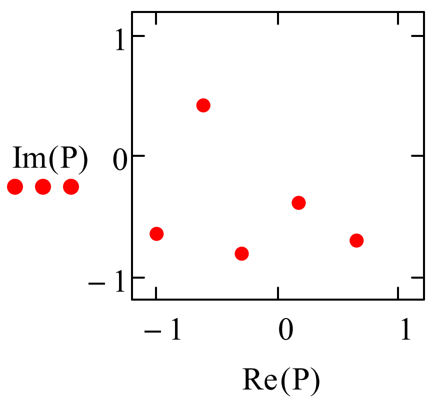
This file looks at some features of Valery Ochkov's very interesting problem about the conic defined by five points randomly distributed in a square

I learned from Alex Bogomolny's Cut-the-Knot site that one can find the center of a conic from five points on the conic just by intersecting lines through pairs of known points (adepts of projective geometry knew this, of course, but not I). The relevant articles are at
www.cut-the-knot.org/Curriculum/Geometry/TangentToEllipse.shtml
and
www.cut-the-knot.org/Curriculum/Geometry/TangentTriangleToEllipse.shtml

Since translating our conic to the origin reduces the system we need to solve to three equations for the three coefficients that determine the discriminant, it is worth writing some ugly Mathcad code to see how this works out. For the convenience of writing a point with one number this part of the file is done with complex coordinates.

A global definition (below, next to the solution graph) draws five points in [-1, 1] x [-1, 1]. Here they are:



The basic operation is the intersection of lines through pairs of points: The function K returns the complex coordinates of the intersection of the lines through A,B and through C,D.

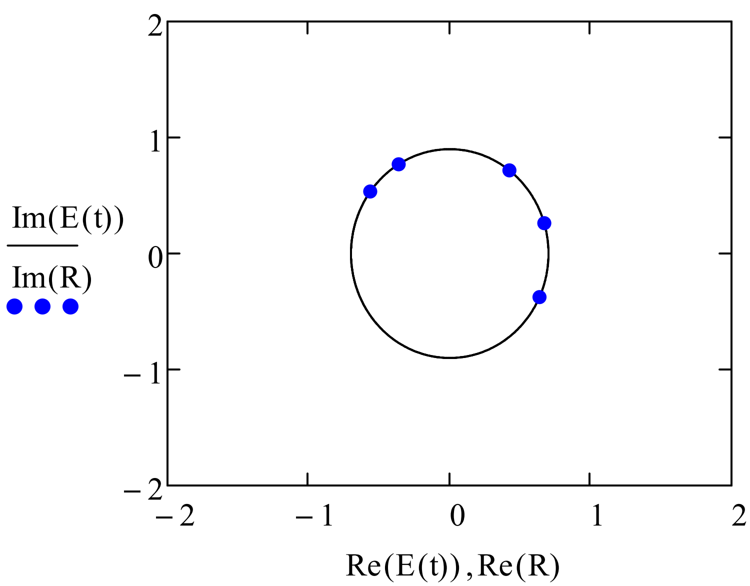
$$K(A,B,C,D) := C + \frac{\mathrm{Im}\left[(A-C)\cdot\left[\overline{(A-B)}\right]\right]}{\mathrm{Im}\left[(D-C)\cdot\left[\overline{(A-B)}\right]\right]}\cdot(D-C)$$

Per Bogomolny's procedure, the first thing we need is three pairs of tangents to the conic. We'll generate one tangent using an ellipse for an example, but note that the beauty of the construction for our purposes is that it works *without our needing to know what kind of conic our points are on*; in the projective world the conics are indistinguishable.

To illustrate we'll take five points on an ellipse

$$E(t) := 0.7\cos(t) + 0.9i\cdot\sin(t)$$

$$R := \begin{pmatrix} 0.67 + 0.262i \\ 0.424 + 0.716i \\ -0.362 + 0.771i \\ -0.563 + 0.535i \\ 0.636 - 0.375i \end{pmatrix}$$



$$L(A,B) := \begin{pmatrix} -9\cdot A + 10\cdot B \\ 11\cdot A - 10\cdot B \end{pmatrix}$$

this draws a line through A and B

$$p' := K(R_0, R_1, R_3, R_4)$$

point of intersection of the 0,1 and 3,4 lines

$$q' := K(R_0, R_4, R_2, R_3)$$

point of intersection of the 0,4 and 2,3 lines

$$r' := K(p', q', R_1, R_2)$$

point of intersection of the p,q and 1,2 lines

$$T := L(r', R_4)$$

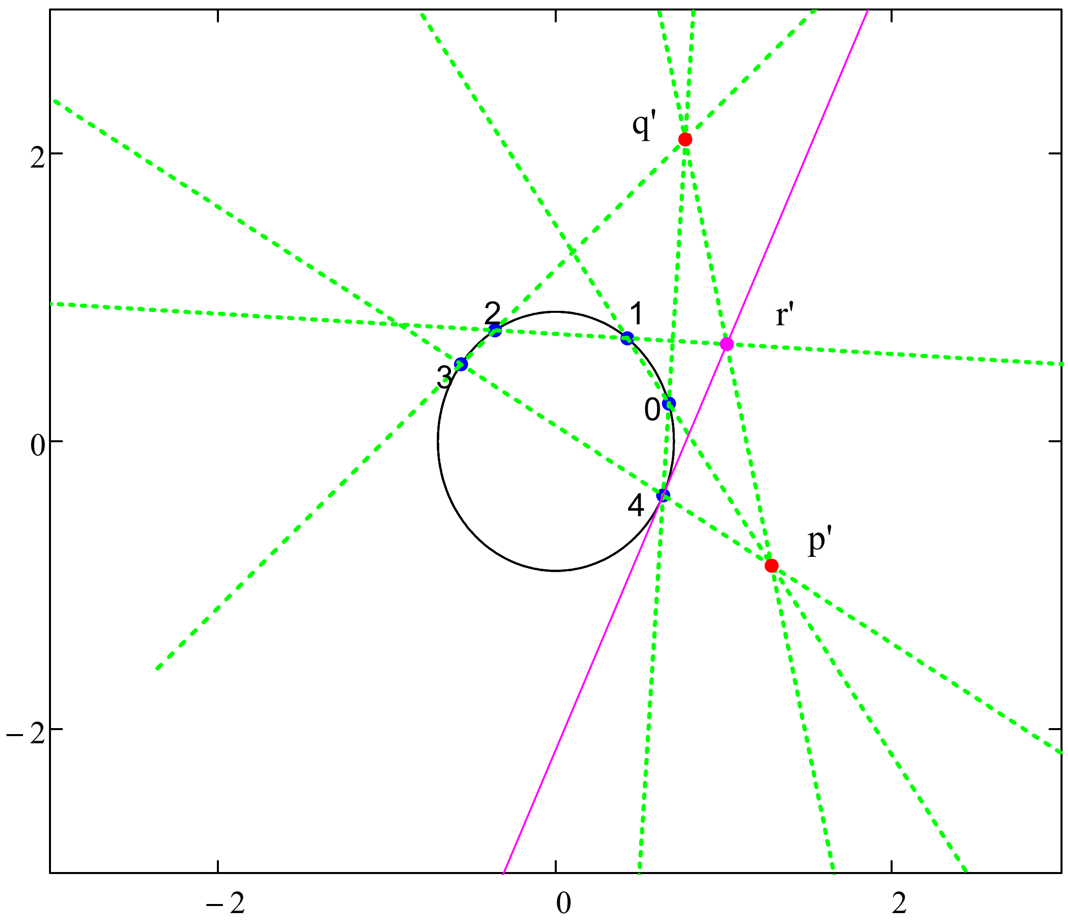
the tangent is the line through r prime and point 4

$$I1 := \text{augment}(L(R_0, R_1), L(R_3, R_4), L(R_0, R_4), L(R_2, R_3), L(p', q'), L(R_1, R_2))$$

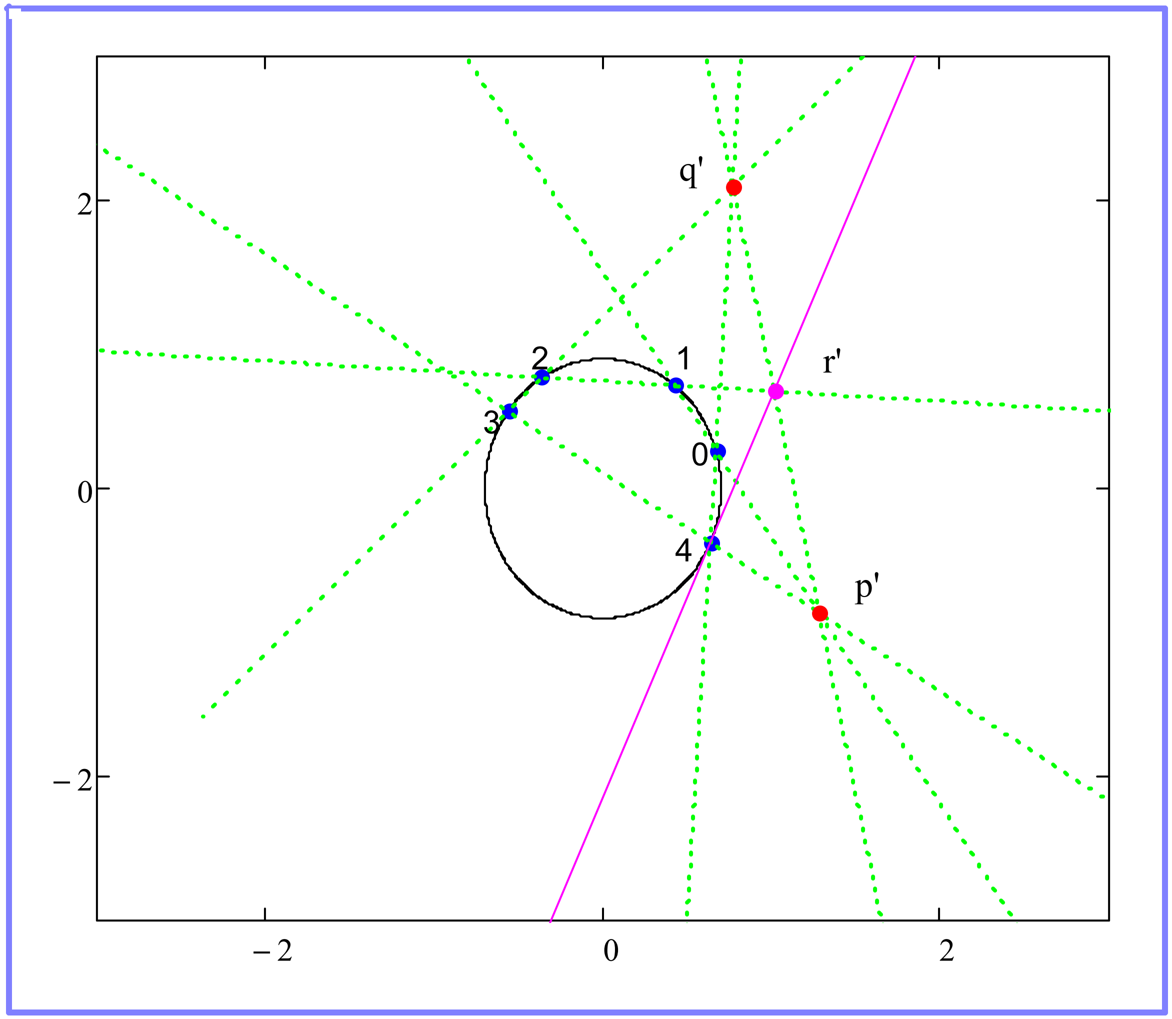
all the construction lines

$$P1 := \begin{pmatrix} p' \\ q' \end{pmatrix}$$

the first two intersections



q'
r'
p'
0
1
2
3
4
2
0
−2
−2
0
2



The construction of a tangent to an ellipse given five points on the ellipse. [Frozen in case formatting changes the label positions above.]

So there is one tangent, the pink line through the final intersection and point 4. Now we carry out this construction for the random points in the vector P:

$p := K(P_0, P_1, P_3, P_4) = -1.09 - 0.911i$ point of intersection of the 0,1 and 3,4 lines

$q := K(P_0, P_4, P_2, P_3) = -0.147 - 0.679i$ point of intersection of the 0,4 and 2,3 lines

$r := K(p, q, P_1, P_2) = 0.333 - 0.562i$ point of intersection of the p,q and 1,2 lines

The line tangent to the conic at P4 is then the line through P4 and r

We now need tangents (given by two points) at two other points on the ellipse: Tangent at P3

$p1 := K(P_0, P_1, P_2, P_3)$

$q1 := K(P_0, P_3, P_4, P_2)$

$r1 := K(p1, q1, P_1, P_4)$ tangent at P3 is line through P3 and r1

$p2 := K(P_4, P_0, P_1, P_2)$

$q2 := K(P_4, P_2, P_3, P_1)$

$r2 := K(p2, q2, P_0, P_3)$ tangent at P2 is line through P2 and r2

Now we find the intersection of each pair of tangents (one tangent is in common, which is OK)

$z1 := K(P_4, r, P_3, r1)$ $\quad$ $z2 := K(P_3, r1, P_2, r2)$

Finally we find the intersection of the lines through these tangent intersections and the midpoints of the chords between the points of tangency. This is the center.

$w1 := \dfrac{P_4 + P_3}{2}$ $\quad$ $w2 := \dfrac{P_3 + P_2}{2}$

$center := K(z1, w1, z2, w2) = 0.381 - 0.835i$

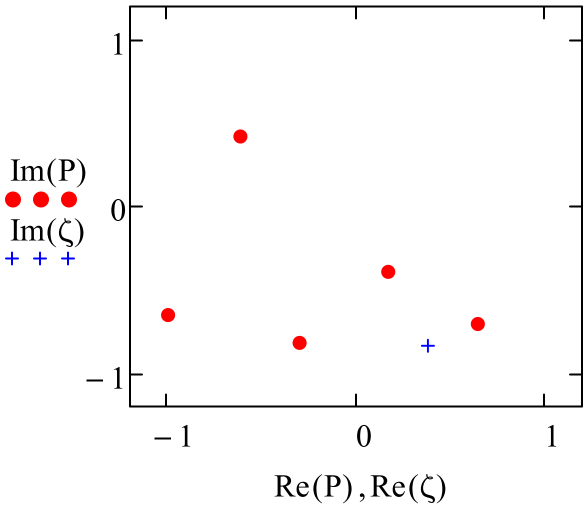
Putting this all in one function we have:

$$
\mathrm{cen}(P) := \left|
\begin{array}{l}
\begin{pmatrix} p \\ q \end{pmatrix} \leftarrow \begin{pmatrix} K(P_0, P_1, P_3, P_4) \\ K(P_0, P_4, P_2, P_3) \end{pmatrix} \\
r \leftarrow K(p, q, P_1, P_2) \\
\begin{pmatrix} p1 \\ q1 \end{pmatrix} \leftarrow \begin{pmatrix} K(P_0, P_1, P_2, P_3) \\ K(P_0, P_3, P_4, P_2) \end{pmatrix} \\
r1 \leftarrow K(p1, q1, P_1, P_4) \\
\begin{pmatrix} p2 \\ q2 \end{pmatrix} \leftarrow \begin{pmatrix} K(P_4, P_0, P_1, P_2) \\ K(P_4, P_2, P_3, P_1) \end{pmatrix} \\
r2 \leftarrow K(p2, q2, P_0, P_3) \\
\begin{pmatrix} z1 \\ z2 \end{pmatrix} \leftarrow \begin{pmatrix} K(P_4, r, P_3, r1) \\ K(P_3, r1, P_2, r2) \end{pmatrix} \\
\begin{pmatrix} w1 \\ w2 \end{pmatrix} \leftarrow \frac{1}{2} \cdot \begin{pmatrix} P_4 + P_3 \\ P_3 + P_2 \end{pmatrix} \\
K(z1, w1, z2, w2)
\end{array}
\right.
$$

With K defined as above, this computes the center of the conic for both ellipse solutions and hyperbolas. Expanded, this center would be some monstrous expression in the five real parts and five imaginary parts of the complex points P. Even K is beyond displaying via symbolics. The computation generates 12 intersection points of lines and uses two midpoints. An attempt to expand the function **cen** of a general set five complex points runs out of memory on my machine.

$\zeta := \text{cen}(P)$ This is the center of the conic.

$pts := \text{augment}(\text{Re}(P - \zeta), \text{Im}(P - \zeta), \text{Re}(P - \zeta) \cdot 0)$ are the translated points



Here are random points and the computed center.

Now we set up for real solving by turning P into real vectors X and Y. Since after translation we only have three coefficients to find we need only three points; it doesn't matter which, so we pick the first three. But of course the center "knows about" all five original points, and therefore so do our chosen three points.

Translate P: $p := \mathrm{submatrix}(P - \zeta, 0, 2, 0, 0)$

$$X := \mathrm{Re}(p) = \begin{pmatrix} -1.378 \\ -0.994 \\ -0.211 \end{pmatrix} \qquad Y := \mathrm{Im}(p) = \begin{pmatrix} 0.184 \\ 1.256 \\ 0.443 \end{pmatrix}$$

The system is now:

$$M \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \text{with} \qquad M := \begin{bmatrix} (X_0)^2 & 2 \cdot X_0 \cdot Y_0 & (Y_0)^2 \\ (X_1)^2 & 2 \cdot X_1 \cdot Y_1 & (Y_1)^2 \\ (X_2)^2 & 2 \cdot X_2 \cdot Y_2 & (Y_2)^2 \end{bmatrix}$$

Using lsolve we find

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} := \mathrm{lsolve}\left[M, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right] = \begin{pmatrix} 2.621 \\ 8.706 \\ 12.769 \end{pmatrix} \qquad a \cdot c - b^2 = -42.327$$

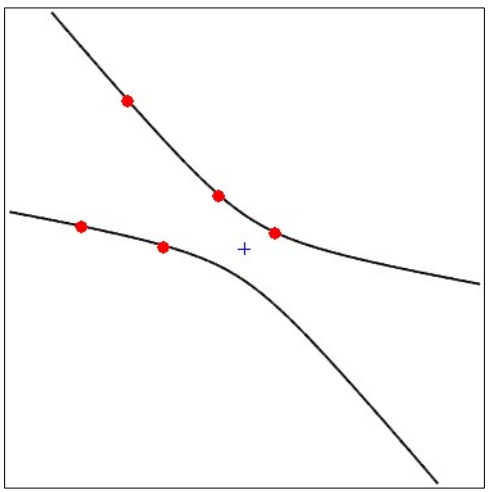
$$f(x,y) := a \cdot x^2 + b \cdot 2 \cdot x \cdot y + c \cdot y^2 - 1$$

The function f will be 0 at our points, so we can graph the conic found above by contouring.

$$m := 0..400 \qquad n := 0..400$$

$$x_{m,n} := \frac{m-200}{100} \qquad y_{m,n} := \frac{n-200}{100}$$

$$C := \overrightarrow{f(x,y)} \qquad cr := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



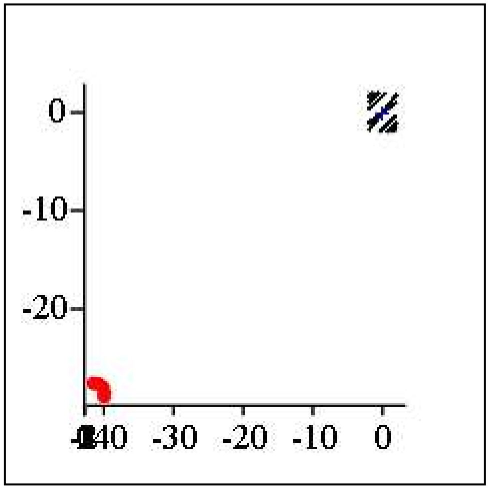
The window here is [-2, 2] x [-2, 2]

Here, finally is the global definition that draws the random points. Recompute P to see different sets of points.

$$P \equiv (\text{runif}(5,-1,1) + i \cdot \text{runif}(5,-1,1))$$

As noted above, the solution is drawn just by contouring so you may see some extra contours along with the solution.

Sometimes the center is very far away from the points, as in this example (remember, the points have been translated to put the center of the conic at the origin). See below for one implication of this fact.



$$P = \begin{pmatrix} 0.756 - 0.027i \\ -0.809 + 0.983i \\ 0.727 - 0.362i \\ 0.563 + 0.563i \\ -0.092 + 0.921i \end{pmatrix}$$

The center is $\begin{pmatrix} 40.676 \\ 28.571 \end{pmatrix}$

$$a \cdot c - b^2 = -1.081 \times 10^{-5}$$

The hyperbola is nearly a pair of lines!

Since the system has now only three equations we can abandon lsolve and solve directly for the discriminant in terms of the three translated points.

To set up the back substitution we do a few row operations by matrix multiplication.

Add a multiple of row j to row k:

$$\mathrm{add}(a,j,k) := \left| \begin{array}{l} \mathrm{add} \leftarrow \mathrm{identity}(3) \\ \mathrm{add}_{k,j} \leftarrow a \\ \mathrm{add} \end{array} \right.$$

With this we can encode the setup for backsubstitution, where the augmented matrix Π is our system of three equations.

$$\Pi := \mathrm{augment}\left[M, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right]$$

$$\Pi1 := \mathrm{add}\left(-\frac{\Pi_{2,0}}{\Pi_{0,0}}, 0, 2\right) \cdot \mathrm{add}\left(-\frac{\Pi_{1,0}}{\Pi_{0,0}}, 0, 1\right) \cdot \Pi$$

$$\Pi2 := \mathrm{add}\left(-\frac{\Pi1_{2,1}}{\Pi1_{1,1}}, 1, 2\right) \cdot \Pi1 = \begin{pmatrix} 1.9 & -0.506 & 0.034 & 1 \\ 0 & -2.235 & 1.561 & 0.48 \\ 0 & 0 & 0.074 & 0.939 \end{pmatrix}$$

Now back substitution

$$c := \frac{\Pi2_{2,3}}{\Pi2_{2,2}} = 12.769$$

$$b := \frac{\Pi2_{1,3} - c \cdot \Pi2_{1,2}}{\Pi2_{1,1}} = 8.706$$

$$a := \frac{\Pi2_{0,3} - c \cdot \Pi2_{0,2} - b \cdot \Pi2_{0,1}}{\Pi2_{0,0}}$$

Write this solution as a single procedure called **sol**:

$$\mathrm{sol}(\Pi) := \left| \begin{array}{l} \Pi1 \leftarrow \mathrm{add}\left(-\frac{\Pi_{2,0}}{\Pi_{0,0}}, 0, 2\right) \cdot \mathrm{add}\left(-\frac{\Pi_{1,0}}{\Pi_{0,0}}, 0, 1\right) \cdot \Pi \\ \Pi2 \leftarrow \mathrm{add}\left(-\frac{\Pi1_{2,1}}{\Pi1_{1,1}}, 1, 2\right) \cdot \Pi1 \\ \chi \leftarrow \frac{\Pi2_{2,3}}{\Pi2_{2,2}} \\ \beta \leftarrow \frac{\Pi2_{1,3} - \chi \cdot \Pi2_{1,2}}{\Pi2_{1,1}} \\ \alpha \leftarrow \frac{\Pi2_{0,3} - \chi \cdot \Pi2_{0,2} - \beta \cdot \Pi2_{0,1}}{\Pi2_{0,0}} \\ \begin{pmatrix} \alpha \\ \beta \\ \chi \end{pmatrix} \end{array} \right.$$

$$\mathrm{sol}(\Pi) = \begin{pmatrix} 2.621 \\ 8.706 \\ 12.769 \end{pmatrix} \qquad \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2.621 \\ 8.706 \\ 12.769 \end{pmatrix}$$

So these solutions match what we got from lsolve
(By the way, this illustrates the value (for me) of the Mathcad interface with its easy swapping between symbolic and numeric answers: Without numerics to check my symbolic expressions I could never construct functions like cen or sol: my own internal symbolics is just not that good.)

Now apply this to get an expression for the discriminant in terms of the coordinates of three of the five points (shifted so the center is at the origin).

$$X := X \qquad Y := Y \qquad Z := Z$$

$$\Pi := \begin{bmatrix} (X_0)^2 & 2 \cdot X_0 \cdot Y_0 & (Y_0)^2 & 1 \\ (X_1)^2 & 2 \cdot X_1 \cdot Y_1 & (Y_1)^2 & 1 \\ (X_2)^2 & 2 \cdot X_2 \cdot Y_2 & (Y_2)^2 & 1 \end{bmatrix}$$

$$\Xi := \mathrm{sol}(\Pi) \ \Bigg| \begin{matrix} \text{expand} \\ \text{simplify} \end{matrix} \rightarrow \begin{bmatrix} \dfrac{X_2 \cdot (Y_0)^2 \cdot Y_2 - X_1 \cdot (Y_0)^2 \cdot Y_1 + X_0 \cdot Y_0 \cdot (Y_1)^2 - X_0 \cdot Y_0 \cdot (Y_2)^2 - X_2 \cdot (Y_1)^2 \cdot Y_2 + X_1 \cdot Y_1 \cdot (Y_2)^2}{(X_0 \cdot Y_1 - X_1 \cdot Y_0) \cdot (X_0 \cdot Y_2 - X_2 \cdot Y_0) \cdot (X_1 \cdot Y_2 - X_2 \cdot Y_1)} \\ -\dfrac{(X_0)^2 \cdot (Y_1)^2 - (X_0)^2 \cdot (Y_2)^2 - (X_1)^2 \cdot (Y_0)^2 + (X_1)^2 \cdot (Y_2)^2 + (X_2)^2 \cdot (Y_0)^2 - (X_2)^2 \cdot (Y_1)^2}{2 \cdot (X_0 \cdot Y_1 - X_1 \cdot Y_0) \cdot (X_0 \cdot Y_2 - X_2 \cdot Y_0) \cdot (X_1 \cdot Y_2 - X_2 \cdot Y_1)} \\ -\dfrac{Y_2 \cdot (X_0)^2 \cdot X_2 - Y_1 \cdot (X_0)^2 \cdot X_1 + Y_0 \cdot X_0 \cdot (X_1)^2 - Y_0 \cdot X_0 \cdot (X_2)^2 - Y_2 \cdot (X_1)^2 \cdot X_2 + Y_1 \cdot X_1 \cdot (X_2)^2}{(X_0 \cdot Y_1 - X_1 \cdot Y_0) \cdot (X_0 \cdot Y_2 - X_2 \cdot Y_0) \cdot (X_1 \cdot Y_2 - X_2 \cdot Y_1)} \end{bmatrix}$$

The denominators are uniform except for the sign, so we can simplify by writing
(copying the symbolic answer above to turn it into a function):

$$\Omega(X,Y) := \begin{bmatrix} X_2\cdot(Y_0)^2\cdot Y_2 - X_1\cdot(Y_0)^2\cdot Y_1 + X_0\cdot Y_0\cdot(Y_1)^2 - X_0\cdot Y_0\cdot(Y_2)^2 - X_2\cdot(Y_1)^2\cdot Y_2 + X_1\cdot Y_1\cdot(Y_2)^2 \\ -\frac{(X_0)^2\cdot(Y_1)^2 - (X_0)^2\cdot(Y_2)^2 - (X_1)^2\cdot(Y_0)^2 + (X_1)^2\cdot(Y_2)^2 + (X_2)^2\cdot(Y_0)^2 - (X_2)^2\cdot(Y_1)^2}{2} \\ -\left[Y_2\cdot(X_0)^2\cdot X_2 - Y_1\cdot(X_0)^2\cdot X_1 + Y_0\cdot X_0\cdot(X_1)^2 - Y_0\cdot X_0\cdot(X_2)^2 - Y_2\cdot(X_1)^2\cdot X_2 + Y_1\cdot X_1\cdot(X_2)^2\right] \end{bmatrix}$$

$$\Omega(X,Y)_0\cdot\Omega(X,Y)_2 - \left(\Omega(X,Y)_1\right)^2 \;\Big|\begin{array}{l}\text{expand}\\ \text{simplify}\end{array} \rightarrow -\frac{(X_0\cdot Y_1 - X_1\cdot Y_0 + X_0\cdot Y_2 - X_2\cdot Y_0 + X_1\cdot Y_2 - X_2\cdot Y_1)\cdot(X_0\cdot Y_1 - X_1\cdot Y_0 +}{}$$

The discriminant is then:

$$\frac{\Omega(X,Y)_0\cdot\Omega(X,Y)_2 - \left(\Omega(X,Y)_1\right)^2}{\left[(X_0\cdot Y_1 - X_1\cdot Y_0)\cdot(X_0\cdot Y_2 - X_2\cdot Y_0)\cdot(X_1\cdot Y_2 - X_2\cdot Y_1)\right]^2} = -42.327$$

The numerator factors, and each factor is a sum of 2-by-two determinants, so define

$$\alpha := \left|\begin{pmatrix} X_0 & X_1 \\ Y_0 & Y_1 \end{pmatrix}\right|$$

$$\beta := \left|\begin{pmatrix} X_0 & X_2 \\ Y_0 & Y_2 \end{pmatrix}\right|$$

$$\gamma := \left|\begin{pmatrix} X_1 & X_2 \\ Y_1 & Y_2 \end{pmatrix}\right|$$

Then we can write the discriminant as

$$\frac{(\alpha + \beta + \gamma)\cdot(-\alpha + \beta + \gamma)\cdot(\alpha - \beta + \gamma)\cdot(\alpha + \beta - \gamma)}{4\cdot(\alpha\cdot\beta\cdot\gamma)^2} = -42.327$$

Since we only care about the sign of this, we can just look at the numerator: these are the signed areas (times 2) of four triangles: the one made by our original three points, and the other three made by reflecting one of these points through the origin and leaving the other two in place. So graphically we can answer the "ellipse or hyperbola?" question by looking at the order in which each of these four triangles is traced when we follow the order of the points in the vectors X and Y. We record a minus for counterclockwise and a plus for clockwise: the product of these four signs is the sign of the discriminant.

The convenience of this triangle-tracing solution is in fact illusory: The center can be very far from the points, so as I sit at my desk in Chicago reflecting points, I may find that one of them lands somewhere in central Illinois (say 140 miles away in Urbana-Champaign) and I have no way to tell ccw from cw. The blue-highlighted example above illustrates a milder but still problematic case.

Though we only care about the sign of the discriminant, it is interesting to look at the cumulative distribution, which we can get from a version of Valery's simulation:

$$N := 10000$$

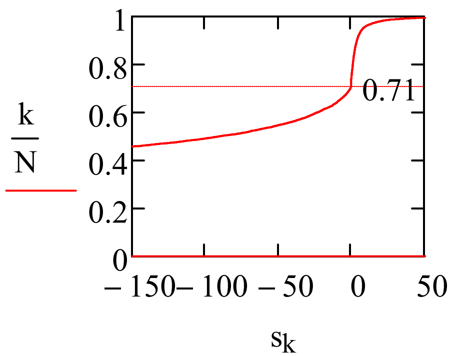
$$U := \left| \begin{array}{l} Z_{N-1} \leftarrow 1 \\ \text{for } k \in 0..N-1 \\ \quad \left| \begin{array}{l} x \leftarrow \text{runif}(5,-1,1) \\ y \leftarrow \text{runif}(5,-1,1) \\ c \leftarrow \text{cen}(x + i \cdot y) \\ X \leftarrow \text{submatrix}(x - \text{Re}(c),0,2,0,0) \\ Y \leftarrow \text{submatrix}(y - \text{Im}(c),0,2,0,0) \\ \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \leftarrow \begin{bmatrix} \left| \begin{pmatrix} X_0 & X_1 \\ Y_0 & Y_1 \end{pmatrix} \right| \\ \left| \begin{pmatrix} X_0 & X_2 \\ Y_0 & Y_2 \end{pmatrix} \right| \\ \left| \begin{pmatrix} X_1 & X_2 \\ Y_1 & Y_2 \end{pmatrix} \right| \end{bmatrix} \\ Z_k \leftarrow \dfrac{(\alpha + \beta + \gamma) \cdot (-\alpha + \beta + \gamma) \cdot (\alpha - \beta + \gamma) \cdot (\alpha + \beta - \gamma)}{4 \cdot (\alpha \cdot \beta \cdot \gamma)^2} \end{array} \right. \\ Z \end{array} \right.$$

$s := sort(U)$

$k := 0 .. N - 1$

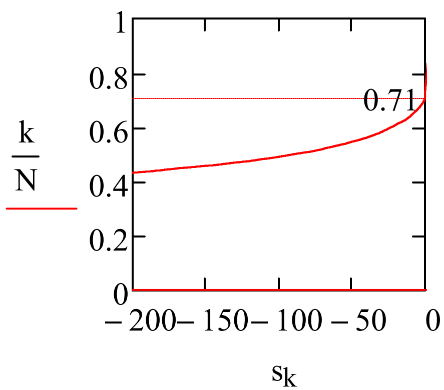
$max(s) = 1.225 \times 10^{3}$ $min(s) = -1.143 \times 10^{12}$

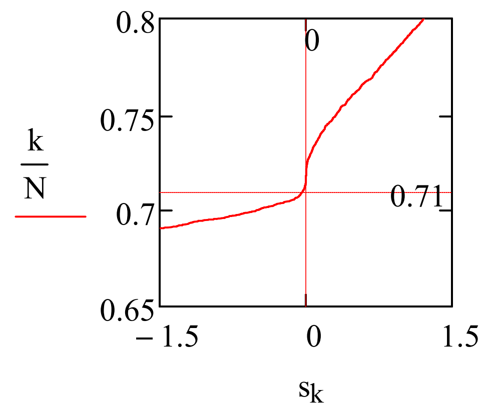
The enormous absolute value of the minimum presumably reflects a near-zero alpha, beta, or gamma.



So the cumulative distribution has a kink and a vertical derivative exactly at the transition from negative to positive (hyperbola to ellipse), at which point we have used up the hyperbola fraction of the values of the discriminant, leaving something like $1 - 0.71 = 0.29$ for the ellipses, as we expect from Valery's original simulation.

A few other views:

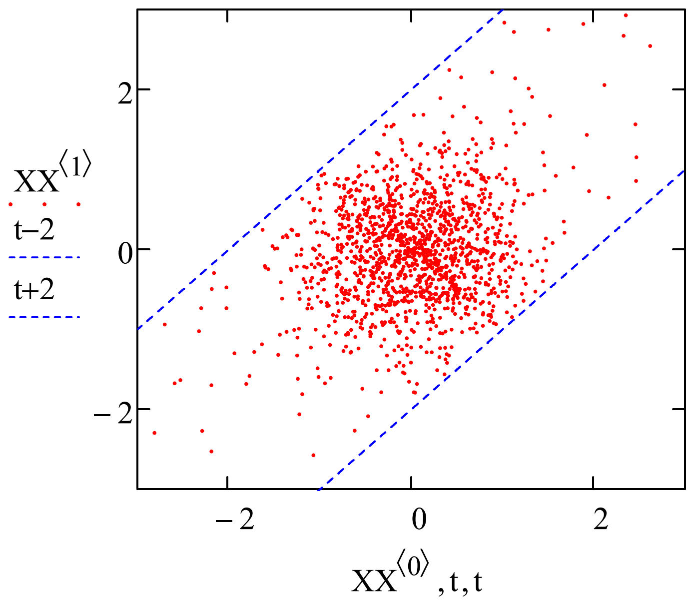




Let us see how the three points are correlated: The six coordinates started out as independent uniformly distributed random variables, but subtracting the center introduces a correlation.

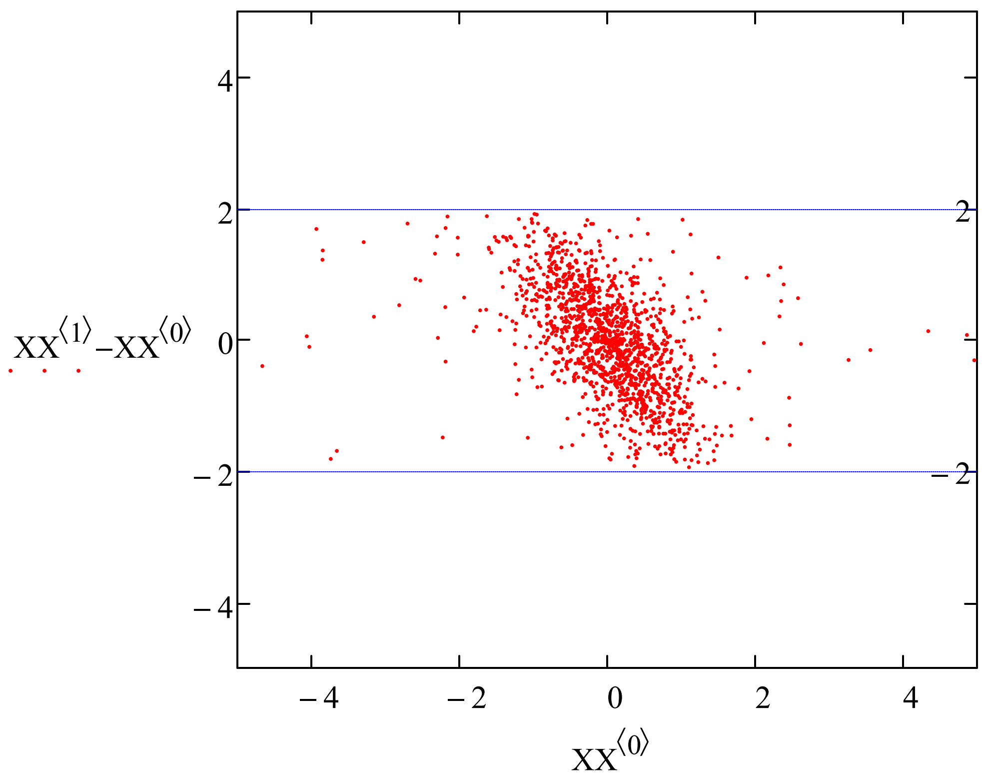
$N := 1500$

$$\begin{pmatrix} XX \\ YY \end{pmatrix} := \left| \begin{array}{l} X^{\langle N-1 \rangle} \leftarrow \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ Y^{\langle N-1 \rangle} \leftarrow \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \text{for } k \in 0 .. N-1 \\ \quad \left| \begin{array}{l} x \leftarrow \text{runif}(5, -1, 1) \\ y \leftarrow \text{runif}(5, -1, 1) \\ c \leftarrow \text{cen}(x + i \cdot y) \\ X^{\langle k \rangle} \leftarrow \text{submatrix}(x - \text{Re}(c), 0, 2, 0, 0) \\ Y^{\langle k \rangle} \leftarrow \text{submatrix}(y - \text{Im}(c), 0, 2, 0, 0) \end{array} \right. \\ \begin{pmatrix} X^T \\ Y^T \end{pmatrix} \end{array} \right.$$



$XX^{\langle 1 \rangle}$
t−2
t+2
2
0
−2
−2
0
2
$XX^{\langle 0 \rangle}$, t, t

There certainly is a correlation: for example, the plot suggests that $-2 \le XX^{\langle 1\rangle} - XX^{\langle 0\rangle} \le 2$). If we transform accordingly we see....



... but this is deceptive. Since

$$\min\left(XX^{\langle 0\rangle}\right) = -18.625$$

$$\max\left(XX^{\langle 0\rangle}\right) = 190.603$$

the actual correlation of this scatter is driven by outliers, not the nice negative-sloping cloud we see in the middle.

The next step might be to look at the distributions induced in the original points by subtracting the center, to see if any marginal information can be extracted. We're perhaps no closer here to solving Valery's problem, but leads in interesting directions.

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