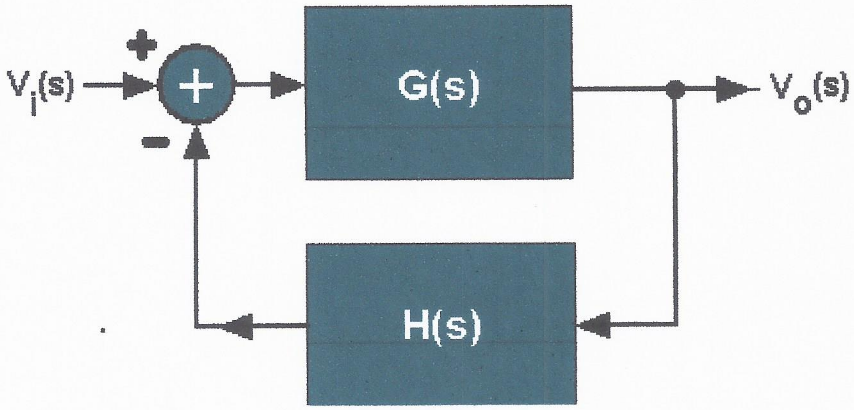
Nichols Chart

Consider a generic L.T.I. feedback system here below depicted:



as known, the transfer function is:

$$G_{fNch}(j\cdot\omega) = \frac{V_o(j\cdot\omega)}{V_i(j\cdot\omega)} = \frac{G_{Nch}(j\cdot\omega)}{1 + G_{Nch}(j\cdot\omega)\cdot H_{Nch}(j\cdot\omega)}$$

The following products be represented in exponential forms

$$G_{fNch}(j\cdot\omega)\cdot H_{Nch}(j\cdot\omega) = M_{Nch}\cdot e^{j\cdot\phi}$$

open loop gain: $G_{Nch}(j\cdot\omega)\cdot H_{Nch}(j\cdot\omega) = Q_{Nch}\cdot e^{j\cdot\alpha}$

it follows that:

$$G_{fNch}(j\cdot\omega)\cdot H_{Nch}(j\cdot\omega) = \frac{G_{Nch}(j\cdot\omega)\cdot H_{Nch}(j\cdot\omega)}{1 + G_{Nch}(j\cdot\omega)\cdot H_{Nch}(j\cdot\omega)} = \frac{Q_{Nch}\cdot e^{j\cdot\alpha}}{1 + Q_{Nch}\cdot e^{j\cdot\alpha}}$$

$$Q_{Nch} = \left|G_{Nch}(j\cdot\omega)\cdot H_{Nch}(j\cdot\omega)\right|$$

hence must be worth the equality:

$$M_{Nch}\cdot e^{j\cdot\phi} = \frac{Q_{Nch}\cdot e^{j\cdot\alpha}}{1 + Q_{Nch}\cdot e^{j\cdot\alpha}}$$

In addition put:

$$M_{dB} = 20\cdot\log\left(\left|G_{fNch}(j\cdot\omega)\cdot H_{Nch}(j\cdot\omega)\right|\right) = 20\cdot\log\left(\left|\frac{G_{Nch}(j\cdot\omega)\cdot H_{Nch}(j\cdot\omega)}{1 + G_{Nch}(j\cdot\omega)\cdot H_{Nch}(j\cdot\omega)}\right|\right)$$

or rather:

$$M_{dB} = 20 \cdot \log\left[\frac{Q_{Nch}}{\sqrt{(Q_{Nch} \cdot \cos(\alpha) + 1)^2 + (Q_{Nch} \cdot \sin(\alpha))^2}}\right] \qquad \phi = \arg\left(\frac{\mathbf{G_{Nch}}(j \cdot \omega) \cdot \mathbf{H_{Nch}}(j \cdot \omega)}{1 + \mathbf{G_{Nch}}(j \cdot \omega) \cdot \mathbf{H_{Nch}}(j \cdot \omega)}\right)$$

inverting:

$$10^{\frac{M_{dB}}{20}} = \frac{Q_{Nch}}{\sqrt{(Q_{Nch} \cdot \cos(\alpha) + 1)^2 + (Q_{Nch} \cdot \sin(\alpha))^2}}$$

placing $K = 10^{\frac{M_{dB}}{20}}$ can write $K = \frac{Q_{Nch}}{\sqrt{(Q_{Nch} \cdot \cos(\alpha) + 1)^2 + (Q_{Nch} \cdot \sin(\alpha))^2}}$

$$K^2 = \frac{Q_{Nch}^2}{(Q_{Nch} \cdot \cos(\alpha) + 1)^2 + (Q_{Nch} \cdot \sin(\alpha))^2}$$

which is a quadratic equation with constant coefficients in the unknown Q_{Nch}.

Results that (representing Q_{Nch} in dB):

$$Q_{Nch}(\alpha, M_{dB}) := 20 \cdot \log\left[\left[\begin{matrix} 10^{\frac{M_{dB}}{20}} \cdot \dfrac{-\cos(\alpha) \cdot 10^{\frac{M_{dB}}{20}} + \sqrt{1 - \left(10^{\frac{M_{dB}}{20}}\right)^2 \cdot \sin(\alpha)^2}}{\left(10^{\frac{M_{dB}}{20}}\right)^2 - 1} \\ -10^{\frac{M_{dB}}{20}} \cdot \dfrac{\cos(\alpha) \cdot 10^{\frac{M_{dB}}{20}} + \sqrt{1 - \left(10^{\frac{M_{dB}}{20}}\right)^2 \cdot \sin(\alpha)^2}}{\left(10^{\frac{M_{dB}}{20}}\right)^2 - 1} \end{matrix}\right]\right]$$

Moreover, expanding the following relation

$$\frac{Q_{Nch} \cdot e^{j \cdot \alpha}}{1 + Q_{Nch} \cdot e^{j \cdot \alpha}} = \frac{Q_{Nch} \cdot (Q_{Nch} + \cos(\alpha) + j \cdot \sin(\alpha))}{Q_{Nch}^2 + 2 \cdot \cos(\alpha) \cdot Q_{Nch} + 1} = M \cdot e^{j \cdot \phi}$$

results: $M \cdot e^{j \cdot \phi} = M \cdot (\cos(\phi) + j \cdot \sin(\phi)) = M \cdot \cos(\phi) \cdot \left(1 + j \cdot \frac{\sin(\phi)}{\cos(\phi)}\right)$

or even $M \cdot \cos(\phi) \cdot (1 + j \cdot \tan(\phi)) = \frac{Q_{Nch} \cdot (Q_{Nch} + \cos(\alpha) + j \cdot \sin(\alpha))}{Q_{Nch}^2 + 2 \cdot \cos(\alpha) \cdot Q_{Nch} + 1}$

so that: $$M\cdot\cos(\phi)\cdot(1+j\cdot\tan(\phi)) = \frac{Q_{Nch}\cdot\left(Q_{Nch}+\cos(\alpha)\right)\cdot\left[1+j\cdot\frac{\sin(\alpha)}{\left(Q_{Nch}+\cos(\alpha)\right)}\right]}{Q_{Nch}^{\;2}+2\cdot\cos(\alpha)\cdot Q_{Nch}+1}$$

Follows from this that:

$$\tan(\phi) = \frac{\sin(\alpha)}{Q_{Nch}+\cos(\alpha)} \qquad M\cdot\cos(\phi) = \frac{Q_{Nch}\cdot\left(Q_{Nch}+\cos(\alpha)\right)}{Q_{Nch}^{\;2}+2\cdot\cos(\alpha)\cdot Q_{Nch}+1}$$

solving the first one respect to Q_{Nch}, you get:

$$Q_{Nch}(\alpha,\phi) = \frac{\sin(\alpha)}{\tan(\phi)} - \cos(\alpha) = \frac{\sin(\alpha-\phi)}{\sin(\phi)}$$

$$A_{Nch}(\alpha,\phi) = 20\cdot\log\left(Q_{Nch}(\alpha,\phi)\right) \qquad \alpha\left(\phi,Q_{Nch}\right) := \phi + \operatorname{asin}\left(Q_{Nch}\cdot\sin(\phi)\right)$$

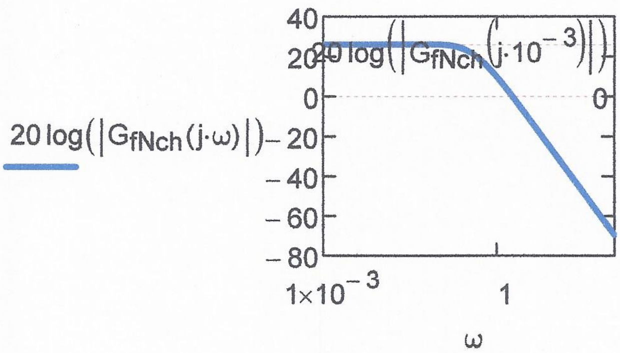
$$A_{Nch}(\alpha,\phi) = 20\cdot\log\left[\frac{-(\tan(\phi)\cdot\cos(\alpha)-\sin(\alpha))}{\tan(\phi)}\right]$$

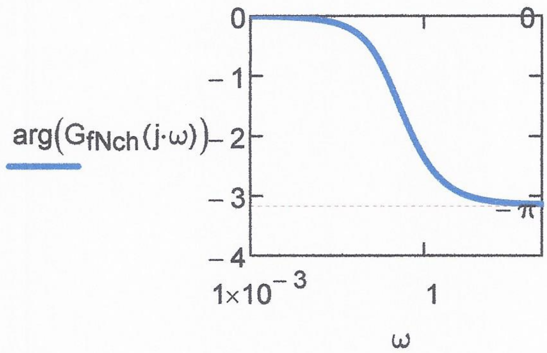
and ultimately: $$A_{Nch}(\alpha,\phi) := 20\cdot\log[\sin(\alpha)\cdot(\cot(\phi)-\cot(\alpha))] \qquad \tan(\phi) < \tan(\alpha)$$

About the Nichols Chart, it is noted that in abscissa there is the angle α while in ordinate there are $Q_{Nch}(\alpha,MdB)$ in dB and $A_{Nch}(\alpha,\varphi)$. The graph then shows two families of curves, one with parameter MdB a the other with parameter φ.

Example: $$G_{fNch}(s) := \frac{20\cdot(1+s\cdot 8)}{(1+s\cdot 2)\cdot(1+s\cdot 4)\cdot(1+s\cdot 6)}$$ ($|G_{fNch}(j\boldsymbol{\omega})|_{dB}$ blue line)

Bode plots:

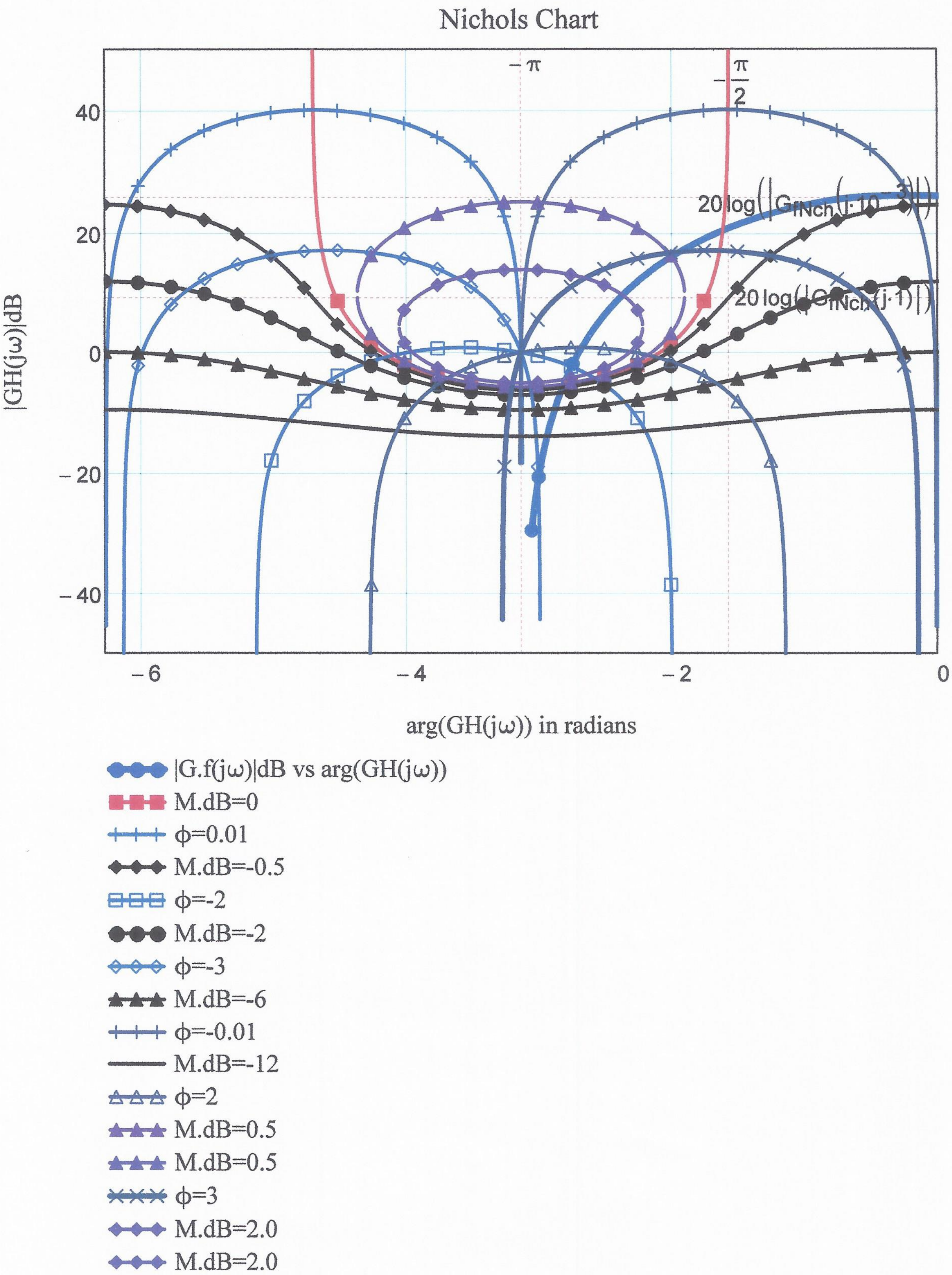


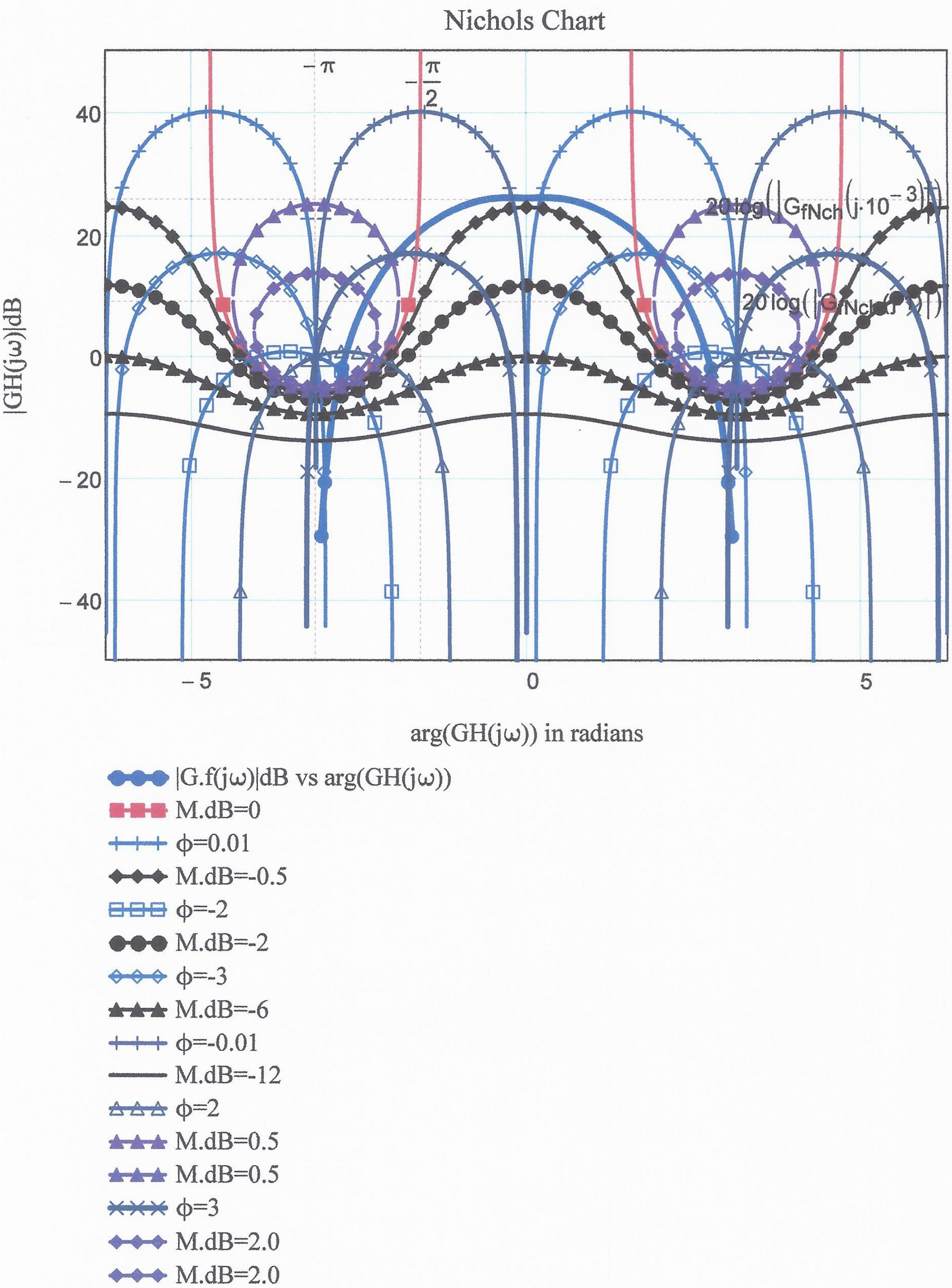


$$m := 2 \qquad \alpha := -\pi\cdot m, -\pi\cdot m + \frac{2\cdot\pi\cdot m}{10000} .. \pi\cdot m$$

$Q_{Nch}(\alpha, M_{dB})$ MdB is a parameter $\qquad A_{Nch}(\alpha,\phi)$ ϕ is a parameter

From the observation of the graph, can be deduced the values of the *gain margin* and the *phase margin* and therfore have a measure of the degree of stability of the system with feedback.





Nichols Chart
−π
−π/2
40
20
0
−20
−40
−5
0
5
|GH(jω)|dB
arg(GH(jω)) in radians
20log(|G_fNch(j·10^−3)|)
20log(|G_fNcl(j·1)|)
|G.f(jω)|dB vs arg(GH(jω))
M.dB=0
ϕ=0.01
M.dB=-0.5
ϕ=-2
M.dB=-2
ϕ=-3
M.dB=-6
ϕ=-0.01
M.dB=-12
ϕ=2
M.dB=0.5
M.dB=0.5
ϕ=3
M.dB=2.0
M.dB=2.0