Von Mises (revisited)

10/17/2020, FWK

Note to Sam Fares: What you asked would have been simpler two years ago; I was still working and had ready access to Mathcad 15. Fortunately for me but unfortunately for you I've retired, and all I have left is Prime Express and a very nice Mathcad Viewer from Luc that allows me to see the older sheets. What you posted had things overlapping some regions, so the program couldn't be seen completely. Rather than abandon you I've undertaken a Von Mises tutorial; ALL of this information you could harvest from the internet.

Reference: https://www.simscale.com/docs/simwiki/fea-finite-element-analysis/what-is-von-mises-stress/#:~:text=Von%20Mises%20stress%20is%20a,ductile%20materials%2C%20such%20as%20metals.

The von Mises stress is a criterion for yielding, widely used for metals and other ductile materials. It states that yielding will occur in a body if the components of stress acting on it are greater than the criterion:

Cauchy stress tensor

$$\frac{(\tau_{11}-\tau_{22})^2+(\tau_{22}-\tau_{33})^2+(\tau_{33}-\tau_{11})^2+6\cdot(\tau_{12}^2+\tau_{23}^2+\tau_{13}^2)}{6}=k^2 \qquad \tau=\begin{bmatrix}\tau_{11} & \tau_{12} & \tau_{13}\\ \tau_{21} & \tau_{22} & \tau_{23}\\ \tau_{31} & \tau_{32} & \tau_{33}\end{bmatrix}$$

The constant k is defined through experiment and τ is the stress tensor. Common experiments for defining k are made from uniaxial stress, where the above expression reduces to: $\frac{\tau_y^2}{3}=k^2$.

If τ_y reaches the simple tension elastic limit, S_y, then the above expression becomes: $\frac{S_y^2}{3}=k^2$

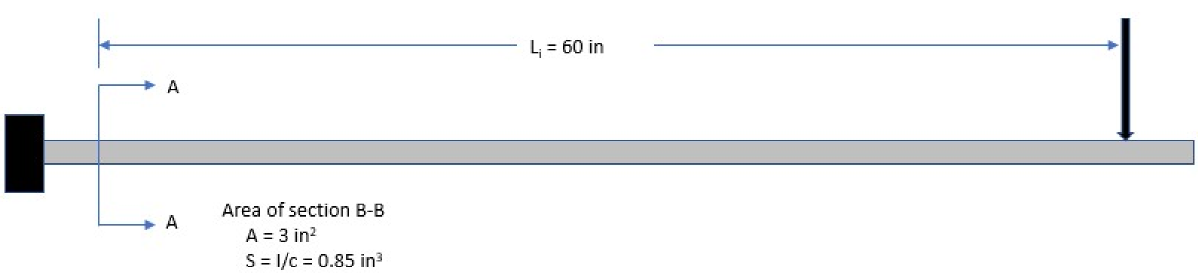
Substituting:
$$\sqrt{\frac{(\tau_{11}-\tau_{22})^2+(\tau_{22}-\tau_{33})^2+(\tau_{33}-\tau_{11})^2+6\cdot(\tau_{12}^2+\tau_{23}^2+\tau_{13}^2)}{2}}=S_y$$

The von Mises stress, τ_y, is defined as: $\tau_y^2=3\cdot k^2$. Therefore, the von Mises yield criterion is also commonly rewritten as: $\tau_y \ge S_y$

Application to the problem in the sheet:

The problem was never defined, but we can make a reasonable guess:

$A:=3\cdot in^2$ $\quad S:=0.85\cdot in^3$ $\quad L_1:=60.0\ in$



We'll assume a cantilever beam with a concentrated load a fixed distance from the vertical plane of interest.

taking the 11 direction as along the beam, and looking at an element on the top surface, the stress tensor becomes:

$$\tau(p) := \begin{bmatrix} \frac{p \cdot L_1}{S} & \frac{-p}{A} & 0 \\ \frac{p}{A} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \tau(1\ kip) = \begin{bmatrix} 70.588 & -0.333 & 0 \\ 0.333 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ksi \qquad \tau = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix}$$

The Von Mises stress formula. To make the subscript numbers consistent: ORIGIN ≡ 1

$$\sigma_v(\tau) := \sqrt{\frac{\left(\tau_{1,1}-\tau_{2,2}\right)^2 + \left(\tau_{2,2}-\tau_{3,3}\right)^2 + \left(\tau_{3,3}-\tau_{1,1}\right)^2 + 6 \cdot \left(\tau_{1,2}^{\ 2} + \tau_{2,3}^{\ 2} + \tau_{1,3}^{\ 2}\right)}{2}}$$

$\sigma_v(\tau(1\ kip)) = 70.591\ ksi$ If the bar has a yield strength less than 70 ksi, it is yielding.

Okay, what happens if we rotate the tensor plane from vertical?

$$R_z(\theta) := \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Consider now a second coordinate system, with base vectors $\mathbf{e}'_1, \mathbf{e}'_2$, obtained from the first by a rotation θ. The components of the transformation matrix are

$$Q_{ij} = \mathbf{e}_i \cdot \mathbf{e}'_j = \begin{bmatrix} \mathbf{e}_1 \cdot \mathbf{e}'_1 & \mathbf{e}_1 \cdot \mathbf{e}'_2 \\ \mathbf{e}_2 \cdot \mathbf{e}'_1 & \mathbf{e}_2 \cdot \mathbf{e}'_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & \cos(90+\theta) \\ \cos(90-\theta) & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$T_{ij} = Q_{ip}Q_{jq}T'_{pq}, \quad T'_{ij} = Q_{pi}Q_{qj}T_{pq}$ **Tensor Transformation Formulae** (1.13.5)

the components of **S** in the second coordinate system are $[\mathbf{S}'] = [\mathbf{Q}^{\mathrm{T}}][\mathbf{S}][\mathbf{Q}]$, so

or, in matrix form,

$$[\mathbf{T}] = [\mathbf{Q}][\mathbf{T}'][\mathbf{Q}^{\mathrm{T}}], \qquad [\mathbf{T}'] = [\mathbf{Q}^{\mathrm{T}}][\mathbf{T}][\mathbf{Q}] \qquad (1.13.6)$$

$$\begin{bmatrix} S'_{11} & S'_{12} \\ S'_{21} & S'_{22} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

So the rotated stress tensor becomes: $T(p,\theta) := R_z(\theta)^{\mathrm{T}} \cdot \tau(p) \cdot R_z(\theta)$

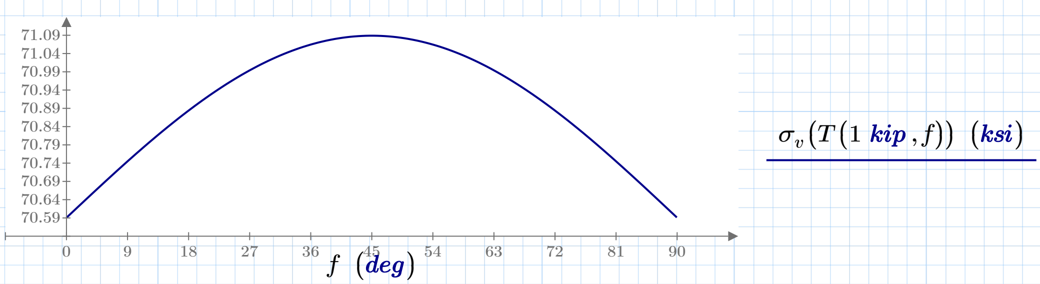
$$\tau(1\ kip) = \begin{bmatrix} 70.588 & -0.333 & 0 \\ 0.333 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ksi \qquad T(1\ kip, 0\ deg) = \begin{bmatrix} 70.588 & -0.333 & 0 \\ 0.333 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ksi$$

$$T(1\ kip, 90\ deg) = \begin{bmatrix} 0 & -0.333 & 0 \\ 0.333 & 70.588 & 0 \\ 0 & 0 & 0 \end{bmatrix} ksi \qquad T(1\ kip, 45\ deg) = \begin{bmatrix} 35.294 & -35.627 & 0 \\ -34.961 & 35.294 & 0 \\ 0 & 0 & 0 \end{bmatrix} ksi$$

$$\overrightarrow{\sigma_v\left(T\left(1\ kip, \begin{bmatrix} 0 \\ 90 \\ 45 \end{bmatrix} deg\right)\right)} = \begin{bmatrix} 70.591 \\ 70.591 \\ 71.089 \end{bmatrix} ksi$$

$f := 0, 0.01 .. \frac{\pi}{2}$

Looks HORRIBLE but look at the vertical scale!



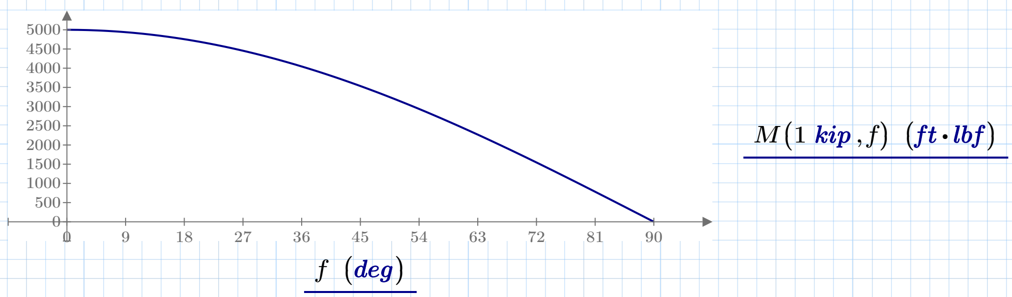
Just for fun, let's rotate the load vector. That's going to change a bunch of stuff

$$P(p,\theta) := p \cdot \begin{bmatrix} \sin(\theta) \\ \cos(\theta) \\ 0 \end{bmatrix}$$

θ is the angle of the vector from vertical (measured clockwise)

$$R := \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} \qquad P(1\ kip, 45\ deg) = \begin{bmatrix} 0.707 \\ 0.707 \\ 0 \end{bmatrix} kip$$

$$M(p,\theta) := (R \times P(p,\theta))_3 \qquad M(1\ kip, 0\ deg) = 5000\ ft \cdot lbf \qquad 1\ kip \cdot L_1 = 5000\ ft \cdot lbf$$



$$G := Grid$$

$$\tau_F(p,\theta) := \begin{bmatrix} \dfrac{M(p,\theta)}{S} & \dfrac{-P(p,\theta)_2}{A} & 0 \\ \dfrac{P(p,\theta)_2}{A} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad TT(t,\theta) := R_z(\theta)^{T} \cdot t \cdot R_z(\theta)$$

We see that the reference frame angle (0,45,90 degrees) doesn't change the Von Mises stress but the angle of the applied load does! Q E D