Playing with a chain
Or
Physical and Mathematical Informatics

The article describes an educational laboratory work within the framework of interdisciplinary connections at the intersection of informatics, mathematics and physics: the study of the sagging of a closed chain with different support points. A technology of computer processing of photos and videos of a physical experiment with the subsequent processing of the media files on the computer is illustrated. We present the π chain number (1.258... − the optimal ratio of the chain length to the distance between its attachment points) and its relation to the previously unexplored problem of the shape of the sag of a closed chain hanging on two nails. The analytic expression for this constant was first found. A new physical and mathematical constant is found − the critical angle of the sagging of a closed chain on the "hangers". The problem of the sagging of a closed chain on a cone is detailed. The applicability of the computer tool "optimization with constraints" for solving problems of theoretical mechanics is investigated. Three main tools of mathematical computer tools are described: numerical mathematics, symbolic mathematics and graphics. The importance of using units of measurement when solving physical problems on a computer is emphasized. The problem of publishing mathematical formulas in articles and books is discussed.

Keywords: Theoretical mechanics, sagging chain, chain function, derivative, integral, potential energy, kinetic energy, Lagrange-Dirichlet principle, Newton's law, constrained optimization, system of algebraic equations, computer graphics, animation, Mathcad.

Introduction

Before *spinners* (turntables, twists) appeared, people who did not know what to do with their hands turned a coin, a box of matches or... a chain on their fingers.

Let's also turn the chain, but not on the fingers, but on ... two nails, on "hangers", on a cone ... And we will do this not for idleness, but for scientific and cognitive purposes. In passing, we will discover a couple of new physical and mathematical constants.

Theoretical, implementation and experimental parts

Usually articles of this kind are clearly divided into theoretical, implementation and experimental parts. But in this article it's impossible to do it − they mixed up with each other. The author hopes that the reader will understand why this happened, and will take this not quite an ordinary technique.

1. The chain on two nails

In [1] it is described how a chain is suspended at two ends (see Figure 1) and how it is all photographed and sent to the computer, where it is processed appropriately. The fingers of two schoolchildren were used as "nails". In this school class (excellent laboratory work for two classes − physical and computer) at the junction of mathematics, physics and computer science, it was once again shown that the chain is not sagging along the parabola (and this is a very common misconception, and not only for schoolchildren and students), but on the *catenary*, whose formula [2, 3] will be used in the calculations below.

Fig. 1. Experiment with the sagging chain

Let us consider some modifications that have not yet been investigated for a classical problem of the calculus of variations, the sagging chain problem.

Take a closed chain[1] and hang it not on a finger, but on two nails (on two pins), driven into the wall [2] – see Fig. 2. How will it sag? The problem is extremely idealized: the chain itself is "classical" (absolutely flexible and inextensible), and the frictional forces of the chain around the nails are absent. The first condition is easy to be met in an application, while the second is almost impossible, given the fact that the chain with its joints at the attachment points clings fairly tightly to the nails. For all this construction it is necessary, for example, to tap, so that it will come to the expected equilibrium. On nails it is possible to fasten a block with bearing and then to throw a chain on it. This will complicate the geometry of the problem, but it will allow calculations to be compared with an application in the same way as was done in [1].

[1] Circular in the sense that it has neither beginning nor end. Such chains without locks are hung around the neck, if the length of the chain allows it to pass through the head. In the third part of the article, we'll talk about putting the chain on a "neck without a head" − on a cone. By the way, about the locks. They can be regarded as certain point weights on the chain, complicating its calculation and making the task more interesting (the problem of a pendant on the chain, a cabin of the cableway). By "circularity" we can also mean the absolute smoothness of the chain − the absence of friction at the points of its attachment.

[2] A sheet of millimeter (graph) paper is attached to the wall, with which you can measure the coordinates of individual points of the sagging chain and compare them with those that were obtained as a result of calculation. Such a work is described in [1]. In our computer age millimeter paper is almost out of use. The author recently raked the rubble in his office and came across rolls of millimeter paper − ordinary, semi-logarithmic and logarithmic in both coordinates. There was an idea to paste it in place of wall paper in the room, where there is the laboratory works with the chain. But this idea of mathematical design of the premises is not new: the floor in the lobby of the headquarters of the Mathematical Association of America (IAA) in Washington is lined with tiles in the form of pentagonal parquet. A search and classification of polygonal parquets is a visual and interesting task of the theory of paving modern combinatorial geometry.

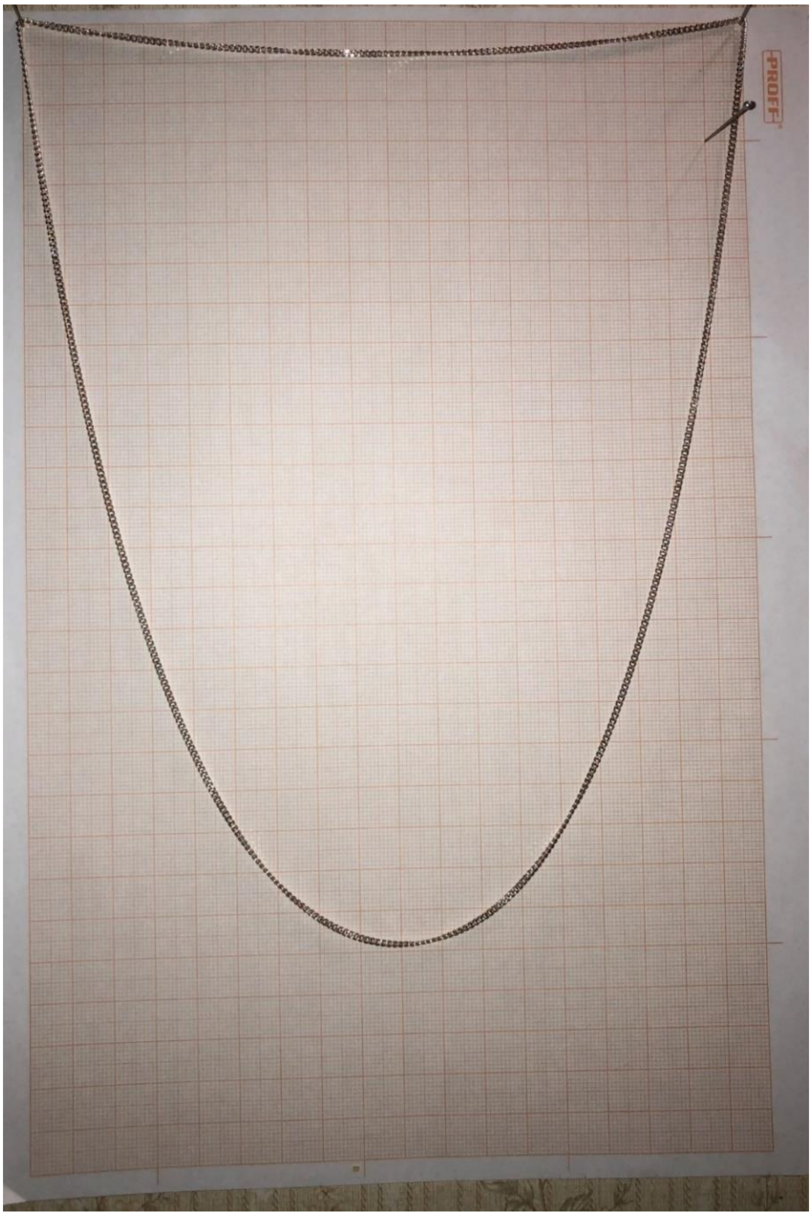


Fig. 2. Real closed chain suspended on two nails

Figure 3 shows the beginning of calculation of the sagging of the closed chain: input of the initial data [3] – length of the chain (60 cm) and coordinates of the attachment points of the nails, on which the closed chain is thrown (0-28 and 18-15 cm).

[3] As a rule, such problems are solved in dimensionless quantities. But the Mathcad package has a unit of measurement tool and it's a sin not to use it. This, in particular, will avoid mistakes of incorrect input of formulas. The user gram-force unit (gf) is introduced into the calculation – one thousandth of the built-in kilogram-force unit (kgf). Our calculations are tuned to a system of physical quantities cm-g-s, so the default values of forces will be output in dynes, which we will change to gram-force.

$L := 60cm$ Length of the circular chain

$x_1 := 0cm$ $y_1 := 28cm$ Left chain fastening point

$x_2 := 18cm$ $y_2 := 15cm$ Right chain fastening point

$$y(x,h,a,x_0) := h + a\cdot\left(\cosh\left(\frac{x - x_0}{a}\right) - 1\right)$$ Catenary

$$y'(x,a,x_0) := \frac{d}{dx}y(x,h,a,x_0) \rightarrow \sinh\left(\frac{x - x_0}{a}\right)$$ Derivative of the catenary

$m_c := 70\,\frac{mg}{cm}$ Linear mass of the chain $gf := \frac{kgf}{1000}$

$$\Delta := \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = 22.2\,cm$$ Distance between the points

$$PE(h_D, h_U, a_D, a_U, x_{0D}, x_{0U}) := \left| \begin{array}{l} L_D \leftarrow \int_{x_1}^{x_2} \sqrt{1 + \left(y'(x, a_D, x_{0D})\right)^2}\,dx \\ y_D \leftarrow \dfrac{\int_{x_1}^{x_2} y(x, h_D, a_D, x_{0D})\cdot\sqrt{1 + \left(y'(x, a_D, x_{0D})\right)^2}\,dx}{L_D} \\ y_U \leftarrow \dfrac{\int_{x_1}^{x_2} y(x, h_U, a_U, x_{0U})\cdot\sqrt{1 + \left(y'(x, a_U, x_{0U})\right)^2}\,dx}{L - L_D} \\ g\cdot m_c\cdot\left[y_D\cdot L_D + y_U\cdot(L - L_D)\right] \end{array} \right.$$

Fig. 3. Beginning of the Mathcad calculation of the sagging of a closed chain on two nails

Important note. The authors do not give individual formulas in this article, but place in it the texts of the solution of problems in the physical and mathematical program Mathcad. At the same time, the authors perfectly understand that this prevents some readers, accustomed to traditional publications with separate numbered formulas, to understand the essence of the tasks being solved. What can you say about this!? First, all the formulas of this article are fairly well read in the texts because Mathcad uses general mathematical notation employing WYSIWIG technology (What You See Is What You Get). Secondly, this style of publication meets the modern world trend, where formulas in books and articles are typed not in the text (mathematical) editors of LaTeX, Word Equation, MathML, etc., but in Maple, Mathematica, Mathcad, etc. By such "living" formulas, you can consider, it is permissible for them to build graphics and create animations, which significantly reduces the likelihood of errors and typos in them. Moreover, more and more often the chapters of books and journal articles are printouts of solving problems in mathematical programs. If such texts are not placed on paper but in electronic form, then they can be "live": the reader is allowed to change the original data and receive a new answer. The example in Fig. 4: author's settlement site for Mathcad Calculation Server technology, with which you can calculate the length of the chain by specifying the coordinates of the points of attachment of its ends h_1 and h_2, and also its clearance – the minimum distance from the "floor" h.

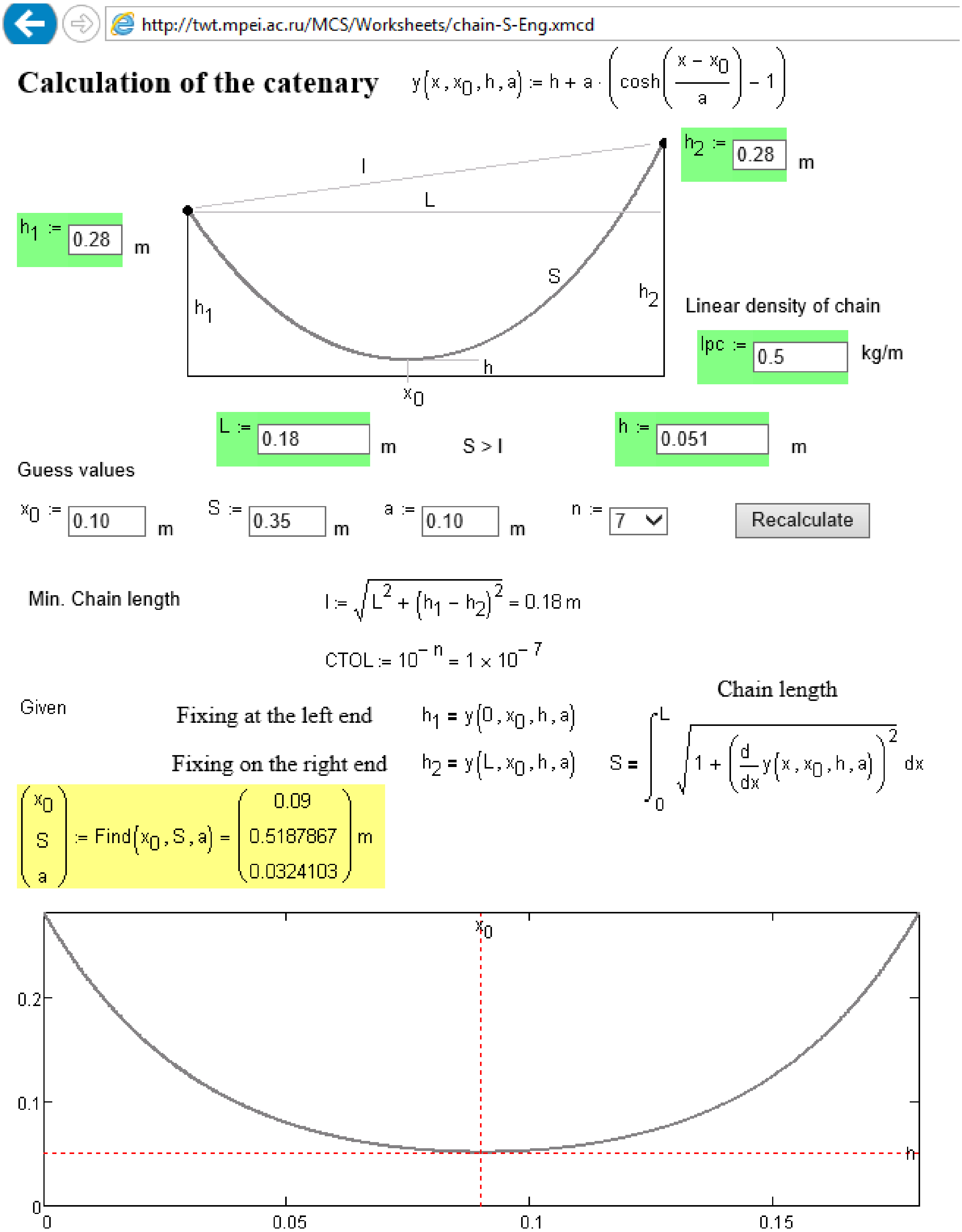


Fig. 4. Online-calculation of the sagging chain

Figure 4 shows the calculation of the length of the lower part of the sagging closed chain − 51.9 cm. If in this online calculation the value of h is changed from 5.1 to 27.1 cm, the length of the upper short part of the chain is calculated to be equal to 18.1 cm (one millimetre larger than the distance between the attachment points L). In total, this gives 70 cm − this is approximately the length of the chain shown in Fig. 2 after it was removed from the nails and its length was measured straight (using a ruler), and not by an indirect (calculated) method. So the theory is verified by practice!

Figure 3 also shows the user functions − the formula of the catenary[4] (y) and its derivative[5] (y'). And another function called PE with six arguments returns the value of the potential energy sagging on two nails of the closed chain, depending on the parameters of the two segments of the catenary, on which the chain is divided − see Fig. 2. We will conditionally denote these sites by indices as D (down, lower) and U (up, upper). But this division is conventional − the lower part of the chain can appear at the top, and the upper one at the bottom.

The law of mechanics (Lagrange principle [4]) states that the chain sags so that its potential energy becomes minimal. Here looms a typical *optimization* problem with *constraints*, that can be solved with the Mathcad built-in function Minimize, along with the Given keyword − see Fig. 5.

[4] The canonical formula of the catenary, used in paper and electronic mathematical reference books and textbooks, has the form $a \cdot \cosh(x / a)$. But in the calculations we will use a non-canonical form of the catenary with two additional parameters h and x_0: $h + a \cdot \cosh(((x-x_0) / a) - 1)$. In this formula, a specific point of this curve is clearly defined − the catenary has a minimum (sagging chain – $a > 0$) or maximum (arch − $a < 0$) at the point with coordinates $h - x_0$, which will need to be found in the course of our calculations.

[5]The formula of the derivative of the catenary function is generated by the Mathcad package itself using the symbolic mathematics tool (right arrow operator). Formulas for the length of this curve and for the ordinates of its center of gravity that use a certain integral and are also involved in calculations have been found on the Internet, although they can also be easily derived using the Mathcad symbolic mathematics or "in the mind", based on our own knowledge of the basics of mathematical analysis.

$$\begin{pmatrix} h_D \\ h_U \\ a_D \\ a_U \\ x_{0D} \\ x_{0U} \end{pmatrix} := \begin{pmatrix} 10 \\ 15 \\ 3 \\ 1 \\ 10 \\ 10 \end{pmatrix} cm \qquad \text{Guess values}$$

Given

$$\int_{x_1}^{x_2} \sqrt{1 + \left(y'\left(x, a_D, x_{0D}\right)\right)^2}\, dx + \int_{x_1}^{x_2} \sqrt{1 + \left(y'\left(x, a_U, x_{0U}\right)\right)^2}\, dx = L$$

$$y\left(x_1, h_D, a_D, x_{0D}\right) = y_1 \qquad y\left(x_2, h_D, a_D, x_{0D}\right) = y_2$$

$$y\left(x_1, h_U, a_U, x_{0U}\right) = y_1 \qquad y\left(x_2, h_U, a_U, x_{0U}\right) = y_2$$

$$\begin{pmatrix} h_D \\ h_U \\ a_D \\ a_U \\ x_{0D} \\ x_{0U} \end{pmatrix} := \text{Minimize}\left(PE, h_D, h_U, a_D, a_U, x_{0D}, x_{0U}\right) = \begin{pmatrix} 14.64 \\ 7.356 \\ 10.408 \\ 4.403 \\ 15.27 \\ 10.672 \end{pmatrix} cm$$

$$L_D := \int_{x_1}^{x_2} \sqrt{1 + \left(y'\left(x, a_D, x_{0D}\right)\right)^2}\, dx = 24.13\, cm \qquad L_U := L - L_D = 35.87\, cm$$

Fig. 5. Minimizing the potential energy of a hanging closed chain

The following calculation steps are executed in Fig. 5:

– input of the initial assumption for the numerical solution of the minimization problem;

– entering constraints after the word Given; these are: the length L of the chain remains constant, and the two sections of the closed chain – the lower (D) and the upper (U) are fixed at the ends at two given points (1 and 2);

– execution of the Minimize function, that returned as answers: the parameters a, h and x_0 of the two parts of the catenary (D and U) so that their potential energy PE is minimal and the constraints are satisfied. This answer (the vector of constants) can be transferred to the initial assumptions and assures that the answer does not change. However we should always remember that this answer is *approximate*, depending on the value of the given accuracy of the numerical solution of the problem.

The value m_c is the linear mass of the chain. It does not affect the shape of its chain, but it will help us, first, to check the correctness of the calculation shown in Fig. 5 from the point of view of the balance of forces, and secondly, to calculate the forces that stretch the chain in correspondence of its attachment on the two nails. This new problem can be reduced to the numerical solution of an overdetermined system of eight nonlinear algebraic equations with six unknowns describing the "mechanics" of a closed chain sagging on two nails with a minimum potential energy. The calculation uses the Mathcad built-in function Find, which returns the values of unknown equations that convert them into identities with accuracy determined by the method and by the parameters of the numerical solution of the problem.

$$\begin{pmatrix} F_{y1D} \\ F_{y2D} \\ F_{y1U} \\ F_{y2U} \\ F_{xD} \\ F_{xU} \end{pmatrix} := \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} N$$

Given

$$F_{y1D} + F_{y2D} = g \cdot m_c \cdot L_D \qquad F_{y1U} + F_{y2U} = g \cdot m_c \cdot L_U$$

$$F_{y1D} = -y'(x_1, a_D, x_{0D}) \cdot F_{xD} \qquad F_{y1U} = -y'(x_1, a_U, x_{0U}) \cdot F_{xU}$$

$$F_{y2D} = y'(x_2, a_D, x_{0D}) \cdot F_{xD} \qquad F_{y2U} = y'(x_2, a_U, x_{0U}) \cdot F_{xU}$$

$$F_{y1D}^2 + F_{xD}^2 = F_{y1U}^2 + F_{xU}^2$$

$$F_{y2D}^2 + F_{xD}^2 = F_{y2U}^2 + F_{xU}^2$$

$$\begin{pmatrix} F_{y1D} \\ F_{y2D} \\ F_{y1U} \\ F_{y2U} \\ F_{xD} \\ F_{xU} \end{pmatrix} := \mathrm{Find}(F_{y1D}, F_{y2D}, F_{y1U}, F_{y2U}, F_{xD}, F_{xU}) = \begin{pmatrix} 1.54365 \\ 0.11098 \\ 1.74238 \\ 0.80299 \\ 0.83945 \\ 0.30399 \end{pmatrix} \cdot gf$$

$$F_1 := 2\sqrt{F_{y1D}^2 + F_{xD}^2} = 3.514 \cdot gf \qquad F_2 := 2\sqrt{F_{y2D}^2 + F_{xD}^2} = 1.694 \cdot gf$$

Fig. 6. Calculation of tension forces of a chain on two nails

Physical laws, shown in the calculation in Fig. 6, such as:

- the sum of the values of the vertical projections of the forces of fastening the two parts of the chain on two nails are equal to the values of the weight of the two parts of the chain; The name variable F_{y1D} means that it is the vertical (y) projection of the force (F) with which the lower (D) part of the chain pulls the left (1) nail down;
- the values of the horizontal projections of the forces of fastening the parts of the chain on the two nails (F_{xD} and F_{xU}) are equal to each other and are related to the values of the vertical projections of these forces through the values of the derivative of the catenary at these two points: $y'(x_1, a_D, x_{0D})$, $y'(x_2, a_D, x_{0D})$, $y'(x_1, a_U, x_{0U})$ and $y'(x_2, a_U, x_{0U})$;
- the tension forces of the lower and upper parts of the chain at the points of their fastening are equal: the chain parts do not overstretch each other and are in equilibrium.

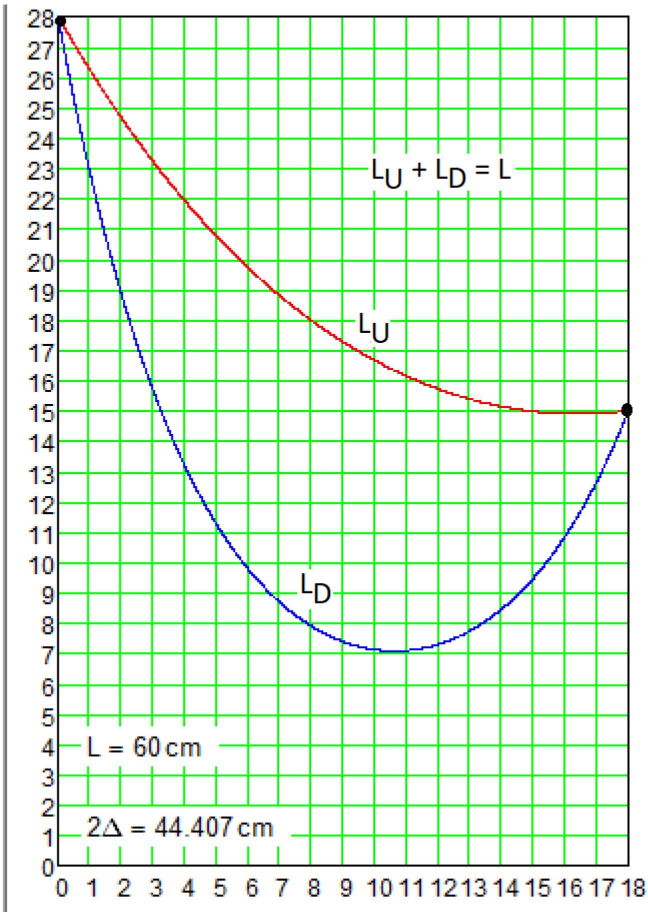
The calculation shown in Fig. 6, is verified. We were convinced that the values of the parameters of the catenary (the variables a_D, a_U, h_D, h_U, x_{0D} and x_{0U}), found through minimization of the potential energy, correspond to the values of the forces ensuring the equilibrium conditions of our mechanical system.

But what did the numerical computer experiment show us?!

The fact that the closed chain, depending on the location of the two nails can sag in two ways − see Figures 7 and 8.

Case 1. The chain is divided into two unequal sections − see Fig. 7.

Case 2. The chain does not bifurcate and sags in two equal halves − see Fig. 8.



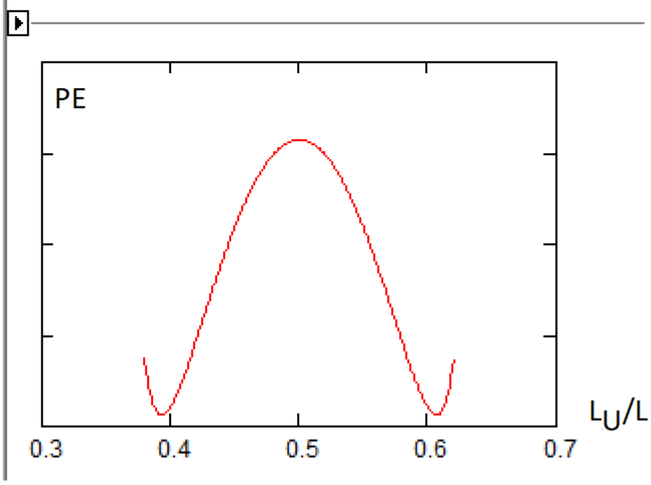


Fig. 7. The closed chain hanging on two nails is divided into two unequal sections

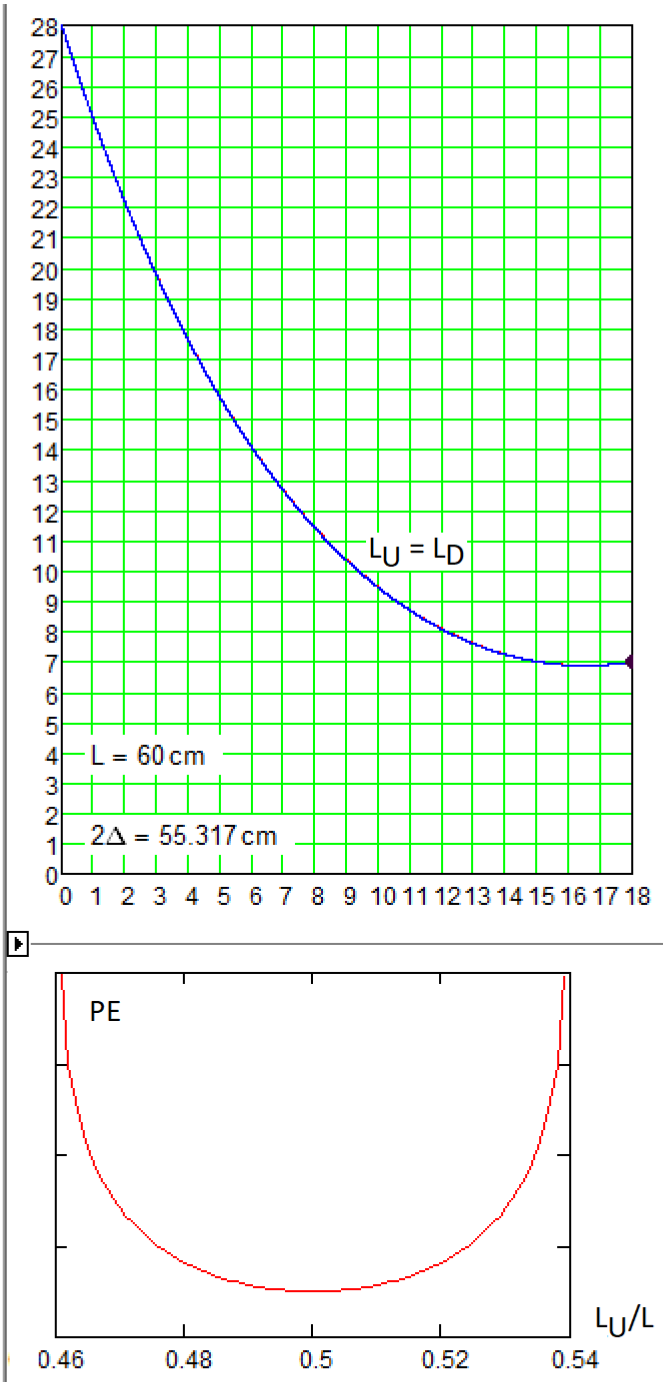


Fig. 8. The closed chain hanging on two nails is divided into two equal sections

At the bottom of Figures 7 and 8, the graphical dependences of the potential energy value of two parts of the closed chain are shown depending on the ratio of their lengths − L_U/L. The following special points are visible on the graphs:

- Fig. 7 − two minima (stable equilibrium) and one maximum (unstable equilibrium − the chain can in principle be of this form with two equal halves (see Figure 8), but at the slightest external disturbance it will slip into the left or right potential wells);

- Fig. 8 − one minimum (stable equilibrium − the chain sags in two equal parts).

Figure 9 shows a Mathcad document that defines a user function that returns the value of the potential energy (PE) of the chain sagging on two nails, depending on the ratio of the length of one part to the total length of the chain (LU/L). In this calculation case the Mathcad package built-in function Find does not return a constant vector as in Fig. 6, but a vector of user-defined functions − dependent on the parameters of the two parts of the catenary, as function of the variable (argument of the function) LU/L. These vector functions named AnsD and AnsU are decomposed into separate scalar functions with names h_D, a_D, etc. This technique allowed us to create a function named PE and then build its graphs, depending on the ratio of the length of the L_U to the length L for different coordinates of the points of attachment of the closed chain − see Figures 7 and 8.

Given

$$y(x_1, h_D, a_D, x_{0D}) = y_1 \qquad y(x_2, h_D, a_D, x_{0D}) = y_2 \qquad \int_{x_1}^{x_2} \sqrt{1 + \left(y'(x, a_D, x_{0D})\right)^2}\, dx = L \cdot LU/L$$

$$AnsD(LU/L) := Find(h_D, a_D, x_{0D})$$

$$h_D(LU/L) := AnsD(LU/L)_0 \qquad a_D(LU/L) := AnsD(LU/L)_1 \qquad x_{0D}(LU/L) := AnsD(LU/L)_2$$

Given

$$y(x_1, h_U, a_U, x_{0U}) = y_1 \qquad y(x_2, h_U, a_U, x_{0U}) = y_2 \qquad \int_{x_1}^{x_2} \sqrt{1 + \left(y'(x, a_U, x_{0U})\right)^2}\, dx = L - L \cdot LU/L$$

$$AnsU(LU/L) := Find(h_U, a_U, x_{0U})$$

$$h_U(LU/L) := AnsU(LU/L)_0 \qquad a_U(LU/L) := AnsU(LU/L)_1 \qquad x_{0U}(LU/L) := AnsU(LU/L)_2$$

$$PE(LU/L) := \int_{x_1}^{x_2} y(x, h_D(LU/L), a_D(LU/L), x_{0D}(LU/L)) \cdot \sqrt{1 + \left(y'(x, a_D(LU/L), x_{0D}(LU/L))\right)^2}\, dx \ldots$$
$$+ \int_{x_1}^{x_2} y(x, h_U(LU/L), a_U(LU/L), x_{0U}(LU/L)) \cdot \sqrt{1 + \left(y'(x, a_U(LU/L), x_{0U}(LU/L))\right)^2}\, dx$$

Fig. 9. The dependence of the potential energy of the two parts of the sagging closed chain on the ratio of their lengths[6]

To find the boundary separating the two forms of sagging of a closed chain on two nails, we will be helped ... by an interlude.

Interlude. The chain number π

Problem. How much longer should the chain (non-closed, conventional with two ends) be than the distance between the two points of its suspension, which are on the same level, so that the tension at the ends of the chain is minimal?

[6] In the formula for the potential energy, in Figure 9, the constant $m_c \cdot g$ (the linear weight of the chain material − see Figure 5) was removed and other simplifications were made that accelerate the calculation. However it's still quite long. The author, on his rather fast computer, drew the energy curves shown in Fig. 7 and 8, in several tens of minutes. The calculation of the coordinates of each point of this curve requires the numerical solution of a rather complex system of nonlinear algebraic equations with high accuracy. And this task itself is very time-consuming in addition to the other resources of the computer.

This is a typical problem of finding a minimum: if you increase the length of the chain, then this force will enhance due to the growth of its vertical projection due to the increase in the weight of the chain. If the chain is shortened, the force of its tension at the attachment points will increase due to the growth of its horizontal projection. When the value of the chain length approaches the value of the distance between the points of its attachment, it ceases to be just a chain and becomes ... a tightly stretched string. The calculation of the string, its vibrations − is a separate very interesting physical and mathematical problem, where the chain, pardon, string cannot be considered inextensible. Here it is necessary to take into account the modulus of elasticity of the string material and other parameters.

Figure 10 shows the solution of this problem using Mathcad.

To solve the problem, we need to relate the force F and the length of the chain L, i.e. to define a function F(L) − the optimization *objective function*. In the calculation in Figure 10, two user functions are routinely defined: a catenary whose minimum / maximum falls on the Y axis (the X axis passes through the two points of the chain attachments), and its derivative with respect to x. The problem is reduced to the solution of a system of two non-linear algebraic equations (the equality of the heights of the attachment points of the chain and the invariance of the value of the chain length) using the Given-Find block, where the Find function returns not two concrete numbers (the numerical solution of the system of two equations), but defines a user function with name Ans (see also this technique in Figure 9). Then this function-vector (vector because the function returns two numbers) is decomposed into two separate functions a(L) and h(L). Next, the optimization objective function F(L) is formed, along which the graph is plotted and in which the minimum is specified − the point where the derivative value is zero.

$$y(x,a,h) := h + a\cdot\left(\cosh\left(\frac{x}{a}\right) - 1\right)$$ Catenary

$$y'(x,a) := \frac{d}{dx}y(x,a,h) \rightarrow \sinh\left(\frac{x}{a}\right)$$ Catenary Derivative

$$X := 0.5m \qquad m_c := 1\frac{kg}{m}$$ Input Data

$$F_y(L) := \frac{m_c\cdot g\cdot L}{2}$$ Vertical force on the suspended chain as function of its length

$$a := 1m \qquad h := -0.2m$$ First approximation

Given

$$y(-X,a,h) = y(X,a,h) = 0m$$ Height Equality at the attachment points

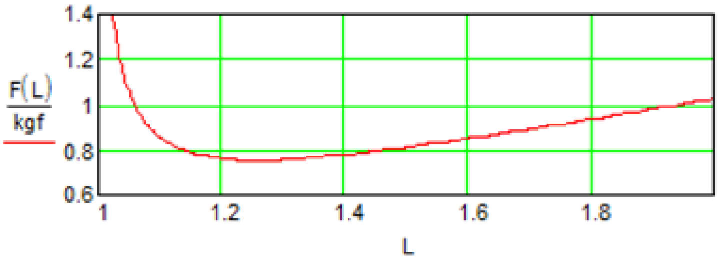
$$L = \int_{-X}^{X}\sqrt{1 + y'(x,a)^2}\,dx$$ Chain length

$$Ans(L) := Find(a,h) \qquad a(L) := Ans(L)_0 \qquad h(L) := Ans(L)_1$$

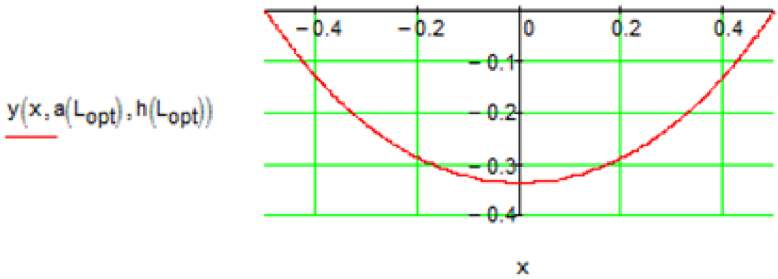
$$\alpha(L) := atan(y'(X,a(L)))$$ Chain angle at the right attachment point

$$F(L) := \frac{F_y(L)}{\sin(\alpha(L))}$$ Optimization Objective Function

$$L := 2\cdot X + \frac{X}{20}, 2\cdot X + \frac{X}{20} + \frac{X}{300} .. 4X$$



$$L_{opt} := root\left(\frac{d}{dL}F(L), L, 2\cdot X + \frac{X}{20}, 4X\right) = 1.258m$$



$$\alpha(L_{opt}) = 56.466\,deg$$

Fig. 10. Determination of the optimal chain length

The number 1.258 ... (the ratio of the length of the optimal chain to the distance, between the points of its attachment at the same level) can be regarded as some new physical and mathematical constant and give it the name π_c (chain number π – catenary) – the ratio of the arc length of the optimal catenary to its "diameter" – the distance between the attachment points. The usual number π is, as everyone knows, the ratio of the arc (the whole circle) to its diameter. It is possible to start a competition − to calculate the maximum number of digits in this constant, similar to how it is done with respect to the "circular" number π: 3.142.

Discussion of this problem with its analytical solution can be found here: https://community.ptc.com/t5/PTC-Mathcad-Questions/Symbolic-solution-of-one-optimization-problem/td-p/130281. The author of the analytical solution is Luc Meekes from Netherland – see the Fig. 11.

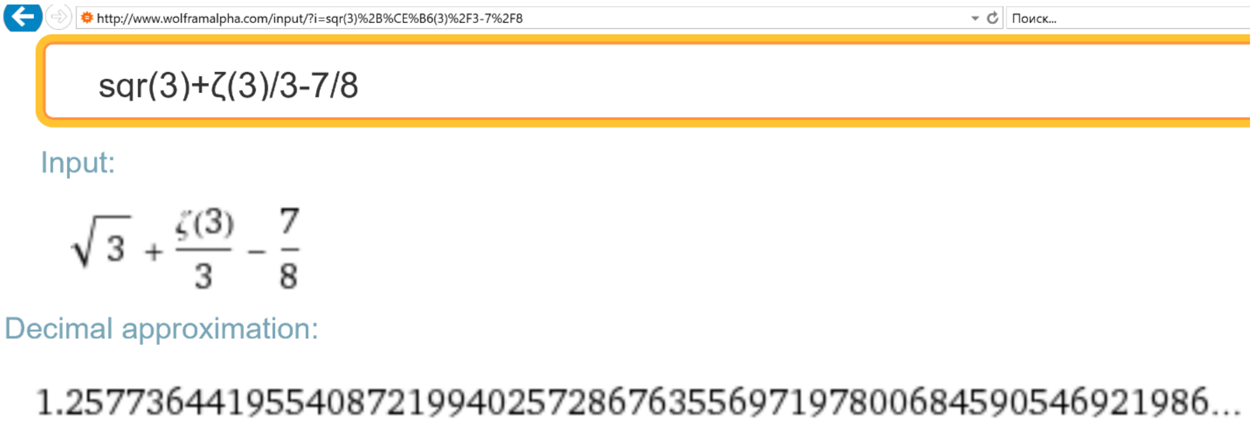


Fig. 11. Symbolical calculation of the optimal chain length in WolframAlpha site

The function ζ(x)’ is the Riemann Zeta function. You might want to check if, with sufficient numerical accuracy, your π_c as approximated, reaches this number, or differs from it.

Hence a practical advice. If you want to hang a chain, rope or cable between two posts or walls at the same height, then, in the absence of any special requirements, it will be sufficient to choose a chain length that is greater by about a quarter of the distance between the attachment points, which are naturally on the same level. In this case ... see above. If the points of attachment of the chain are at different levels, then this constant will become different (and be no longer a constant, but a variable instead).

After the publication in the author's book [5] of this research and the emergence of the hope for the discovery of a new physico-mathematical constant, a search was made for the Internet on the key "1.258 catenary", which led to the publication "Optimal form of a sagging cable" [6], in which already appeared the number 1.258. So we are a little late for the priority on this constant, but we consider ourselves to be his co-author.

(End of the interlude)

So that's it! Let's sketch a chain on two nails as shown in Fig. 7, and we will gradually reduce its length. A numerical experiment shows that at the moment of the fusion of two sections of the chain (Figure 8), the force applied to the nails acquires a minimum value. Further, this force will only increase. This is a certain boundary between the two forms of sagging of the chain shown in Fig. 7 and 8.

A closed chain can be rotated not only on the fingers, but also in a different way: by hanging it on the left and right index fingers and rotating them around the circumference. Figure 12 shows the animation frames of this hand exercise in its mathematical implementation: one point of attachment of the chain is fixed, while the second is rotating in a circle around it. Rather, only one quarter of a circle, but on the other three quarters the result will be identical (symmetrical).

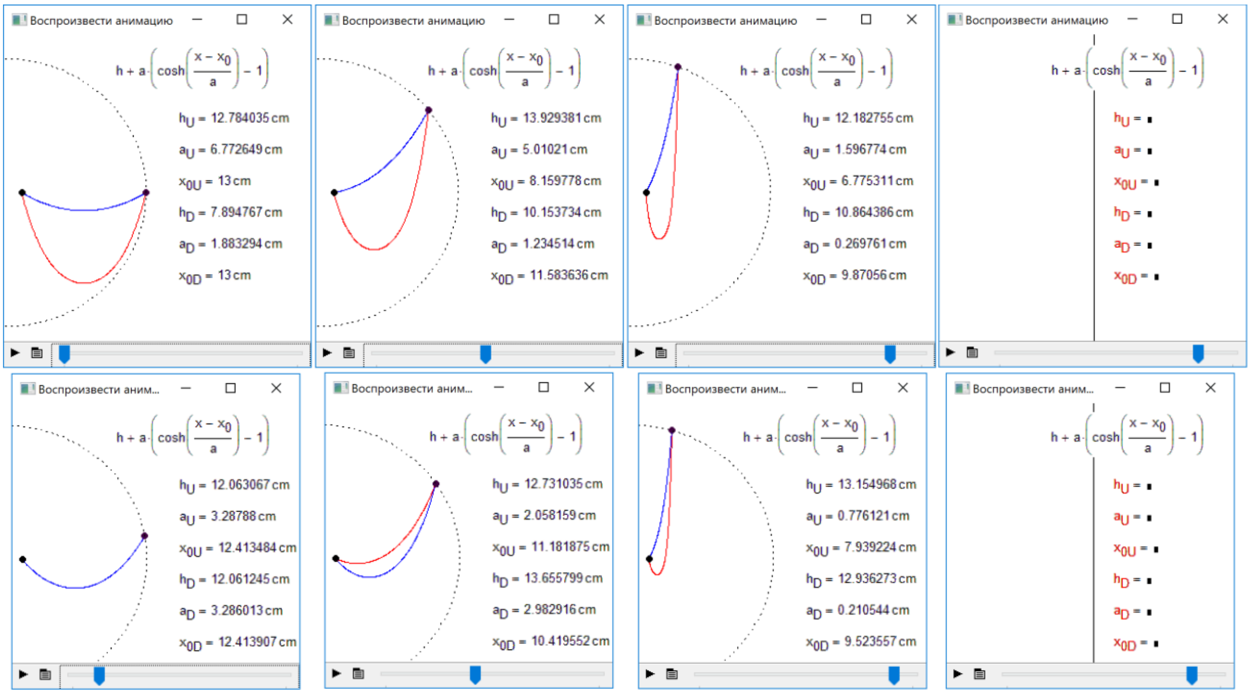


Fig. 12. Frames of animation of rotation of a closed chain between two nails: the top row is a long chain, the bottom row is a short chain.

What conclusions can be drawn by analysing Fig. 12?

Firstly, the distance between the points of suspension of the chain is not the only criterion determining the form of its sagging. This criterion is unique for the case when these points are on the same level – see Fig. 10.

Secondly, two frames of animation on the right edge of Fig. 12 once again underline the limits of the numerical solution of the problem: under certain initial conditions, there is no such solution. This is connected with the very essence of numerical methods and with the fact that a vertically sagging chain requires a different formula for calculation, rather than the one shown in the animation frames in Fig. 12.

2. Chain on "hangers"

A hanger, human shoulders designed, is in the form of an obtuse-angled isosceles triangle with a hook at the top. It is hung up with clothes in the wardrobe[7].

Some things (women's dresses, for example) often slip off from these "shoulders (hangers)" and fall down. Or vice versa, the straps are assembled at the bottom of the hook – at the top of the triangle. Which, too, is not good in terms of keeping things safe and undisturbed. So our research will have not only purely scientific, but also some applied value!

Let's see how this hanger will hold not lingerie underwear, but our closed chain.

[7] Conditionally all hotels in the world can be divided into two different groups. In the closets of some of them (and these, alas, are a minority) hang ordinary "home" hangers. In others, one might say, not "homemade" but "wild" hangers, which consist of two parts – the "hook", which is tightly tied to the rod in the closet, and actually the hanger itself without a hook, that is necessary to cunningly combine with the hook. This is very inconvenient, but it is done to counter the theft of hangers by hotel guests. The author saw in one hotel also a third type. There the hangers were attached to the closet ... by a chain – the object of our investigation. In the old days, in public place, a chain often attached a mug to a pot of drinking water. Now it is often possible to see in offices a chain fastened to a pen for writing so that visitors do not accidentally or intentionally take it away.

Problem. Take the already known closed chain, which is not hung on two nails (Figure 2), but on two straight segments ("on the hangers" − see Figure 13). The forces of friction (a stiction), as in the previous problem, are not taken into account, and the chain is inextensible and completely flexible. How will this chain hang?

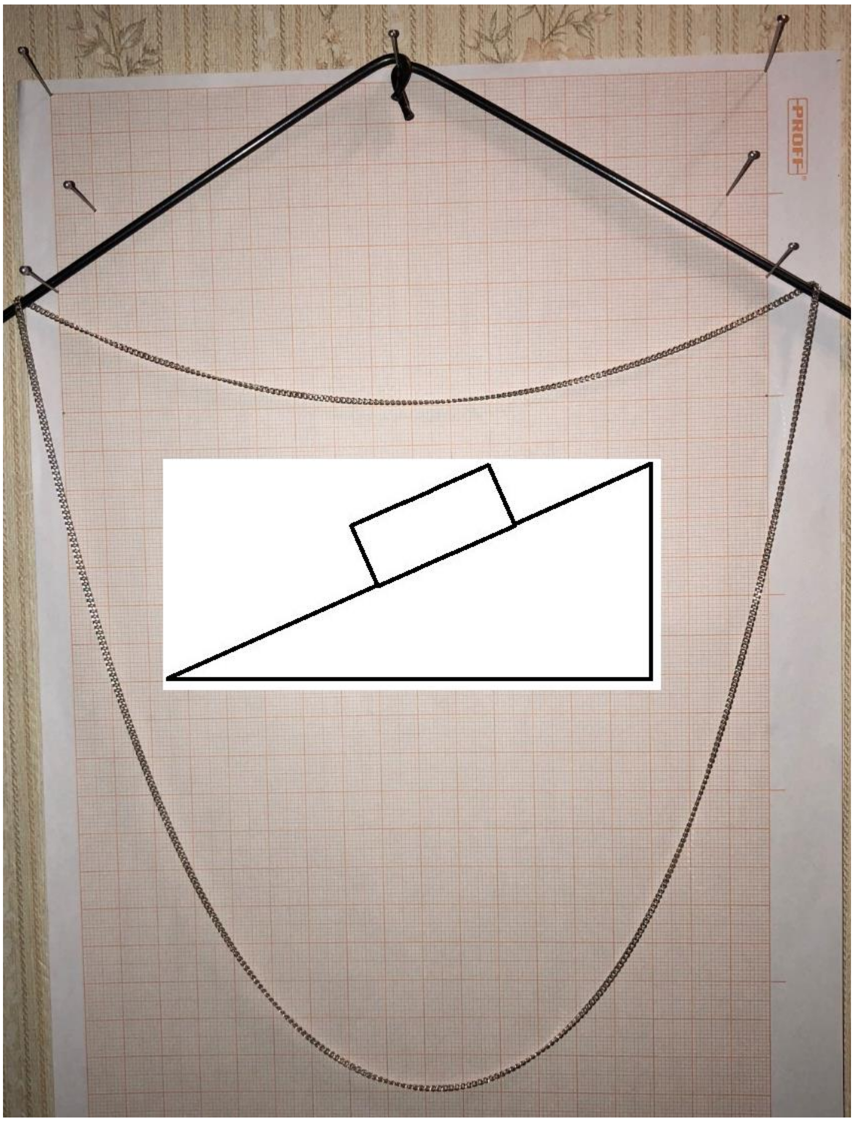


Fig. 13. Closed chain on hangers

Figure 14 shows the protocol for the numerical solution of this problem in Mathcad. The already familiar Minimize function is used, minimizing the potential energy of the closed chain under certain restrictions. The task is similar to the previous one, but the contact points of the chain are not fixed at two points, but "slide" along the segments of the inclined lines − along the "hangers".

It is assumed that the chain on the "hangers" can also hang in an asymmetrical position (obliquely − "a crooked smile"). Therefore, two unknown quantities x_L and x_R are entered in the calculation: the distances from the vertical axis where the hook is located to the left and right suspension points.

$$y(x, h, a, x_0) := h + a \cdot \left(\cosh\left(\frac{x - x_0}{a} \right) - 1 \right)$$

$$y'(x, a, x_0) := \frac{d}{dx} y(x, h, a, x_0) \rightarrow \sinh\left(\frac{x - x_0}{a} \right)$$

$$PE(x_L, x_R, h_D, h_U, a_D, a_U, x_{0D}, x_{0U}) := \int_{x_L}^{x_R} y(x, h_D, a_D, x_{0D}) \cdot \sqrt{1 + y'(x, a_D, x_{0D})^2} \, dx \ldots$$

$$+ \int_{x_L}^{x_R} y(x, h_U, a_U, x_{0U}) \cdot \sqrt{1 + y'(x, a_U, x_{0U})^2} \, dx$$

$$\left(x_L \;\; x_R \;\; h_D \;\; h_U \;\; a_D \;\; a_U \;\; x_{0D} \;\; x_{0U} \right) := (-1 \;\; 2 \;\; 2.5 \;\; 2 \;\; 5 \;\; 7 \;\; -1 \;\; 1)\, m \qquad \text{Guess values}$$

Given

$$\int_{x_L}^{x_R} \sqrt{1 + y'(x, a_D, x_{0D})^2} \, dx + \int_{x_L}^{x_R} \sqrt{1 + y'(x, a_U, x_{0U})^2} \, dx = L$$

$$y(x_L, h_D, a_D, x_{0D}) = x_L \cdot \tan\left(\frac{\pi}{2} - \alpha \right) \qquad y(x_R, h_D, a_D, x_{0D}) = -x_R \cdot \tan\left(\frac{\pi}{2} - \beta \right)$$

$$y(x_L, h_U, a_U, x_{0U}) = x_L \cdot \tan\left(\frac{\pi}{2} - \alpha \right) \qquad y(x_R, h_U, a_U, x_{0U}) = -x_R \cdot \tan\left(\frac{\pi}{2} - \beta \right)$$

$$\begin{pmatrix} x_L \\ x_R \\ h_D \\ h_U \\ a_D \\ a_U \\ x_{0D} \\ x_{0U} \end{pmatrix} := \text{Minimize}(PE, x_L, x_R, h_D, h_U, a_D, a_U, x_{0D}, x_{0U}) = \begin{pmatrix} -1.66502 \\ 1.66501 \\ -3.35281 \\ -3.35269 \\ 3.03114 \\ 3.03188 \\ -0.00002 \\ -0.00002 \end{pmatrix} m$$

Fig. 14. Calculation of the position of the closed chain on the "hangers"

The chain position shown in the photo in Fig. 13, is not an equilibrium one. It is an artificial, determined by the friction force (a stiction) of the chain on the "hangers". Analogy: the bar on the inclined plane (see the centre of Figure 14) is stationary, although it must slide downward, reducing its potential energy. The cause of immobility is the force of friction. And not the frictional force of sliding, which is calculated by a simple formula, but the frictional force of rest, the calculation of which is much more complicated. If these forces would not exist, the chain would either sag with two identical halves symmetrical about the Y axis, or slip off the hangers and sagged vertically, catching the base of the hook of the hanger. And this "either" does not depend on the length of the chain, but is only related to the angle of disclosure of the "hangers". Here there is a qualitative difference between the "hangers" (Figure 13) and the two nails (Figure 2), on which two parts of the closed chain can sag either in two unequal sections (see Figure 7) or symmetrically (Figure 8) to achieve the minimum potential energy. The chain on the "hangers" always hangs in half. It begins, however, to divide into two unequal parts at the moment when the slip begins before slipping off the "hangers".

With the help of a numerical experiment the critical opening angle of the "hangers", in which the chain jumps off the hanger, was determined. To determine this angle, an animation was created, two frames of which are shown in Fig. 15. The animation itself can be seen here https://community.ptc.com/t5/PTC-Mathcad-Questions/Round-chain-on-the-corner/td-p/553624 (site with discussion of the task). With an increase in the angle α (the angle of the "opening" of the hangers) by an amount greater than 50 angular degrees, the chain jumps off the "hangers" and hangs on the hook attachment point − at the apex of the obtuse angle.

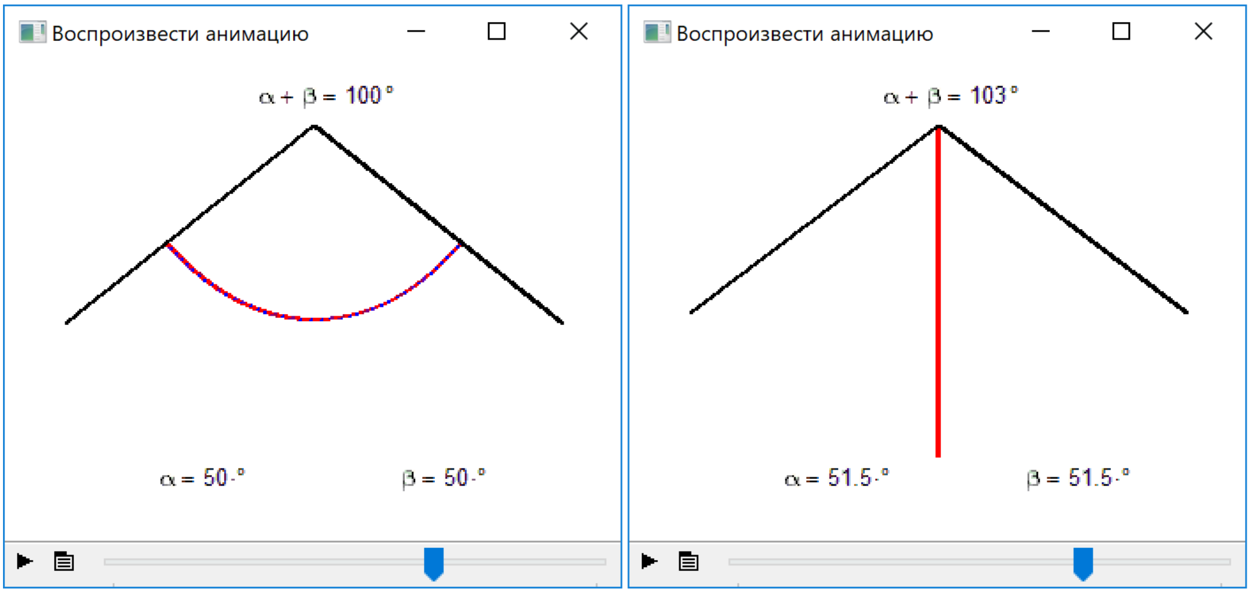


Fig. 15. Two frames of the animation of the closed chain jump from the hangers

Figure 16 shows the calculation of this critical angle - the numerical determination of the value of α, at which the potential energy of a chain sagging along two identical arcs of a chain line (PE), is equal to the potential energy of a chain hanging vertically ($-L^2/4$) and hooked to the top of the "hangers"[8]. This angle α_{cr} was slightly more than 50 degrees, which correlates with the data obtained when viewing the animation (Figure 15).

[8] First the catenary parameters X, Y (right attachment point), h (clearance) and a (catenary scaling parameter) are determined as function of the angle α through 4 equations. Then the critical angle is found by solving the equation equating the potential energy of two identical arcs of a chain line (PE_c) to the potential energy of a chain hanging vertically (PE_v). Conditionally by combining the equation of the arc length and the condition on the derivative we obtain the so-called Whewell equation of the catenary (i.e. L/2=a tan(α)).

$$L := 9m \qquad \begin{pmatrix} X \\ h \\ a \end{pmatrix} := \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix} m \qquad \text{Guess values}$$

Given

$$2\int_{-X}^{X} \sqrt{1 + \sinh\left(\frac{x}{a}\right)^2}\,dx = L \qquad (1)$$

$$h + a\cdot\left(\cosh\left(\frac{X}{a}\right) - 1\right) = -X\cdot\cot(\alpha) \qquad (2)$$

$$\sinh\left(\frac{X}{a}\right) = \tan(\alpha) \qquad (3)$$

$$Ans(\alpha) := Find(X, h, a)$$

$$X(\alpha) := Ans(\alpha)_0 \qquad h(\alpha) := Ans(\alpha)_1 \qquad a(\alpha) := Ans(\alpha)_2$$

$$PE(\alpha) := 2\int_{-X(\alpha)}^{X(\alpha)} \left[h(\alpha) + a(\alpha)\cdot\left(\cosh\left(\frac{x}{a(\alpha)}\right) - 1\right)\right]\cdot\sqrt{1 + \sinh\left(\frac{x}{a(\alpha)}\right)^2}\,dx$$

$$\alpha_{cr} := root\left(PE(\alpha) - \frac{-L}{4}\cdot L, \alpha, 45°, 55°\right) = 50.3395°$$

Fig. 16. Calculation of the critical angle of a closed chain sagging on the hangers

Most likely, an analytical rather than a numerical approach to solve the chain problem on the "hangers" will give an answer not in angular degrees, but in the form of a fraction $n\cdot\pi / m$, where n and m are natural numbers greater than unity. Mathematicians are always somewhat upset when it is said that a sine of 60 angular degrees[9] is equal to 0.866 ... The orthodox mathematician here will say differently − see the second line of calculation in Fig. 17.

$$\sin(60°) = 0.866$$

$$\sin\left(\frac{\pi}{3}\right) \rightarrow \frac{\sqrt{3}}{2}$$

Fig. 17. Numerical and symbolic calculation of the sine in Mathcad

You can try to find analytically (symbolically) the value of α_{cr} − the values of n and m in the expression $n\cdot\pi / m$, but you can do it in the way shown in Fig. 18 − sorting out the values of n and m in the range

[9] An air balloon burst from the clouds. From the basket the flyer shouts to a man on the ground: "Excuse me, where are we?" Answer from the man: "You are in a basket of a balloon!". He was a mathematician. Only from a mathematician can we hear an absolutely accurate but completely useless answer. There is a little-known continuation of this old joke. The flyer in response shouts: "You do not understand! Our navigator’s batteries are depleted, and we cannot determine our coordinates! ". The mathematician looks at his smartphone and answers: "Zero point ninety-six hundredths radians of northern latitude and zero sixty-four hundredth radians of eastern longitude!" Only mathematicians measure angles in radians, and not in more familiar degrees!

from 1 to 30, and remembering those (n = 7, m = 25), which most correspond to the numerical response in Fig. 16.

```
(n)  :=  | Δmin ← 1                                  = ( 7)
(m)      | for  n ∈ 1 .. 50                             (25)
         |   for  m ∈ 1 .. 50
         |      | Δ ← |50.34deg − (n/m)·π|
         |      | if  Δ < Δmin
         |      |    | Δmin ← Δ
         |      |    | n1 ← n
         |      |    | m1 ← m
         | (n1)
         | (m1)
```

$$\frac{7}{25}\pi = 50.4\,\mathrm{deg}$$

Fig. 18. Refinement of the value of the critical angle of the opening of the "hangers" with the chain

The calculation shown in Fig. 18, of course, can be considered a kind of curiosity − an imitation of an analytical approach to the solution of the problem. But the authors emphasize this approach not in vain, since it is very common in our computer age. In some cases, it is very productive, while in others it is counterproductive. We entrust those readers who are strong in analytical transformations, to confirm or refute the assertion that $\alpha_{cr} = 7\pi / 25$.

The reader can also replace rectilinear hangers by round ones or in the form of a segment of a parabola, a hyperbola or even a segment of a chain line (a catenary) and analyze the behaviour of a closed chain on such "designer hangers." On the chain, you can hang a pendant (point mass) and evaluate the properties of such a mechanical system.

3. Chain on a cone

The behaviour of a closed chain on a circular, straight cone is interesting. The reader can either conduct such experiment by himself, or read the report on this work in [7], the acquaintance with this report moved authors to write this divertissement.

A few words about the chain on the cone. It may, depending on the angle of the opening of the cone, either jump off it, or remain on the cone (the frictional forces are again not taken into account). But there is also a third equilibrium − an unstable one, which is qualitatively displayed on the lower graph of Figure 7. The chain at the slightest external jerk can either jump off the cone, or take a stable equilibrium, enveloping the cone. This was proved by the authors of [4]. It is also interesting to investigate the oblique cone, too, and how a closed chain will behave on it. By the way, indirect − "oblique hangers" with a closed chain can be investigated using the program shown in Fig. 14. The result in Figure 19.

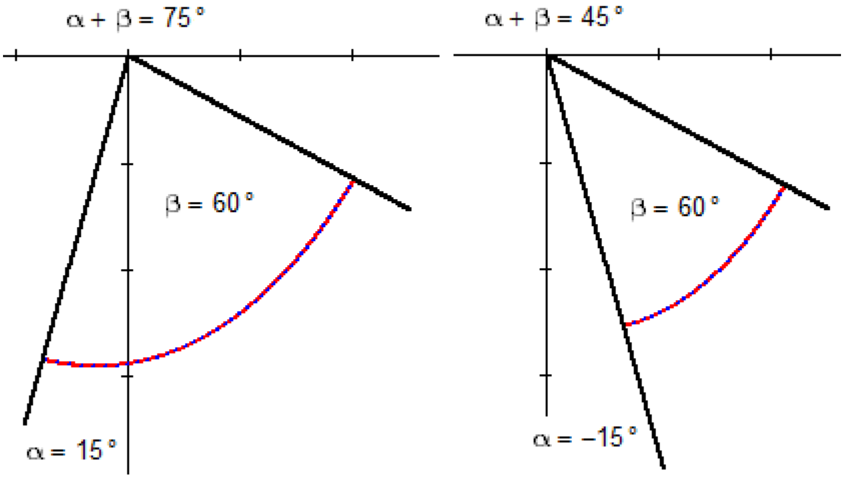


Fig. 19. Chain on "oblique hangers"

At any angle of inclination of the "hangers", a closed chain wrapped around them does not bifurcate, the angle of connection of the ends of the chain with the "hangers" remains straight, which eliminates the sliding of the chain along the "hangers".

Conclusions:

A chain could not only be worn around neck or twisted on the fingers – it is also possible to conduct interesting physical and mathematical experiments with it – real and on the computer! For educational and scientific purposes.

Afterword: the history of writing divertissement and more.

The first author of this article is a thermal and power engineer by education, while as practitioner – an IT specialist in the field of heat engineering and heat power engineering. He is especially interested in issues related to the usage of modern mathematical computer packages for solving heat engineering problems. This can be seen by going to his site with a list of works in this area – http://twt.mpei.ac.ru/ochkov/work1.htm. But the author with his knowledge and skills of working with mathematical packages "climbed" also into related sciences (see http://twt.mpei.ac.ru/ochkov/work2.htm), in particular, in mechanics and mathematics. For example, the possibility was investigated of using the Mathcad built in function Minimize to solve some problems of theoretical mechanics. Traditionally, the solution of such problems was to search for the root of a system of algebraic equations describing the balance of forces and moments of forces in a mechanical system. But if such a system has a sufficient number of degrees of freedom, then it can be reduced to an optimization problem with constraints, which is practically not described in classical textbooks and problem books on theoretical mechanics that are oriented toward a manual rather than a computer solution. These researches resulted in the article [4], which the author sent to the leading journal on theoretical mechanics, published at the Faculty of Mechanics and Mathematics of Moscow State University. The author expected from the journal either no answer, or the answer described by the saying "Do not bother with a pig's snout in a cloth row!". But the main thing that the author expected was the experts' assessments of his work in an area where the author does not consider himself an expert. To the author's surprise, the editorial board replied an answer in the sense that the style and extension of the article does not allow publication in the journal, but it was recommended under the title "Mechanics and Mathcad" for the annual collection of Moscow State University on theoretical mechanics [7]. However the main thing is that in the article there was found no mechanical-mathematical. The article was published next to the article "Chain on the cone" [7], which served as the impetus for writing this divertissement. And that's why!

In [7], quite complex mathematical calculations are presented, that remain a "Chinese puzzle" for many who do not have the corresponding mathematical education and inclination. For the authors of this article, for example. However similar tasks could be solved with the help of a "mathematical accelerator" − using a computer with mathematical programs.

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