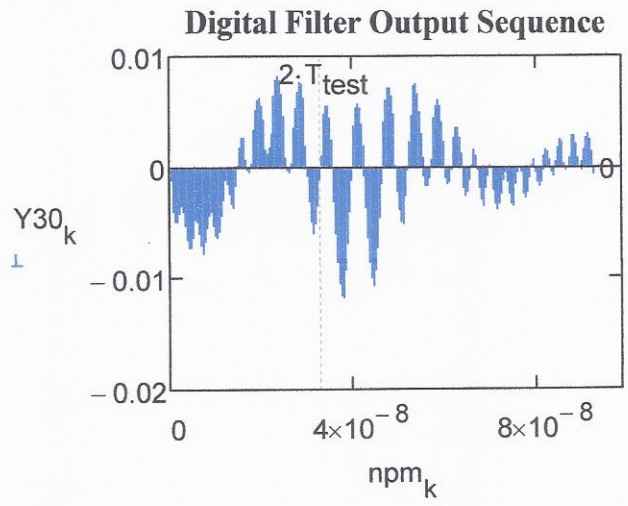


fig.:5.7.7.3

$$A_5 \cdot A_{\text{pm}} = -0.2V$$

## 5.8

### **Analytical search of the output sequence by means of the residues method (considering the bilinear transformation).**

Search of the sequence corresponding to the filter's t. f. using the residues method.

Numerator and denominator's coefficients of the z t. f.:

$$\begin{aligned}
 & \boxed{T_s := T_{sstp}} \\
 \alpha_2 := & \begin{cases} \frac{A_5 \cdot \omega_5^2 \cdot T_s^2}{T_s^2 \cdot \omega_5^2 - 4 \cdot \zeta \cdot T_s + 4} & \text{if } \zeta \neq \omega_5 \\ \frac{A_5 \cdot T_s^2 \cdot \omega_5^2}{(T_s \cdot \omega_5 - 2)^2} & \text{otherwise} \end{cases}, \beta_2 := \begin{cases} \frac{2 \cdot T_s^2 \cdot \omega_5^2 - 8}{T_s^2 \cdot \omega_5^2 - 4 \cdot \zeta \cdot T_s + 4} & \text{if } \zeta \neq \omega_5 \\ \frac{T_s^2 \cdot \omega_5^2 - 4}{2 \cdot (T_s \cdot \omega_5 - 2)^2} & \text{otherwise} \end{cases}, \\
 \gamma_2 := & \begin{cases} \frac{T_s^2 \cdot \omega_5^2 + 4 \cdot \zeta \cdot T_s + 4}{T_s^2 \cdot \omega_5^2 - 4 \cdot \zeta \cdot T_s + 4} & \text{if } \zeta \neq \omega_5 \\ \frac{(T_s \cdot \omega_5 + 2)^2}{(T_s \cdot \omega_5 - 2)^2} & \text{otherwise} \end{cases}
 \end{aligned}$$

$$\zeta - \omega_5 = -173.884 \cdot \frac{\text{Mrads}}{\text{sec}}$$

$$\alpha_2 = -0.248$$

$$\beta_2 = -1.959$$

$$\gamma_2 = 1.058$$

The z t. f. is:

$$H_{lp}(z) := \alpha_2 \cdot \frac{1 + 2 \cdot z^{-1} + z^{-2}}{z^{-2} + \beta_2 \cdot z^{-1} + \gamma_2}, \quad (5.8.1)$$

**First case**  $\boxed{\zeta \neq \omega_5}$  Define the

$$F10(z, n) = z^{n-1} \cdot H_{lp}(z) \quad (5.8.2)$$

function:

$$\text{namely: } F10(z, n) := \alpha_2 \cdot \frac{z^n \cdot (z + 1)^2}{\gamma_2 \cdot z^3 + \beta_2 \cdot z^2 + z} \quad (5.8.3)$$

$$\zeta = 0.018 \cdot \frac{\text{Grads}}{\text{sec}}$$

$$\omega_5 = 0.192 \cdot \frac{\text{Grads}}{\text{sec}}$$

$$z := z$$

$$\beta_2 := \beta_2$$

$$\gamma_2 := \gamma_2$$

Poles calculation:  $\text{poles1} := (\gamma_2 \cdot z^3 + \beta_2 \cdot z^2 + z) \text{ solve, } z \rightarrow$

$$\left( \begin{array}{c} 0 \\ \frac{\beta_2 + \sqrt{\beta_2^2 - 4 \cdot \gamma_2}}{2} \\ \frac{\beta_2 - \sqrt{\beta_2^2 - 4 \cdot \gamma_2}}{2} \\ \gamma_2 \end{array} \right) \quad (5.8.4)$$

$$z := z \quad p0 := p0 \quad p1 := p1 \quad p2 := p2$$

$$\zeta \neq \omega_5 \quad p0 := \text{poles1}_0 \quad p1 := \text{poles1}_1 \quad p2 := \text{poles1}_2$$

$$p1 := -\left( \frac{\beta_2 + \sqrt{\beta_2^2 - 4 \cdot \gamma_2}}{2 \cdot \gamma_2} \right) \quad p2 := -\left( \frac{\beta_2 - \sqrt{\beta_2^2 - 4 \cdot \gamma_2}}{2 \cdot \gamma_2} \right)$$

$$p0 = 0 \quad p1 = \blacksquare \quad p2 = 0.926 + 0.297j$$

$$t := t \quad r := \text{ceil}(\max(|\text{poles1}|)) \cdot 1.0 \quad r = 2$$

$$\xi(t) := r \cdot \cos(t)$$

$$\psi(t) := r \cdot \sin(t)$$

$$\phi(t) := \xi(t) + j \cdot \psi(t)$$

$$t_0 := 0 \quad t_{\text{fin}} := 2 \cdot \pi \quad t := t_0, t_0 + \frac{t_{\text{fin}} - t_0}{1000} \dots t_{\text{fin}} \quad n = 2$$

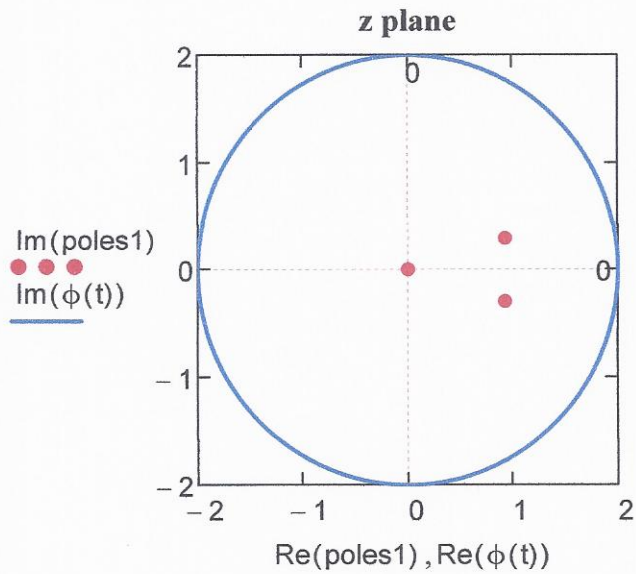


fig.:5.8.1

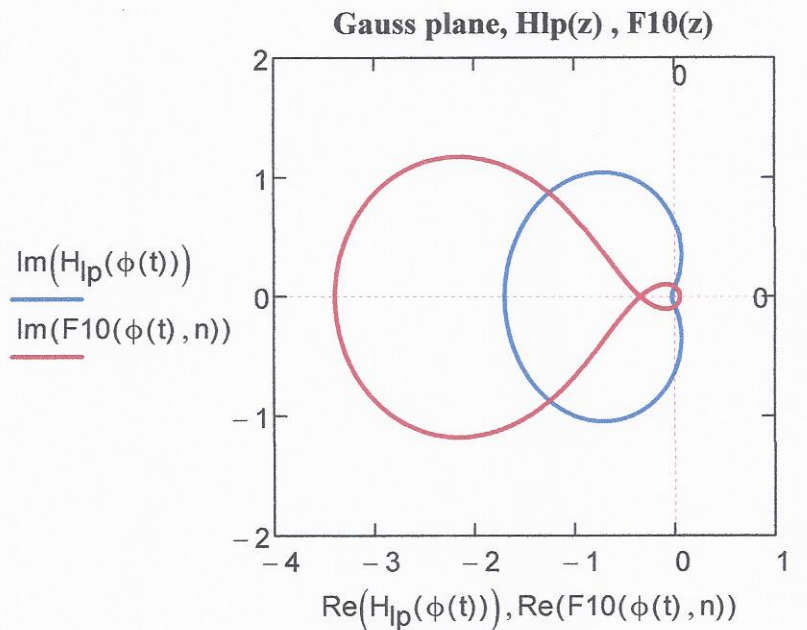


fig.:5.8.2

Knowing the poles of the function  $F10(z, n)$ , it can be written too:



$$F10(z, n) = \frac{\alpha_2}{\gamma_2} \cdot \frac{z^n \cdot (z+1)^2}{(z-p_0) \cdot (z-p_1) \cdot (z-p_2)} \quad (5.8.5)$$

The corresponding sequence is given by:

$$h10_n = \sum_{j=0}^2 \lim_{z \rightarrow p_j} [(z-p_j) \cdot F10(z)] \quad (5.8.6)$$

hence: 
$$h10_k = \frac{\alpha_2}{\gamma_2} \cdot \sum_{j=0}^2 \lim_{z \rightarrow p_j} \left[ (z-p_j) \cdot \frac{z^k \cdot (z+1)^2}{(z-p_0) \cdot (z-p_1) \cdot (z-p_2)} \right] \quad (5.8.7)$$

or: 
$$h10_k = \frac{\alpha_2}{\gamma_2} \cdot \left[ \lim_{z \rightarrow p_0} \left[ (z-p_0) \cdot \frac{z^k \cdot (z+1)^2}{(z-p_0) \cdot (z-p_1) \cdot (z-p_2)} \right] \dots \right. \\ \left. + \lim_{z \rightarrow p_1} \left[ (z-p_1) \cdot \frac{z^k \cdot (z+1)^2}{(z-p_0) \cdot (z-p_1) \cdot (z-p_2)} \right] \dots \right. \\ \left. + \lim_{z \rightarrow p_2} \left[ (z-p_2) \cdot \frac{z^k \cdot (z+1)^2}{(z-p_0) \cdot (z-p_1) \cdot (z-p_2)} \right] \right] \quad (5.8.8)$$

simplifying: 
$$h10_k = \frac{\alpha_2}{\gamma_2} \cdot \left[ \lim_{z \rightarrow p_0} \left[ \frac{z^k \cdot (z+1)^2}{(z-p_1) \cdot (z-p_2)} \right] \dots \right. \\ \left. + \lim_{z \rightarrow p_1} \left[ \frac{z^k \cdot (z+1)^2}{(z-p_0) \cdot (z-p_2)} \right] \dots \right. \\ \left. + \lim_{z \rightarrow p_2} \left[ \frac{z^k \cdot (z+1)^2}{(z-p_0) \cdot (z-p_1)} \right] \right] \quad (5.8.9)$$

(1)  $p_0 = 0$

Calculation of  $\lim_{z \rightarrow 0} \left[ \frac{z^k \cdot (z+1)^2}{(z-p_1) \cdot (z-p_2)} \right]$  if  $k=0$   $\lim_{z \rightarrow 0} \left[ \frac{z^0 \cdot (z+1)^2}{(z-p_1) \cdot (z-p_2)} \right] = \frac{1}{p_1 \cdot p_2}$

$z := z \quad \beta_2 := \beta_2 \quad \gamma_2 := \gamma_2$

for  $k > 0$   $\lim_{z \rightarrow 0} \left[ \frac{z^k \cdot (z+1)^2}{(z-p_1) \cdot (z-p_2)} \right] = 0 \quad (5.8.10)$

$$l_0 = \frac{1}{p_1 \cdot p_2} \cdot \delta(k, 0) \quad \frac{1}{p_1 \cdot p_2} = \frac{1}{\gamma_2} \quad (5.8.11)$$

(2)  $p_0 = 0$

Calculation of  $\lim_{z \rightarrow p_1} \left[ \frac{z^k \cdot (z+1)^2}{(z-p_0) \cdot (z-p_2)} \right] = \frac{p_1^k \cdot (p_1+1)^2}{p_1 \cdot (p_1-p_2)}$

$$l_1 = \frac{p_1^k \cdot (p_1+1)^2}{p_1 \cdot (p_1-p_2)} \quad (5.8.12)$$

(3)  $p_0 = 0$

$$\text{Calculation of } \lim_{z \rightarrow p_2} \left[ \frac{z^k \cdot (z+1)^2}{(z-p_0) \cdot (z-p_1)} \right] = \frac{p_2^k \cdot (p_2+1)^2}{p_2 \cdot (p_2-p_1)}$$

$$|_2 = \frac{p_2^k \cdot (p_2+1)^2}{p_2 \cdot (p_2-p_1)} \quad (5.8.13)$$

The sequence is:

$$h_{10k} := \frac{\alpha_2}{\gamma_2} \cdot \left[ \frac{1}{p_1 \cdot p_2} \cdot \delta(k,0) + \frac{p_1^k \cdot (p_1+1)^2}{p_1 \cdot (p_1-p_2)} \dots + \frac{p_2^k \cdot (p_2+1)^2}{p_2 \cdot (p_2-p_1)} \right] \quad (5.8.14)$$

▾ Proof (5.8.14)

**Proof of (5.8.14):**

the z transform of (5.8.14) should be the given z transfer function here rewritten:

$$H_{lp}(z) = \alpha_2 \cdot \frac{1 + 2 \cdot z^{-1} + z^{-2}}{z^{-2} + \beta_2 \cdot z^{-1} + \gamma_2} = \frac{\alpha_2}{\gamma_2} \cdot \frac{(z+1)^2}{z^2 + \frac{\beta_2}{\gamma_2} \cdot z + \frac{1}{\gamma_2}} = \frac{\alpha_2}{\gamma_2} \cdot \frac{(z+1)^2}{(z-p_1) \cdot (z-p_2)}$$

z transform of each term of (5.8.14)

$$p_1 := p_1 \quad p_2 := p_2$$

$$\frac{1}{p_1 \cdot p_2} \cdot \delta(k,0) \text{ ztrans, } k \rightarrow \frac{1}{p_1 \cdot p_2}$$

$$\frac{p_1^k \cdot (p_1+1)^2}{p_1 \cdot (p_1-p_2)} \text{ ztrans, } k \rightarrow \frac{z \cdot (p_1+1)^2}{p_1 \cdot (p_1-p_2) \cdot (p_1-z)}$$

$$\frac{p_2^k \cdot (p_2+1)^2}{p_2 \cdot (p_2-p_1)} \text{ ztrans, } k \rightarrow \frac{z \cdot (p_2+1)^2}{p_2 \cdot (p_1-p_2) \cdot (p_2-z)}$$

$$\frac{\alpha_2}{\gamma_2} \cdot \mathcal{Z} \left[ \frac{1}{p_1 \cdot p_2} \cdot \delta(k,0) + \frac{p_1^k \cdot (p_1+1)^2}{p_1 \cdot (p_1-p_2)} \dots + \frac{p_2^k \cdot (p_2+1)^2}{p_2 \cdot (p_2-p_1)} \right] = \frac{\alpha_2}{\gamma_2} \cdot \left[ \frac{1}{p_1 \cdot p_2} - \frac{z \cdot (p_1+1)^2}{p_1 \cdot (p_1-p_2) \cdot (p_1-z)} + \frac{z \cdot (p_2+1)^2}{p_2 \cdot (p_1-p_2) \cdot (p_2-z)} \right]$$

$$\frac{1}{p_1 \cdot p_2} - \frac{z \cdot (p_1+1)^2}{p_1 \cdot (p_1-p_2) \cdot (p_1-z)} + \frac{z \cdot (p_2+1)^2}{p_2 \cdot (p_1-p_2) \cdot (p_2-z)} = \frac{(z+1)^2}{(p_1-z) \cdot (p_2-z)}$$

q.e.d.

▴ Proof (5.8.14)

**Second case:**  $\zeta = \omega_5$ ,  $T_{s\lambda} := 2 \cdot T_{sstp}$

$$\alpha_{21} := \frac{A_5 \cdot T_s^2 \cdot \omega_5^2}{(T_s \cdot \omega_5 - 2)^2} \quad \beta_{21} := \frac{T_s \cdot \omega_5 + 2}{(T_s \cdot \omega_5 - 2)}$$

$$H4(z) := \alpha_{21} \cdot \frac{(z^{-1} + 1)^2}{(z^{-1} + \beta_{21})^2} \quad (5.8.15)$$

$$\alpha_{21} = -2.0982304898 \quad \beta_{21} = -1.9161289188$$

$$z := z \quad \alpha_{21} := \alpha_{21} \quad \beta_{21} := \beta_{21}$$

$$\text{Define the function: } F20(z, k) = z^{k-1} \cdot H4(z) \quad (5.8.16)$$

$$F20(z, k) := \frac{\alpha_{21} \cdot z^k \cdot (z + 1)^2}{z \cdot (\beta_{21} \cdot z + 1)^2} \quad (5.8.17)$$

$$\text{Poles of } F20(z, k) = \frac{\alpha_{21} \cdot z^k \cdot (z + 1)^2}{z \cdot (\beta_{21} \cdot z + 1)^2}$$

$$\zeta = 0.018 \cdot \frac{\text{Grads}}{\text{sec}} \quad \omega_5 = 0.192 \cdot \frac{\text{Grads}}{\text{sec}}$$

$$\zeta = \omega_5 \quad z := z \quad \alpha_{21} := \alpha_{21} \quad \beta_{21} := \beta_{21} \quad \zeta := \zeta$$

$$\text{poles} := z \cdot (\beta_{21} \cdot z + 1)^2 \text{ solve, } z \rightarrow \begin{pmatrix} 0 \\ 1 \\ \beta_{21} \\ 1 \\ \beta_{21} \end{pmatrix}$$

$$\text{poles} = \begin{pmatrix} 0 \\ 0.522 \\ 0.522 \end{pmatrix} \quad p00 := \text{poles}_0 \quad p01 := \text{poles}_1$$

$$\text{First order pole: } p00 := 0 \quad \text{Second order pole: } p01 := \frac{1}{\beta_{21}}$$

$$r := \text{ceil}(\max(|\text{poles}|)) \cdot 1.0 \quad r = 1$$

$$\xi(t) := r \cdot \cos(t)$$

$$\psi(t) := r \cdot \sin(t)$$

$$\phi(t) := \xi(t) + j \cdot \psi(t)$$

$$t_0 := 0 \quad t_{fin} := 2 \cdot \pi$$

$$t := t_0, t_0 + \frac{t_{fin} - t_0}{100} \dots t_{fin}$$

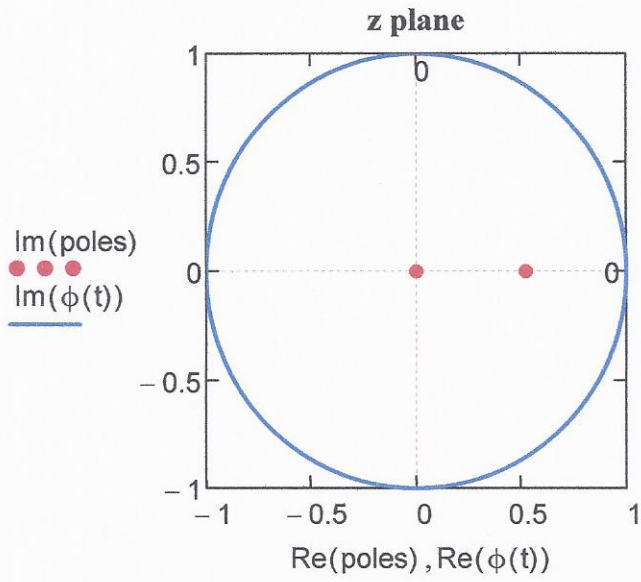


fig.:5.8.3

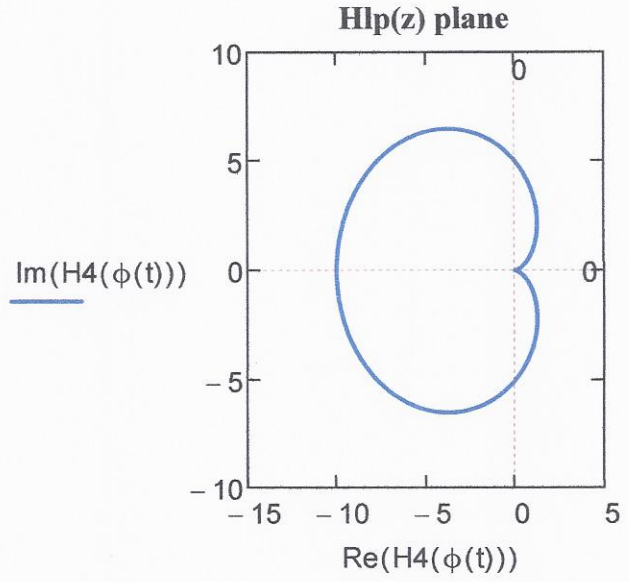


fig.:5.8.4



$$F_{20}(z, k) = \frac{\alpha_{21} \cdot z^k \cdot (z+1)^2}{\beta_{21} \cdot (z-p_{00}) \cdot (z-p_{01})^2} \quad (5.8.19)$$

$$h_{20k} = \text{Res}(F_{20}(z, k), p_{00}) + \text{Res}(F_{20}(z, k), p_{01}) \quad (5.8.20)$$

$$p_{00} = 0 \quad h_{20k} = \frac{\alpha_{21}}{\beta_{21}} \cdot \left[ \lim_{z \rightarrow p_{00}} \left[ (z-p_{00}) \cdot \frac{z^k \cdot (z+1)^2}{(z-p_{00}) \cdot (z-p_{01})^2} \right] \dots \right. \\ \left. + \lim_{z \rightarrow p_{01}} \left[ \frac{\partial}{\partial z} \left[ (z-p_{01})^2 \cdot \frac{z^k \cdot (z+1)^2}{(z-p_{00}) \cdot (z-p_{01})^2} \right] \right] \right] \quad (5.8.21)$$

namely, simplifying:

$$h_{20k} = \frac{\alpha_{21}}{\beta_{21}} \cdot \left[ \lim_{z \rightarrow 0} \left[ \frac{z^k \cdot (z+1)^2}{(z-p_{01})^2} \right] \dots \right. \\ \left. + \lim_{z \rightarrow p_{01}} \left[ \frac{\partial}{\partial z} \left[ \frac{z^k \cdot (z+1)^2}{(z-p_{00})} \right] \right] \right] \quad (5.8.22)$$

(1)  $\text{Res}(F_{20}(z, k), p_{00}) \quad p_{00} = 0$

For  $k=0$  and  $p_{01} \neq 0$

$$\lim_{z \rightarrow 0} \left[ \frac{z^0 \cdot (z+1)^2}{(z-p_{01})^2} \right] = \frac{\delta(k, 0)}{p_{01}^2} \quad (5.8.23)$$

For  $k>0$

$$\lim_{z \rightarrow 0} \left[ \frac{z^k \cdot (z+1)^2}{(z-p_{01})^2} \right] = \lim_{z \rightarrow 0} \left[ \frac{z^k \cdot (z+1)^2}{(z-p_{01})^2} \right] = 0 \quad (5.8.24)$$

$$\text{Res}(F_{20}(z, k), p_{00}) = \frac{\alpha_{21}}{\beta_{21}} \cdot \frac{\delta(k, 0)}{p_{01}^2}$$

(2)  $\text{Res}(F_{20}(z, k), p_{01}) \quad \lim_{z \rightarrow p_{01}} \left[ \frac{\partial}{\partial z} \left[ \frac{z^k \cdot (z+1)^2}{(z-p_{00})} \right] \right] \quad (5.8.25)$

$$p_{01} = -\frac{1}{\beta_{22}} \quad \frac{\partial}{\partial z} \left[ \frac{z^k \cdot (z+1)^2}{z} \right] = z^{k-2} \cdot (z+1) \cdot (k+z+k \cdot z-1) \quad (5.8.26)$$

For  $k=0$  and  $p_{01} \neq 0$

$$\lim_{z \rightarrow p_{01}} \left[ \frac{\partial}{\partial z} \left[ \frac{z^0 \cdot (z+1)^2}{z} \right] \right] = \lim_{z \rightarrow p_{01}} \left[ \frac{\partial}{\partial z} \left[ \frac{(z+1)^2}{z} \right] \right] \\ \frac{\partial}{\partial z} \left[ \frac{(z+1)^2}{z} \right] \text{ simplify, max } \rightarrow 1 - \frac{1}{z^2}$$

$$\lim_{z \rightarrow p01} \left[ \frac{\partial}{\partial z} \left[ \frac{(z+1)^2}{z} \right] \right] = \left( 1 - \frac{1}{p01^2} \right) \cdot \delta(k, 0)$$

$$\text{Res}(F20(z, k), p01) = \frac{\alpha_{21}}{\beta_{21}} \cdot \lim_{z \rightarrow p01} \left[ z^{k-2} \cdot (z+1) \cdot (k+z+k \cdot z - 1) \right] \quad (5.8.27)$$

$$\text{Res}(F20(z, k), p01) = \frac{\alpha_{21}}{\beta_{21}} \cdot p01^{k-2} \cdot (p01+1) \cdot [k + p01 \cdot (1+k) - 1] \quad (5.8.28)$$

$$h20_k = \frac{\alpha_{21}}{\beta_{21}} \cdot \left[ \frac{\delta(k, 0)}{p01^2} + \left( 1 - \frac{1}{p01^2} \right) \cdot \delta(k, 0) + p01^{k-2} \cdot (p01+1) \cdot [k + p01 \cdot (1+k) - 1] \right]$$

$$h20_k = \frac{\alpha_{21}}{\beta_{21}} \cdot [\delta(k, 0) + p01^{k-2} \cdot (p01+1) \cdot [k + p01 \cdot (1+k) - 1]] \quad (5.8.29)$$

▣ Proof (5.8.29)

**Proof of (5.8.29):**

the z transform of (5.8.29) should be the given z transfer function here rewritten:

$$H4(z) = \alpha_{21} \cdot \frac{(z^{-1} + 1)^2}{(z^{-1} + \beta_{21})^2} = \frac{\alpha_{21} \cdot (z+1)^2}{(\beta_{21} \cdot z + 1)^2}$$

$$\frac{\alpha_{21} \cdot (\beta_{21} - 1) \cdot [\beta_{21} \cdot (k-1) - k - 1]}{\beta_{21}^2} \cdot \left( -\frac{1}{\beta_{21}} \right)^k + \alpha_{21} \cdot \delta(k, 0) \quad \left| \begin{array}{l} \text{ztrans, k} \\ \text{simplify, max} \end{array} \right. \rightarrow \frac{\alpha_{21} \cdot (z+1)^2}{(\beta_{21} \cdot z + 1)^2}$$

▣ Proof (5.8.29)

Finally, the result considering both cases:  $\zeta \neq \omega_5$  and  $\zeta = \omega_5$ , is:

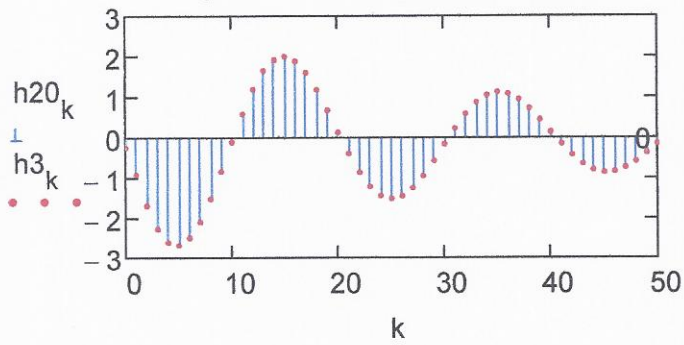
$$h20_k := \begin{cases} \frac{\alpha_2}{\gamma_2} \cdot \left[ \delta(k, 0) + \frac{p1^{k-1} \cdot (p1+1)^2}{(p1-p2)} \dots + \frac{p2^{k-1} \cdot (p2+1)^2}{(p2-p1)} \right] & \text{if } \zeta \neq \omega_5 \\ \frac{\alpha_{21}}{\beta_{21}} \cdot [\delta(k, 0) + p01^{k-2} \cdot (p01+1) \cdot [k + p01 \cdot (1+k) - 1]] & \text{otherwise} \end{cases} \quad (5.8.30)$$

$$\omega_5 = 0.192 \cdot \frac{\text{Grads}}{\text{sec}} \quad \zeta = 0.018 \cdot \frac{\text{Grads}}{\text{sec}}$$

$$\zeta = 0.018 \cdot \frac{\text{Grads}}{\text{sec}} \quad \omega_5 = 0.192 \cdot \frac{\text{Grads}}{\text{sec}}$$

$h20^T =$	0	1	2	3	4	5	6	
	0	-0.22	-0.902	-1.682	-2.262	-2.598	-2.672	...

Digital Pulse Response Sequence



Digital Pulse Response Sequence

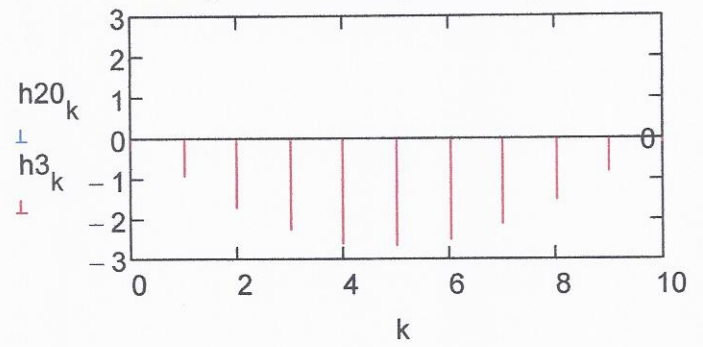


fig.:5.8.5

Stability ( $S31 < \infty$ ):

$$S31 := \sum_{k=0}^{\text{rows}(h1)-1} |h20_k| \quad S31 = 69.057$$

Energy of the sequence h1:

$$E31 := \sum_{k=0}^{\text{rows}(h1)-1} (|h20_k|)^2 \quad E31 = 82.831$$

$$T_s = 3.279 \cdot \text{ns} \quad T_{\text{sstp}} = 1.639 \cdot \text{ns}$$

$$y3_\nu := \sum_{k=0}^{N0-1} (\text{if}(\nu - k \geq 0, h20_k \cdot u1_{\nu-k}, 0)) \quad (5.8.31)$$

Redefine the output waveform :

$$y_{\text{sr}}(t) := A_5 \cdot V_{\text{pp}} \cdot \begin{cases} g_{\text{sr}}(t, A_5, \zeta, \omega_5) \cdot \Phi(t) & \text{if } \zeta \neq \omega_5 \\ [1 - e^{-t \cdot \omega_5} \cdot (t \cdot \omega_5 + 1)] \cdot \Phi(t) & \text{otherwise} \end{cases} \quad (5.2.1.1)$$

$$t := 0 \cdot T_{\text{test}}, 0 \cdot T_{\text{test}} + \frac{20 \cdot T_5 - 0 \cdot T_{\text{test}}}{1000} .. 20 \cdot T_5 \quad V_{\text{pp}} = 5V$$

Input Sequence

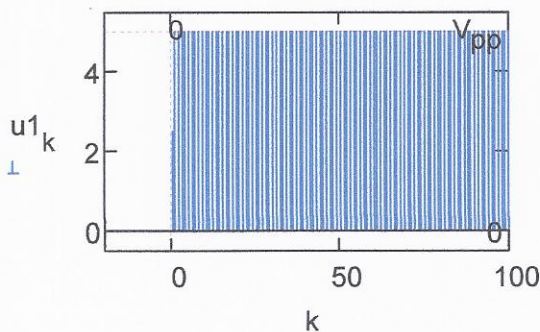


fig.:5.8.6

Output Sequence

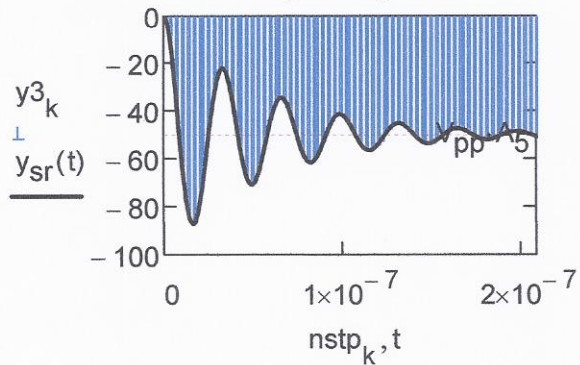
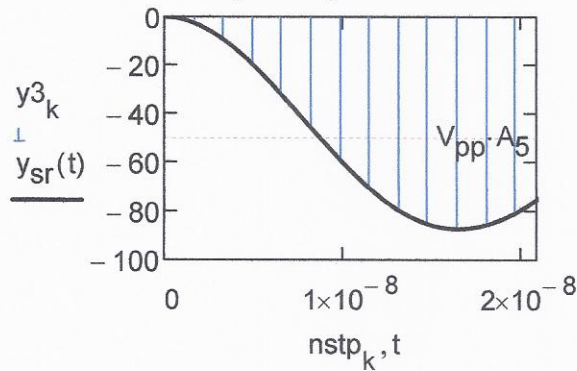


fig.:5.8.7



### Output Sequence Details



$$y3^T = \begin{array}{c|cccccccc|c} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & V \\ \hline 0 & 0 & -3.357 & -9.817 & -19.677 & -31.827 & -45.001 & -57.91 & -69.359 & \dots & \end{array}$$

### Calculation of the filter output as the inverse z transform of $H(z)V_f(z)$

$$\text{Filter's } z \text{ transfer function } H_{lp}(z) = \alpha_2 \cdot \frac{1 + 2 \cdot z^{-1} + z^{-2}}{z^{-2} + \beta_2 \cdot z^{-1} + \gamma_2} = \frac{\alpha_2}{\gamma_2} \cdot \frac{(z+1)^2}{z^2 + \frac{\beta_2}{\gamma_2} \cdot z + \frac{1}{\gamma_2}} \quad (5.8.32)$$

$$\zeta \neq \omega_5 \quad \zeta = 0.018 \cdot \frac{\text{Grads}}{\text{sec}} \quad \omega_5 = 0.192 \cdot \frac{\text{Grads}}{\text{sec}}$$

$$z := z \quad \beta_2 := \beta_2 \quad \gamma_2 := \gamma_2 \quad \zeta := \zeta$$

$$\text{poles2} := z^2 + \frac{\beta_2}{\gamma_2} \cdot z + \frac{1}{\gamma_2} \quad \left| \begin{array}{l} \text{solve, } z \\ \text{simplify} \end{array} \right. \rightarrow \begin{pmatrix} \frac{\beta_2 - \sqrt{\beta_2^2 - 4 \cdot \gamma_2}}{2 \cdot \gamma_2} \\ \frac{\beta_2 + \sqrt{\beta_2^2 - 4 \cdot \gamma_2}}{2 \cdot \gamma_2} \end{pmatrix}$$

$$\text{poles2} = \begin{pmatrix} 0.926 + 0.297j \\ 0.926 - 0.297j \end{pmatrix} \quad p0 := \text{poles2}_0 \quad p1 := \text{poles2}_1$$

$$p0 := -\frac{\beta_2 - \sqrt{\beta_2^2 - 4 \cdot \gamma_2}}{2 \cdot \gamma_2} \quad p1 := -\frac{\beta_2 + \sqrt{\beta_2^2 - 4 \cdot \gamma_2}}{2 \cdot \gamma_2} \quad (5.8.33)$$

$$p0 = 0.926 + 0.297j \quad p1 = 0.926 - 0.297j$$

Step response:

$$Y3(z) = H_{lp}(z) \cdot X(z) \quad (5.8.34)$$

of the l. p. filter:



$$V_{pp} := V_{pp} \quad V_{pp} \text{ ztrans} \rightarrow \frac{V_{pp} \cdot z}{z-1} \quad z \text{ transform of the input } X(z) := \frac{V_{pp} \cdot z}{z-1} \quad (5.8.35)$$

$$\alpha_2 := \alpha_2 \quad \beta_2 := \beta_2 \quad \gamma_2 := \gamma_2 \quad p_0 := p_0 \quad p_1 := p_1$$

$$p_0 = 0.926 + 0.297j$$

$$p_1 = 0.926 - 0.297j$$

First case  $\zeta \neq \omega_5$

$$y_{330k} := \frac{\alpha_2}{\gamma_2} \cdot \frac{(z+1)^2}{(z-p_0) \cdot (z-p_1)} \cdot \frac{V_{pp} \cdot z}{z-1} \quad \left| \begin{array}{l} \text{invztrans, z, k} \\ \text{simplify} \\ \text{factor} \end{array} \right. \rightarrow$$

$$y_{330k} := V_{pp} \cdot \frac{\alpha_2}{\gamma_2} \cdot \left[ \frac{(p_0+1)^2 \cdot p_0^k}{(p_0-p_1) \cdot (p_0-1)} - \frac{p_1^k \cdot (p_1+1)^2}{(p_0-p_1) \cdot (p_1-1)} + \frac{4}{(p_0-1) \cdot (p_1-1)} \right] \quad (5.8.36)$$

Proof

**Proof of (5.8.36):**

the z transform of (5.8.36) should be the given z transform of the response here rewritten:

$$Y_3(z) = V_{pp} \cdot \frac{\alpha_2}{\gamma_2} \cdot \frac{(z+1)^2}{(z-p_0) \cdot (z-p_1)} \cdot \frac{z}{z-1}$$

$$p_0 := p_0 \quad p_1 := p_1$$

$$\frac{(p_0+1)^2 \cdot p_0^k}{(p_0-p_1) \cdot (p_0-1)} \text{ ztrans, k} \rightarrow \frac{z \cdot (p_0+1)^2}{(p_0-p_1) \cdot (p_0-z) \cdot (p_0-1)}$$

$$\mathcal{Z} \left[ \frac{(p_0+1)^2 \cdot p_0^k}{(p_0-p_1) \cdot (p_0-1)} \right] = \frac{z \cdot (p_0+1)^2}{(p_0-p_1) \cdot (p_0-z) \cdot (p_0-1)}$$

$$p_0 := p_0 \quad p_1 := p_1$$

$$\frac{p_1^k \cdot (p_1+1)^2}{(p_0-p_1) \cdot (p_1-1)} \text{ ztrans, k} \rightarrow \frac{z \cdot (p_1+1)^2}{(p_0-p_1) \cdot (p_1-z) \cdot (p_1-1)}$$

$$\mathcal{Z} \left[ \frac{p_1^k \cdot (p_1+1)^2}{(p_0-p_1) \cdot (p_1-1)} \right] = \frac{z \cdot (p_1+1)^2}{(p_0-p_1) \cdot (p_1-z) \cdot (p_1-1)}$$

$$\mathcal{Z} \left[ \frac{4}{(p_0-1) \cdot (p_1-1)} \right] = \frac{4 \cdot z}{(p_0-1) \cdot (p_1-1) \cdot (z-1)}$$

$$Y_3(z) = V_{pp} \cdot \frac{\alpha_2}{\gamma_2} \cdot \left[ \frac{z \cdot (p_0+1)^2}{(p_0-p_1) \cdot (p_0-z) \cdot (p_0-1)} + \frac{z \cdot (p_1+1)^2}{(p_0-p_1) \cdot (p_1-z) \cdot (p_1-1)} + \frac{4 \cdot z}{(p_0-1) \cdot (p_1-1) \cdot (z-1)} \right]$$

$$p0 := p0$$

$$p1 := p1$$

$$\left[ \begin{array}{l} \frac{z \cdot (p0 + 1)^2}{(p0 - p1) \cdot (p0 - z) \cdot (p0 - 1)} \dots \\ + \frac{z \cdot (p1 + 1)^2}{(p0 - p1) \cdot (p1 - z) \cdot (p1 - 1)} \dots \\ + \left[ \frac{4 \cdot z}{(p0 - 1) \cdot (p1 - 1) \cdot (z - 1)} \right] \end{array} \right] \text{ simplify } \rightarrow \frac{z \cdot (z + 1)^2}{(p0 - z) \cdot (p1 - z) \cdot (z - 1)}$$

$$\alpha_2 = -0.248$$

$$Y3(z) = V_{pp} \cdot \frac{\alpha_2}{\gamma_2} \cdot \frac{z \cdot (z + 1)^2}{(p0 - z) \cdot (p1 - z) \cdot (z - 1)}$$

$$\gamma_2 = 1.058$$

q.e.d.

▲ Proof

Second case:  $\zeta = \omega_5$

$$A_5 = -10$$

$$\alpha_2 := \frac{A_5 \cdot T_s^2 \cdot \omega_5^2}{(T_s \cdot \omega_5 - 2)^2}$$

$$\gamma_2 := \frac{T_s \cdot \omega_5 + 2}{(T_s \cdot \omega_5 - 2)}$$

$$\alpha_2 = -2.098$$

$$\gamma_2 = -1.916$$

$$y_{331k} := \left[ \alpha_2 \cdot \frac{(z^{-1} + 1)^2}{(z^{-1} + \gamma_2)^2} \right] \cdot \frac{V_{pp} \cdot z}{z - 1} \begin{array}{l} \text{invztrans, z, k} \\ \text{simplify} \\ \text{factor} \\ \text{collect, } \left( \frac{1}{\gamma_2} \right)^k \end{array} \rightarrow$$

$$y_{331k} := \left[ 4 + \frac{\left( \frac{1}{\gamma_2} \right)^k \cdot [k \cdot [\gamma_2 \cdot [\gamma_2 \cdot (\gamma_2 - 1) - 1] + 1] - \gamma_2 \cdot (3 \cdot \gamma_2 - 2) + 1]}{\gamma_2^2} \right] \cdot \frac{V_{pp} \cdot \alpha_2}{(\gamma_2 + 1)^2} \quad (5.8.37)$$

▼ Proof

**Proof of (5.8.37):**

the z transform of (5.8.37) should be the given z transform of the response for  $\zeta = \omega_5$ , here rewritten:

$$y_{331}(z) = \alpha_2 \cdot \frac{(z^{-1} + 1)^2}{(z^{-1} + \gamma_2)^2} \cdot \frac{V_{pp} \cdot z}{z - 1} = \frac{V_{pp} \cdot \alpha_2 \cdot z \cdot (z + 1)^2}{(\gamma_2 \cdot z + 1)^2 \cdot (z - 1)}$$

$$\gamma_2 := \gamma_2 \quad \alpha_2 := \alpha_2 \quad V_{pp} := V_{pp}$$

$$\left[ 4 + \frac{\left(\frac{-1}{\gamma_2}\right)^k \cdot \left[ k \cdot \left[ \gamma_2 \cdot \left[ \gamma_2 \cdot (\gamma_2 - 1) - 1 \right] + 1 \right] \dots \right]}{\gamma_2^2} \right] \cdot \frac{V_{pp} \cdot \alpha_2}{(\gamma_2 + 1)^2} \text{ztrans, } k \rightarrow \frac{V_{pp} \cdot \alpha_2 \cdot z \cdot (z + 1)^2}{(\gamma_2 \cdot z + 1)^2 \cdot (z - 1)}$$

$$\frac{V_{pp} \cdot \alpha_2}{(\gamma_2 + 1)^2} \cdot z \left[ 4 + \frac{\left(\frac{-1}{\gamma_2}\right)^k \cdot \left[ k \cdot \left[ \gamma_2 \cdot \left[ \gamma_2 \cdot (\gamma_2 - 1) - 1 \right] + 1 \right] \dots \right]}{\gamma_2^2} \right] = \frac{V_{pp} \cdot \alpha_2 \cdot z \cdot (z + 1)^2}{(\gamma_2 \cdot z + 1)^2 \cdot (z - 1)}$$

q.e.d.

□ Proof

Finally, the result, considering both cases:  $\zeta \neq \omega_5$  and  $\zeta = \omega_5$ , is:

$$Q_5 = 5.4 \quad \zeta = 0.018 \cdot \frac{\text{Grads}}{\text{sec}} \quad \omega_5 = 0.192 \cdot \frac{\text{Grads}}{\text{sec}}$$

$$y_{33k} := V_{pp} \cdot \left[ \frac{\alpha_2}{\gamma_2} \cdot \left[ \frac{(p_0 + 1)^2 \cdot p_0^k}{(p_0 - p_1) \cdot (p_0 - 1)} - \frac{p_1^k \cdot (p_1 + 1)^2}{(p_0 - p_1) \cdot (p_1 - 1)} + \frac{4}{(p_0 - 1) \cdot (p_1 - 1)} \right] \text{ if } \zeta \neq \omega_5 \right. \quad (5.8.37)$$

$$\left. \left[ \frac{\left(\frac{-1}{\gamma_2}\right)^k \cdot \left[ k \cdot \left[ \gamma_2 \cdot \left[ \gamma_2 \cdot (\gamma_2 - 1) - 1 \right] + 1 \right] \dots \right]}{4 + \frac{\gamma_2^2}}{\gamma_2^2} \right] \cdot \frac{\alpha_2}{(\gamma_2 + 1)^2} \text{ otherwise} \right]$$

$$V_{pp} \cdot \alpha_2 = -10.491 \text{ V} \quad \zeta = 0.018 \cdot \frac{\text{Grads}}{\text{sec}} \quad \omega_5 = 0.192 \cdot \frac{\text{Grads}}{\text{sec}} \quad Q_5 = 5.4$$

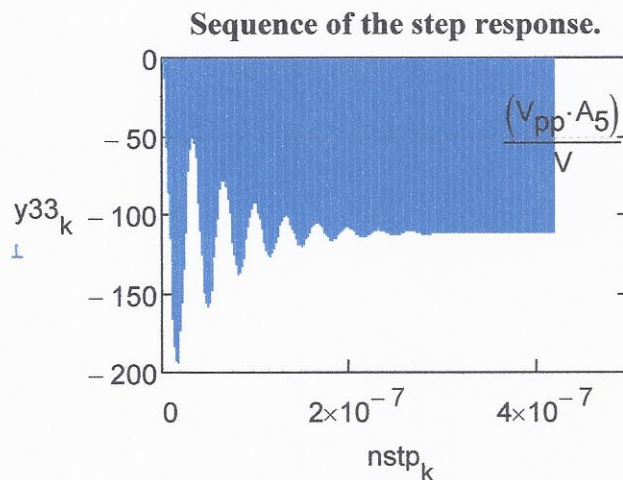


fig.:5.8.8



**Sinusoidal response:**  $Y_{out}(z) = H_{lp}(z) \cdot X(z)$  (5.8.38)

Signal amplitude:  $V_m := V_{pp}$ ,

Signal frequency:  $f_{test} = 60.997 \cdot \text{MHz}$ ,

arbitrary sampling frequency:  $f_s := 10 \cdot f_{test}$ ,  $f_s = 609.968 \cdot \text{MHz}$ , (5.8.)

sampling angular frequency:  $\omega_s := 2 \cdot \pi \cdot f_s$ ,  $\omega_s = 3.833 \cdot \frac{\text{Grads}}{\text{sec}}$ ,

sampling period:  $T_s := \frac{1}{f_s}$ ,  $T_s = 1.639 \cdot \text{ns}$ ,

sampling time step:  $n_k := \frac{k}{f_s}$ ,  $N_0 = 256$  (5.8.)

$\frac{N_0}{f_s} \cdot f_5 = 12.8$   $N_0 = 256$  .

(5.8.)

L. p. filter Input:  $x_{2k} := V_m \cdot \sin(\omega_{test} \cdot n_k)$  (5.8.39)

$n^T =$		0	1	2	3	4	5	6	s
	0	0	$1.639 \cdot 10^{-9}$	$3.279 \cdot 10^{-9}$	$4.918 \cdot 10^{-9}$	$6.558 \cdot 10^{-9}$	$8.197 \cdot 10^{-9}$	...	

$x_{2k}^T =$		0	1	2	3	4	5	V
	0	0	2.939	4.755	4.755	2.939	...	

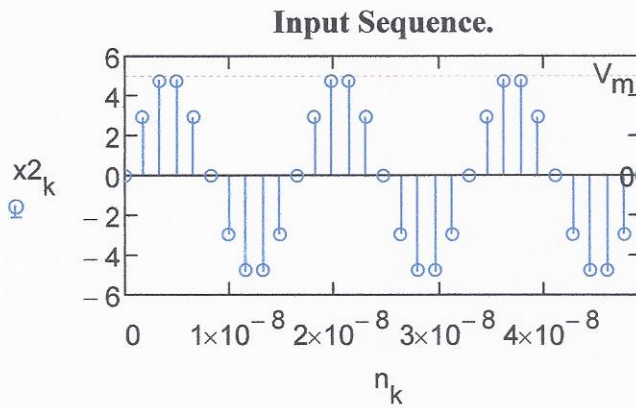


fig.:5.8.9

Z transform of the input signal:

$V_m := V_m$   $\omega_{test} := \omega_{test}$   $T_s := T_s$   $\nu := \nu$

$X(z) := V_m \cdot \sin(\nu \cdot \omega_{test} \cdot T_s) \xrightarrow{\text{ztrans}, \nu} \frac{V_m \cdot z \cdot \sin(T_s \cdot \omega_{test})}{z^2 - 2 \cdot \cos(T_s \cdot \omega_{test}) \cdot z + 1}$   $N1 = 4$



$$X(z) = \frac{V_m \cdot z \cdot \sin(T_s \cdot \omega_{\text{test}})}{z^2 - 2 \cdot \cos(T_s \cdot \omega_{\text{test}}) \cdot z + 1} \quad (5.8.40)$$

$$K := \sin(T_s \cdot \omega_{\text{test}}) \quad \sqrt{1 - K^2} = 0.809 \quad K = 0.588$$

Transfer function:

$$W_{lp}(s) := \begin{cases} \frac{A_5 \cdot \omega_5^2}{s^2 + 2 \cdot \zeta \cdot s + \omega_5^2} & \text{if } \zeta \neq \omega_5 \\ A_5 \cdot \frac{\omega_5^2}{(s + \omega_5)^2} & \text{otherwise} \end{cases} \quad (5.1.7.1)$$

Coefficients of the z transfer function:

$$\alpha_3 := \begin{cases} \frac{A_5 \cdot \omega_5^2 \cdot T_s^2}{T_s^2 \cdot \omega_5^2 - 4 \cdot \zeta \cdot T_s + 4} & \text{if } \zeta \neq \omega_5 \\ \frac{A_5 \cdot T_s^2 \cdot \omega_5^2}{(T_s \cdot \omega_5 - 2)^2} & \text{otherwise} \end{cases}, \quad \beta_3 := \begin{cases} \frac{2 \cdot T_s^2 \cdot \omega_5^2 - 8}{T_s^2 \cdot \omega_5^2 - 4 \cdot \zeta \cdot T_s + 4} & \text{if } \zeta \neq \omega_5 \\ \frac{T_s^2 \cdot \omega_5^2 - 4}{2 \cdot (T_s \cdot \omega_5 - 2)^2} & \text{otherwise} \end{cases}$$

$$\gamma_3 := \begin{cases} \frac{T_s^2 \cdot \omega_5^2 + 4 \cdot \zeta \cdot T_s + 4}{T_s^2 \cdot \omega_5^2 - 4 \cdot \zeta \cdot T_s + 4} & \text{if } \zeta \neq \omega_5 \\ \frac{(T_s \cdot \omega_5 + 2)^2}{(T_s \cdot \omega_5 - 2)^2} & \text{otherwise} \end{cases}$$

$$\alpha_3 = -0.248$$

$$\beta_3 = -1.959$$

$$\gamma_3 = 1.058$$

Z transfer function:

$$H_{lp}(z) := \begin{cases} \alpha_3 \cdot \frac{1 + 2 \cdot z^{-1} + z^{-2}}{z^{-2} + \beta_3 \cdot z^{-1} + \gamma_3} & \text{if } \zeta \neq \omega_5 \\ \alpha_3 \cdot \frac{(z^{-1} + 1)^2}{(z^{-1} + \beta_3)^2} & \text{otherwise} \end{cases}$$

System Output

$$Y_{\text{out}}(z) = \alpha_3 \cdot \begin{cases} \frac{1 + 2 \cdot z^{-1} + z^{-2}}{z^{-2} + \beta_3 \cdot z^{-1} + \gamma_3} \cdot \frac{V_m \cdot z^{-1} \cdot K}{1 - 2 \cdot \sqrt{1 - K^2} \cdot z^{-1} + z^{-2}} & \text{if } \zeta \neq \omega_5 \\ \frac{(z^{-1} + 1)^2}{(z^{-1} + \beta_3)^2} \cdot \left( \frac{V_m \cdot z^{-1} \cdot K}{1 - 2 \cdot \sqrt{1 - K^2} \cdot z^{-1} + z^{-2}} \right) & \text{otherwise} \end{cases} \quad (5.8.41)$$

Search of the sequence corresponding to the output:

**First case:**  $\zeta \neq \omega_5$

$$Y_{\text{out}}(z) = \frac{\alpha_3 \cdot V_m \cdot z^{-1} \cdot K \cdot (1 + 2 \cdot z^{-1} + z^{-2})}{(z^{-2} + \beta_3 \cdot z^{-1} + \gamma_3) \cdot (z^2 - 2 \cdot \sqrt{1 - K^2} \cdot z + 1)} \quad (5.8.42)$$

$$Y_{\text{out}}(z) = \frac{K \cdot V_m \cdot z \cdot \alpha_3 \cdot (z + 1)^2}{\gamma_3 \cdot z^4 + (\beta_3 - 2 \cdot \gamma_3 \cdot \sqrt{1 - K^2}) \cdot z^3 + (\gamma_3 - 2 \cdot \beta_3 \cdot \sqrt{1 - K^2} + 1) \cdot z^2 + (\beta_3 - 2 \cdot \sqrt{1 - K^2}) \cdot z + 1}$$

$$\alpha_3 = -0.248$$

$$\beta_3 = -1.959$$

$$\gamma_3 = 1.058$$

Search of the poles:

$$\text{poles3} := \gamma_3 \cdot z^4 + (\beta_3 - 2 \cdot \gamma_3 \cdot \sqrt{1 - K^2}) \cdot z^3 \dots \text{solve}, z \rightarrow$$

$$+ (\gamma_3 - 2 \cdot \beta_3 \cdot \sqrt{1 - K^2} + 1) \cdot z^2 \dots$$

$$+ (\beta_3 - 2 \cdot \sqrt{1 - K^2}) \cdot z + 1$$

$$\text{poles3} := \begin{pmatrix} \sqrt{1 - K^2} + K \cdot j \\ \sqrt{1 - K^2} - K \cdot j \\ \frac{\beta_3 + \sqrt{\beta_3^2 - 4 \cdot \gamma_3}}{2} \\ \gamma_3 \\ \frac{\beta_3 - \sqrt{\beta_3^2 - 4 \cdot \gamma_3}}{2} \\ \gamma_3 \end{pmatrix}$$

$$p_1 := \sqrt{1 - K^2} + K \cdot j \quad p_2 := \sqrt{1 - K^2} - K \cdot j \quad p_3 := \frac{\beta_3 + \sqrt{\beta_3^2 - 4 \cdot \gamma_3}}{2} \\ p_4 := \frac{\beta_3 - \sqrt{\beta_3^2 - 4 \cdot \gamma_3}}{2}$$

$$p_1 = 0.809 + 0.588j \quad p_2 = 0.809 - 0.588j \quad p_3 = 0.926 - 0.297j \quad p_4 = 0.926 + 0.297j$$

Thanks to the fundamental theorem of Algebra one can write:

$$Y_{\text{out}}(z) = \frac{K \cdot V_m \cdot z \cdot \alpha_3 \cdot (z + 1)^2}{\gamma_3 \cdot (z - p_1) \cdot (z - p_2) \cdot (z - p_3) \cdot (z - p_4)} \quad (5.8.43)$$

In order to calculate the inverse z transform of the output signal, decompose the (5.8.43) in partial fraction

$$\frac{z \cdot (z+1)^2}{(z-p_1) \cdot (z-p_2) \cdot (z-p_3) \cdot (z-p_4)} \text{ parfrac, } z \rightarrow$$

and rewrite the output signal as a linear combination of terms like  $\frac{1}{p_n - z}$ :

$$Y_{\text{out}}(z) = \frac{\alpha_3 \cdot K \cdot V_m}{\gamma_3} \cdot \left[ \begin{array}{l} \frac{p_3 \cdot (p_3 + 1)^2}{(p_3 - p_1) \cdot (p_2 - p_3) \cdot (p_3 - p_4) \cdot (p_3 - z)} \dots \\ + \frac{p_1 \cdot (p_1 + 1)^2}{(p_2 - p_1) \cdot (p_1 - p_3) \cdot (p_1 - p_4) \cdot (p_1 - z)} \dots \\ + \frac{p_2 \cdot (p_2 + 1)^2}{(p_1 - p_2) \cdot (p_2 - p_3) \cdot (p_2 - p_4) \cdot (p_2 - z)} \dots \\ + \frac{p_4 \cdot (p_4 + 1)^2}{(p_4 - p_1) \cdot (p_2 - p_4) \cdot (p_3 - p_4) \cdot (p_4 - z)} \end{array} \right] \quad (5.8.44)$$

the poles are:

$$p_1 = 0.809 + 0.588j \quad p_2 = 0.809 - 0.588j \quad p_3 = 0.926 - 0.297j \quad p_4 = 0.926 + 0.297j$$

$$p_1 := p_1 \quad p_2 := p_2 \quad p_3 := p_3 \quad p_4 := p_4$$

Furthermore, to simplify calculations, define the following constants:

$$G1 := \begin{cases} \frac{p_3 \cdot (p_3 + 1)^2}{(p_3 - p_1) \cdot (p_2 - p_3) \cdot (p_3 - p_4)} & \text{if } \zeta \neq \omega_5 \\ 0 & \text{otherwise} \end{cases}$$

$$G2 := \begin{cases} \frac{p_1 \cdot (p_1 + 1)^2}{(p_2 - p_1) \cdot (p_1 - p_3) \cdot (p_1 - p_4)} & \text{if } \zeta \neq \omega_5 \\ 0 & \text{otherwise} \end{cases}$$

$$G3 := \begin{cases} \frac{p_2 \cdot (p_2 + 1)^2}{(p_1 - p_2) \cdot (p_2 - p_3) \cdot (p_2 - p_4)} & \text{if } \zeta \neq \omega_5 \\ 0 & \text{otherwise} \end{cases}$$

$$G4 := \begin{cases} \frac{p_4 \cdot (p_4 + 1)^2}{(p_4 - p_1) \cdot (p_2 - p_4) \cdot (p_3 - p_4)} & \text{if } \zeta \neq \omega_5 \\ 0 & \text{otherwise} \end{cases}$$

$$Y_{\text{out}}(z) = \frac{\alpha_3 \cdot K \cdot V_m}{\gamma_3} \cdot \left[ \begin{array}{l} \frac{G_1}{(p_3 - z)} \dots \\ + \frac{G_2}{(p_1 - z)} \dots \\ + \frac{G_3}{(p_2 - z)} \dots \\ + \frac{G_4}{(p_4 - z)} \end{array} \right] \quad (5.8.44)$$

The inverse z transform  $z^{-1} \left( \frac{1}{p-z} \right) = \frac{\delta(k,0) - p^k}{p}$

obtaining, after the substitution in (5.8.44):

$$Y_{\text{out}k} = \frac{\alpha_3 \cdot K \cdot V_m}{\gamma_3} \cdot \left[ \begin{array}{l} G_1 \cdot \left( \frac{\delta(k,0) - p_3^k}{p_3} \right) \dots \\ + G_2 \cdot \left( \frac{\delta(k,0) - p_1^k}{p_1} \right) \dots \\ + G_3 \cdot \left( \frac{\delta(k,0) - p_2^k}{p_2} \right) \dots \\ + G_4 \cdot \left( \frac{\delta(k,0) - p_4^k}{p_4} \right) \end{array} \right]$$

and collecting the Kronecker  $\delta$ , finally results:

$$\left( \frac{G_2}{p_1} + \frac{G_1}{p_3} + \frac{G_3}{p_2} + \frac{G_4}{p_4} \right) \cdot \delta(k,0) - \left( G_2 \cdot p_1^{k-1} + G_1 \cdot p_3^{k-1} + G_3 \cdot p_2^{k-1} + G_4 \cdot p_4^{k-1} \right)$$

$$G_5 := \frac{G_2}{p_1} + \frac{G_1}{p_3} + \frac{G_3}{p_2} + \frac{G_4}{p_4}$$

$$Y_{\text{out}k} = \frac{\alpha_3 \cdot K \cdot V_m}{\gamma_3} \cdot \left[ G_5 \cdot \delta(k,0) - \left( G_2 \cdot p_1^{k-1} + G_1 \cdot p_3^{k-1} + G_3 \cdot p_2^{k-1} + G_4 \cdot p_4^{k-1} \right) \right]$$

**Second case:**  $\zeta = \omega_5$

$$Y_{\text{out}}(z) = \frac{(z^{-1} + 1)^2}{(z^{-1} + \beta_3)^2} \cdot \frac{V_m \cdot z^{-1} \cdot K}{1 - 2 \cdot \sqrt{1 - K^2} \cdot z^{-1} + z^{-2}} = \frac{K \cdot V_m \cdot \alpha_3 \cdot (z + 1)^2}{(\beta_3 \cdot z + 1)^2 \cdot (z^2 - 2 \cdot z \cdot \sqrt{1 - K^2} + 1)}$$

$$Y_{\text{out}}(z) = \frac{K \cdot V_m \cdot \alpha_3 \cdot (z + 1)^2}{(\beta_3 \cdot z + 1)^2 \cdot (z^2 - 2 \cdot z \cdot \sqrt{1 - K^2} + 1)} \quad (5.8.48)$$



$$Y_{out}(z) = \frac{K \cdot V_m \cdot \alpha_3 \cdot (z+1)^2}{\beta_3^2 \cdot z^4 + (2 \cdot \beta_3 - 2 \cdot \beta_3^2 \cdot \sqrt{1-K^2}) \cdot z^3 \dots} \quad (5.8.49)$$

$$+ (\beta_3^2 - 4 \cdot \beta_3 \cdot \sqrt{1-K^2} + 1) \cdot z^2 \dots$$

$$+ (2 \cdot \beta_3 - 2 \cdot \sqrt{1-K^2}) \cdot z + 1$$

$$\alpha_3 = -0.248 \quad \beta_3 = -1.959 \quad \gamma_3 = 1.058$$

Search of the poles:

$$\text{poles3} := \beta_3^2 \cdot z^4 + (2 \cdot \beta_3 - 2 \cdot \beta_3^2 \cdot \sqrt{1-K^2}) \cdot z^3 \dots \text{ solve, } z \rightarrow \quad \text{poles4} := \begin{pmatrix} \frac{1}{\beta_3} \\ \frac{1}{\beta_3} \\ \sqrt{1-K^2} + K \cdot j \\ \sqrt{1-K^2} - K \cdot j \end{pmatrix}$$

$$+ (\beta_3^2 - 4 \cdot \beta_3 \cdot \sqrt{1-K^2} + 1) \cdot z^2 \dots$$

$$+ (2 \cdot \beta_3 - 2 \cdot \sqrt{1-K^2}) \cdot z + 1$$

$$p_{11} := -\frac{1}{\beta_3}$$

$$p_{22} := -\frac{1}{\beta_3}$$

$$p_{33} := \sqrt{1-K^2} + K \cdot j$$

$$p_{44} := \sqrt{1-K^2} - K \cdot j$$

Thanks to the fundamental theorem of Algebra one can write:

$$Y_{out}(z) = \frac{\alpha_3 \cdot K \cdot V_m \cdot (z+1)^2}{\beta_3^2 \cdot (z-p_{11})^2 \cdot (z-p_{33}) \cdot (z-p_{44})} \quad (5.8.50)$$

In order to calculate the inverse z transform of the output signal, decompose the (5.8.50) in partial fraction

$$\frac{(z+1)^2}{(z-p_{11})^2 \cdot (z-p_{33}) \cdot (z-p_{44})} \text{ parfrac } \rightarrow$$

and rewrite the z transform of the output signal as a linear combination of terms like  $\frac{1}{p_n - z}$  and

$$\frac{1}{(p_{11}) Y_{out}(z)} = \frac{\alpha_3 \cdot K \cdot V_m}{\beta_3^2} \left[ \begin{aligned} & \frac{(p_{11} + 1)^2}{(p_{11} - p_{33}) \cdot (p_{11} - p_{44}) \cdot (p_{11} - z)^2} \dots \\ & + \frac{(p_{33} + 1)^2}{(p_{11} - p_{33})^2 \cdot (p_{44} - p_{33}) \cdot (p_{33} - z)} \dots \\ & + \frac{(p_{44} + 1)^2}{(p_{11} - p_{44})^2 \cdot (p_{33} - p_{44}) \cdot (p_{44} - z)} \dots \\ & + \frac{(p_{11} + 1) \cdot (p_{33} - 2 \cdot p_{11} + p_{44})^2 \cdot \left[ \begin{aligned} & (-p_{33} - p_{44} - 2) \cdot p_{11} \dots \\ & + p_{33} + p_{44} \cdot (2 \cdot p_{33} + 1) \end{aligned} \right]}{(p_{11} - p_{33})^2 \cdot (p_{11} - p_{44})^2 \cdot (z - p_{11}) \cdot (2 \cdot p_{11} - p_{33} - p_{44})^2} \end{aligned} \right] \quad (5.8.51)$$

Now try to calculate the inverse z transform knowing that:

$$z^{-1} \left( \frac{1}{p - z} \right) = \frac{\delta(k, 0) - p^k}{p}$$

$$z^{-1} \left[ \frac{1}{(p - z)^2} \right] = \frac{\delta(k, 0) + p^k \cdot (k - 1)}{p^2}$$

and defining the following constant coefficients:

$$G_{11} := \frac{(p_{11} + 1) \cdot (p_{33} - 2 \cdot p_{11} + p_{44})^2 \cdot \left[ \begin{aligned} & (-p_{33} - p_{44} - 2) \cdot p_{11} + p_{33} + p_{44} \cdot (2 \cdot p_{33} + 1) \end{aligned} \right]}{(p_{11} - p_{33})^2 \cdot (p_{11} - p_{44})^2 \cdot (2 \cdot p_{11} - p_{33} - p_{44})^2 \cdot p_{11}} \quad (5.8.52)$$

$$G_{12} := \frac{(p_{44} + 1)^2}{(p_{11} - p_{44})^2 \cdot (p_{33} - p_{44}) \cdot p_{44}} \quad (5.8.53)$$

$$G_{22} := \frac{(p_{33} + 1)^2}{(p_{11} - p_{33})^2 \cdot (p_{44} - p_{33}) \cdot p_{33}} \quad (5.8.54)$$

$$G_{33} := \frac{(p_{11} + 1)^2}{(p_{11} - p_{33}) \cdot (p_{11} - p_{44}) \cdot p_{11}^2} \quad (5.8.55)$$

Substituting the previous definition, the function (5.8.51) takes the form:

$$Y_{out}(z) = \frac{\alpha_3 \cdot K \cdot V_m}{\beta_3^2} \cdot \left[ \begin{array}{l} G_{33} \cdot \frac{1}{(p_{11} - z)^2} \dots \\ + G_{22} \cdot \frac{1}{(p_{33} - z)} \dots \\ + G_{12} \cdot \frac{1}{(p_{44} - z)} \dots \\ + G_{11} \cdot \frac{1}{(z - p_{11})} \end{array} \right]$$

Substituting in (5.8.51) the previous coefficients and the inverse z transform found, results:

$$Y_{out_k} = \frac{\alpha_3 \cdot K \cdot V_m}{\beta_3^2} \cdot \left[ \begin{array}{l} G_{33} \cdot \frac{[\delta(k,0) + p_{11}^k \cdot (k-1)]}{p_{11}^2} \dots \\ + G_{22} \cdot \left( \frac{\delta(k,0) - p_{33}^k}{p_{33}} \right) \dots \\ + G_{12} \cdot \left( \frac{\delta(k,0) - p_{44}^k}{p_{44}} \right) \dots \\ + G_{11} \cdot \left( \frac{\delta(k,0) - p_{11}^k}{p_{11}} \right) \end{array} \right] \quad (5.8.56)$$

Collecting the Kronecker deltas,

$$Y_{out_k} = \frac{\alpha_3 \cdot K \cdot V_m}{\beta_3^2} \cdot \left[ \begin{array}{l} \left( \frac{G_{11}}{p_{11}} + \frac{G_{33}}{p_{11}^2} + \frac{G_{22}}{p_{33}} + \frac{G_{12}}{p_{44}} \right) \cdot \delta(k,0) \dots \\ + \frac{G_{33} \cdot p_{11}^k \cdot (k-1)}{p_{11}^2} - \frac{G_{22} \cdot p_{33}^k}{p_{33}} - \frac{G_{12} \cdot p_{44}^k}{p_{44}} - \frac{G_{11} \cdot p_{11}^k}{p_{11}} \end{array} \right] \quad (5.8.57)$$

and after having defined the new constant  $G_{44} := \frac{G_{11}}{p_{11}} + \frac{G_{33}}{p_{11}^2} + \frac{G_{22}}{p_{33}} + \frac{G_{12}}{p_{44}}$

the final result, considering both cases  $\zeta \neq \omega_5$  and  $\zeta = \omega_5$ , is:

$Y_{out_k} := \alpha_3 \cdot K \cdot V_m \cdot$	$\frac{1}{\gamma_3} \cdot \left[ G_5 \cdot \delta(k,0) - \left( G_2 \cdot p_1^{k-1} + G_1 \cdot p_3^{k-1} + G_3 \cdot p_2^{k-1} + G_4 \cdot p_4^{k-1} \right) \right]$ if $\zeta \neq \omega_5$
	$\frac{1}{\beta_3^2} \cdot \left[ \begin{array}{l} G_{44} \cdot \delta(k,0) \dots \\ + \frac{G_{33} \cdot p_{11}^k \cdot (k-1)}{p_{11}^2} - \frac{G_{22} \cdot p_{33}^k}{p_{33}} - \frac{G_{12} \cdot p_{44}^k}{p_{44}} - \frac{G_{11} \cdot p_{11}^k}{p_{11}} \end{array} \right]$ otherwise

(5.8.58)



Numerical sequence of the output:

$Y_{out}^T =$	0	1	2	3	V
0	$-1.222 \cdot 10^{-15}$	$10.265 - 28.557j$	$14.851 - 23.18j$	...	

Energy of the sequence  $Y_{out,k}$ :  $E_{Yout} := \sum_{k=0}^{N0-1} (|Y_{out,k}|)^2 \quad E_{Yout} = 3.747 \times 10^4 V^2$

$$\zeta = 17.743 \cdot \frac{\text{Mrads}}{\text{sec}}$$

$$\omega_5 = 191.627 \cdot \frac{\text{Mrads}}{\text{sec}}$$

$$V_{out} := |W_{Ip}(j \cdot \omega_{test})| V_{pp} \quad V_{out} = 16.541 V$$

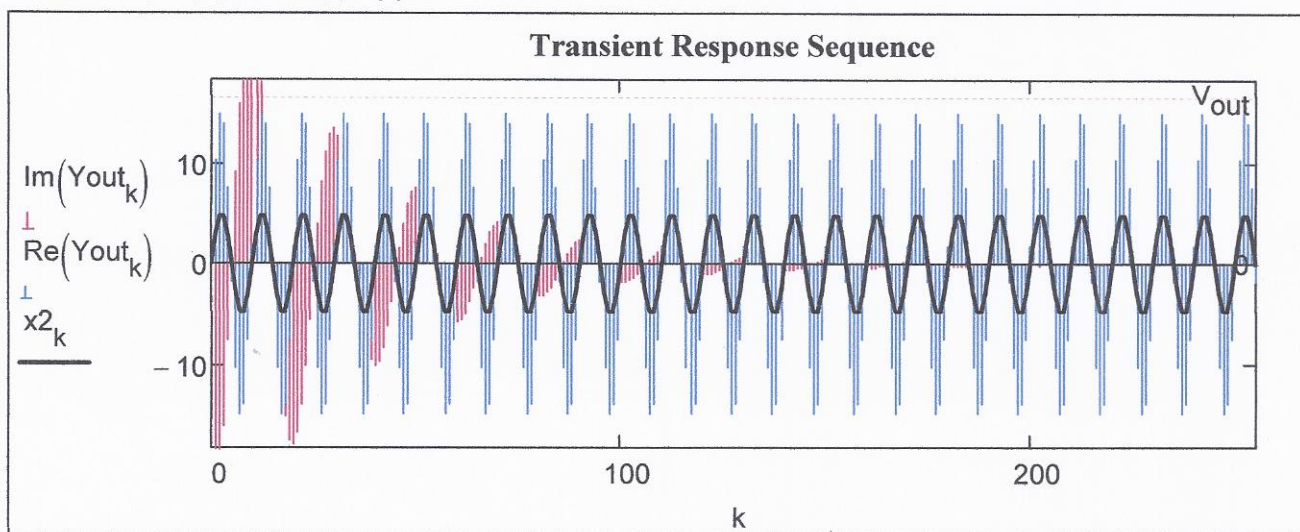


fig.:5.8.10

Pick amplitude of the frequency response if  $Q_5 > 0.5$   $r_{peak} := 20 \cdot \log \left( \left| \frac{A_5 \cdot \omega_5}{2 \cdot \zeta} \right| \right)$

$$20 \cdot \log(|W_{Ip}(j \cdot \omega_{test})|) = 10.392 \cdot \text{dB}$$

$$Q_5 = 5.4$$

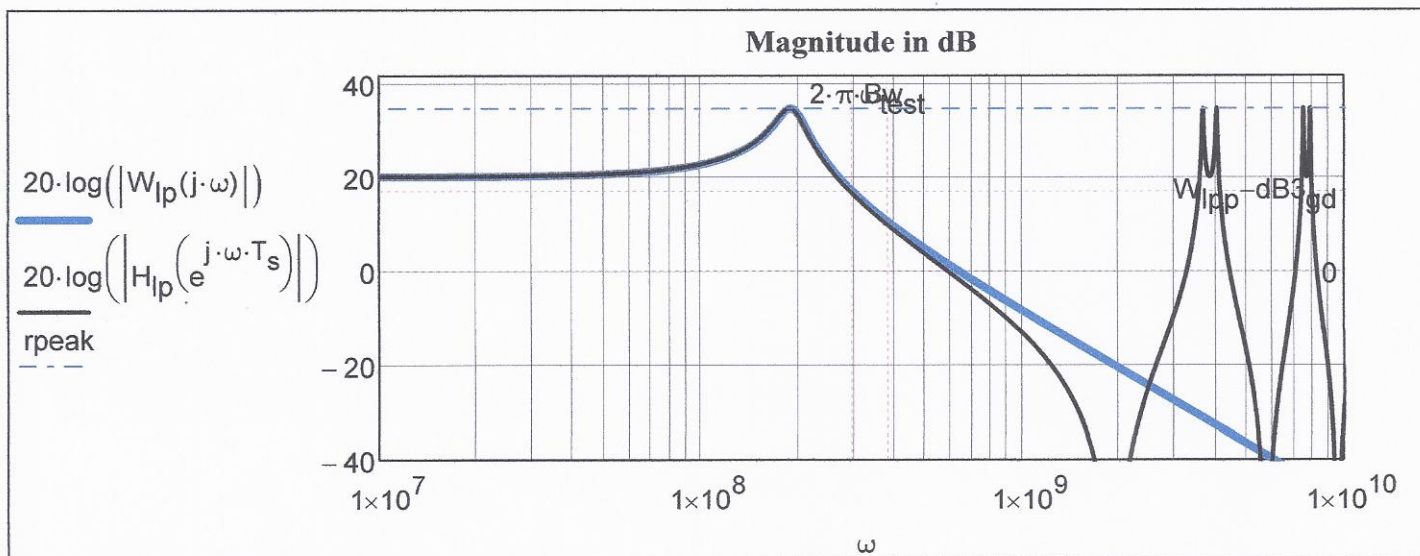




fig.:5.1.8.1

Bw = 47.1·MHz

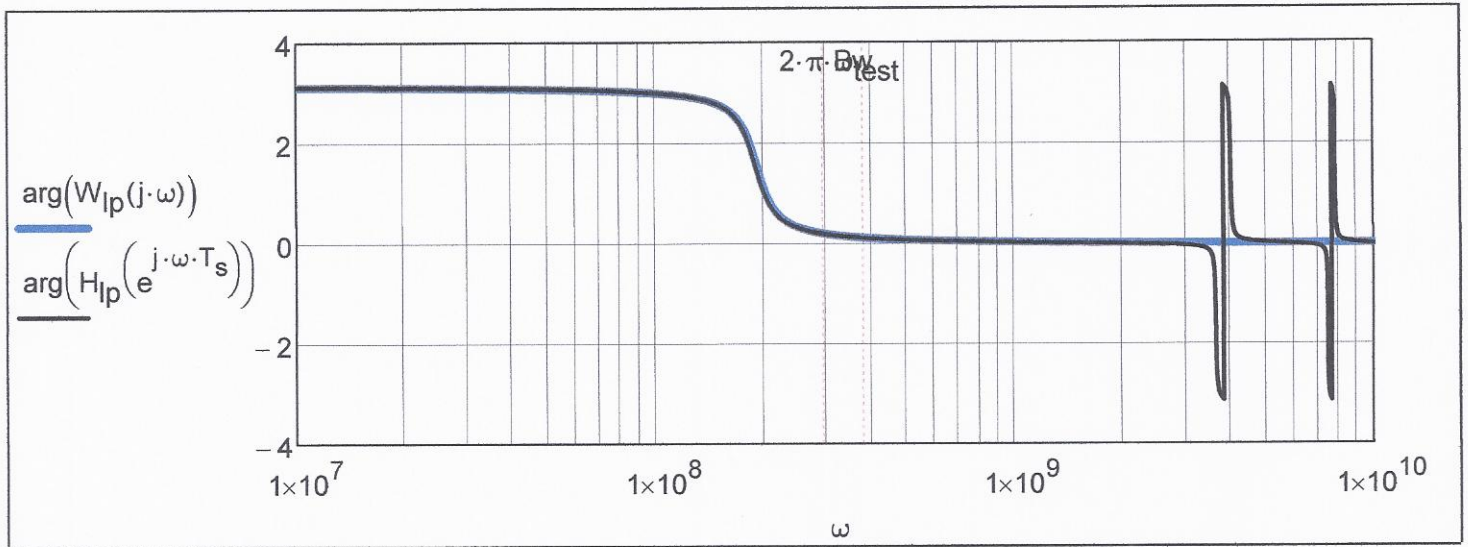


fig.:5.1.8.1'

pick amplitude  $WlpdB_{pick} = 34.685$        $\omega_{pick} = 0.19 \cdot \frac{\text{Grads}}{\text{sec}}$        $Wlpp = 20$        $\omega_5 = 0.192 \cdot \frac{\text{Grads}}{\text{sec}}$

Proof

**Proof: First case:**

$\zeta \neq \omega_5$

$$p_1 := p_1 \quad p_2 := p_2 \quad p_3 := p_3 \quad p_4 := p_4 \quad G_1 := G_1 \quad G_2 := G_2 \quad G_3 := G_3 \quad G_4 := G_4 \quad G_5 := G_5$$

$$G_5 = \frac{G_2}{p_1} + \frac{G_1}{p_3} + \frac{G_3}{p_2} + \frac{G_4}{p_4}$$

$$\left[ \begin{array}{l} \frac{G_2}{p_1} + \frac{G_1}{p_3} + \frac{G_3}{p_2} + \frac{G_4}{p_4} \cdot \delta(k,0) \dots \\ + (-1) \cdot (G_2 \cdot p_1^{k-1} + G_1 \cdot p_3^{k-1} + G_3 \cdot p_2^{k-1} + G_4 \cdot p_4^{k-1}) \end{array} \right] \begin{array}{l} \text{ztrans, k} \\ \text{collect, G1, G2, G3, G4} \\ \text{simplify} \end{array}$$

results: 
$$\frac{G_2}{p_1 - z} + \frac{G_1}{p_3 - z} + \frac{G_3}{p_2 - z} + \frac{G_4}{p_4 - z}$$

Namely 
$$Y_{out}(z) = \frac{\alpha_3 \cdot K \cdot V_m}{\gamma_3} \cdot \left( \frac{G_2}{p_1 - z} + \frac{G_1}{p_3 - z} + \frac{G_3}{p_2 - z} + \frac{G_4}{p_4 - z} \right)$$

**Proof: Second case:**  $\zeta = \omega_5$

$$p_{11} := p_{11} \quad p_{22} := p_{22} \quad p_{33} := p_{33} \quad p_{44} := p_{44}$$

$$G_{11} := G_{11} \quad G_{22} := G_{22} \quad G_{33} := G_{33} \quad G_{44} := G_{44} \quad G_{55} := G_{55}$$

$$\left( \frac{G11}{p11} + \frac{G33}{p11^2} + \frac{G22}{p33} + \frac{G12}{p44} \right) \cdot \delta(k,0) \dots$$

ztrans , k  
 collect , G11 , G22 , G33 , G12 →  
 simplify

$$+ \frac{G33 \cdot p11^k \cdot (k-1)}{p11^2} - \frac{G22 \cdot p33^k}{p33} - \frac{G12 \cdot p44^k}{p44} - \frac{G11 \cdot p11^k}{p11}$$

results:  $\frac{G11}{p11-z} + \frac{G33}{(p11-z)^2} + \frac{G22}{p33-z} + \frac{G12}{p44-z}$

Namely:

$$Yout_k = \frac{\alpha_3 \cdot K \cdot V_m}{\beta_3^2} \cdot \left[ \begin{array}{l} G33 \cdot \frac{1}{(p11-z)^2} \dots \\ + G22 \cdot \frac{1}{(p33-z)} \dots \\ + G12 \cdot \frac{1}{(p44-z)} \dots \\ + G11 \cdot \frac{1}{(z-p11)} \end{array} \right]$$

q.e.d.  
 (Quod Erat Demostrandum)

▲ Proof

