Section 14 Chebyshev Polynomials

The Chebyshev polynomials can be used to construct a polynomial approximation to a given function over a given interval. This document generates an array containing the Chebyshev polynomial coefficients for a given polynomial. You supply:

- **N**, the order of the Chebyshev polynomial
- **f(x)**, the function to approximate
- [**a**, **b**], the range for the function **f(x)**

The document constructs an approximation to a given function. It also shows how to express a given polynomial in terms of Chebyshev polynomials.

References

Richard L. Burden and J. Douglas Faires, *Numerical Analysis*, Prindle, Weber & Schmidt (Boston, 1985). See also Milton Abramowitz and Irene A. Stegun, eds., *Handbook of Mathematical Functions*, Dover Publications, Inc. (New York, 1965), and *Numerical Recipies,* a Mathcad Electronic Book.

Background

The Chebyshev polynomials are used to construct an approximation to a given function over a given interval, for example the desired frequency response curve of a filter over some range of frequencies. Chebyshev approximations have one of the smallest maximum deviations from the original function of all polynomial approximations, and are popular designs for IIR filters.

The general form for a Chebyshev polynomial is:

$$T(N,x) = \cos(N \cdot \operatorname{acos}(x))$$

The terms of a Chebyshev polynomial are orthogonal over the interval [-1, 1] when summed over the set of points

$$\cos\left(n \cdot \frac{\pi}{N}\right), n = 0, 1, \ldots, N$$

with respect to a weighting function, **W**, that assigns weight 1 to each point inside [-1,1] and .5 to the two endpoints. Polynomial approximations of degree **N** can then be constructed by sampling the desired function **f(x)** at **x** = **cos(np/N)**, and summing the product of the function at these **n** points with the corresponding Chebyshev polynomial of order **n**.

Chebyshev filters are either monotonic in the stopband and equiripple in the passband (Type I) or monotonic in the passband and equiripple in the stopband (Type II). A better approximation to the ideal filter will be given by elliptical functions. See **Section 13.2: Analog Elliptic Filter Design**.

Mathcad Implementation

The **n**th Chebyshev polynomial **T(n, x)** is defined as

$$T(n,x) := \cos(n \cdot \mathrm{acos}(x))$$

To see it explicitly for a particular **n**, expand this definition using the symbolic processor. First, position the cursor anywhere inside the region, then select **Mathcad Ribbon**:**Symbolics** button: keyword **expand**:

$$cos(6 \cdot acos(x))$$

$$\cos(6 \cdot \mathrm{acos}(x)) \xrightarrow{expand} 15 \cdot x^4 \cdot (x^2 - 1) + (x^2 - 1)^3 + x^6 + 15 \cdot x^2 \cdot (x^2 - 1)^2$$

$$\cos(6 \cdot \mathrm{acos}(x)) \xrightarrow{expand, collect} 32 \cdot x^6 - 48 \cdot x^4 + 18 \cdot x^2 - 1$$

Defining the Chebyshev Coefficients

First, set **N** to the degree of the highest-order Chebyshev polynomial you will need to approximate your function:

$$N := 6$$

For **n** = 0, 1, ... , **N**, the coefficients for the expression of **T(n, x)** as a polynomial in **x** are given in the columns of the array **C** calculated below.

$$i := 1 .. N \qquad j := 2 .. N \qquad C_{N,N} := 0$$

$$C_{0,0} := 1 \qquad C_{1,1} := 1$$

$$C_{0,j} := \mathbf{if}(\mathrm{mod}(j,2) = 0, \mathbf{if}(\mathrm{mod}(j,4) = 2, -1, 1), 0)$$

$$C_{i,j} := 2 \cdot C_{i-1,j-1} - C_{i,j-2}$$

The coefficients for **T(n, x)** are in column **n** of the array **C**. For example,

$$C^{\langle 6\rangle} = \begin{bmatrix} -1 \\ 0 \\ 18 \\ 0 \\ -48 \\ 0 \\ 32 \end{bmatrix}$$

Where

$$k := 0..N \qquad T(N, x) := \sum_k \left(\left(C^{\langle N\rangle}\right)_k \cdot x^k \right)$$

Define the function **f(x)** to be approximated and the interval [**a**, **b**] over which the approximation is valid:

$$f(x) := \sin(x) \cdot \ln(x)$$

$$a := 1 \qquad b := 5 \qquad x := a, a + .2..b$$

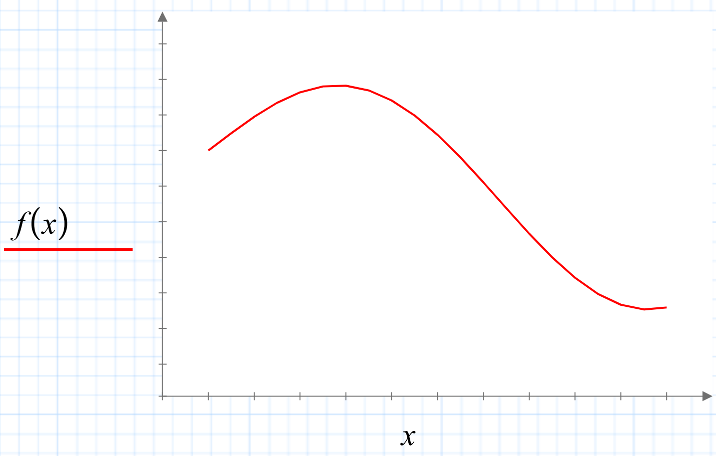

Fig. 14.1 The function to be approximated

Orthogonality

The Chebyshev polynomials **T(**0, **x), T(**1, **x), . . . T(N, x)** are orthogonal over the interval [-1, 1] on the set of **N** + 1 points **cos((n -** 1/2**) p / (N** + 1**))**, **n** = 1, 2, ... , **N** + 1. For example:

$N=6 \qquad n:=1..N+1 \qquad p_n:=\cos\left(\left(n-\frac{1}{2}\right)\cdot\frac{\pi}{N+1}\right)$

$$\sum_n \left(T\left(3,p_n\right)\cdot T\left(4,p_n\right)\right)=1.665\cdot 10^{-16}$$

$$\sum_n \left(T\left(6,p_n\right)\cdot T\left(6,p_n\right)\right)=3.5$$

$$\sum_n \left(T\left(0,p_n\right)\cdot T\left(0,p_n\right)\right)=7$$

Defining the Approximation Coefficients

Products of distinct **T**'s sum to 0 over the points **p**, while products of identical **T**'s sum to either **N** + 1 (for **T** (0, **x**)) or to (**N** + 1)/2. The orthogonality of the polynomials can be used to compute coefficients of **T** in an expression for **f(x)** as a sum of Chebyshev polynomials. If we want, approximately,

$$f(x) = \sum_k \left(c_k\cdot T(k,x)\right)$$

then we can compute the **c terms** by

$$c_k = \sum_n \left(f\left(p_n\right)\cdot T\left(k,p_n\right)\right)$$

The approximation will be exact at **x** = **pn**.

Changing Variables

The formula above is valid only for approximating a function **f** over the interval [-1, 1]. To approximate **f(x)** above, first scale [-1, 1] to [**a**, **b**]:

$$h(x):=.5\cdot(b-a)\cdot x+.5\cdot(a+b)$$

The inverse of the transformation (**y** --> **x**) is:

$r:=\frac{2}{b-a} \qquad s:=-\frac{a+b}{b-a} \qquad g(x):=r\cdot x+s$

The above function, **h(x)**, changes variables to **y** over the range [-1, 1]. The coefficients of the approximation in terms of the Chebyshev coefficients are now calculated as follows. Use the fact that:

$$T\left(k, p_n\right) = cos\left(k \cdot acos\left(cos\left(\left(n - \frac{1}{2}\right) \cdot \frac{\pi}{N+1}\right)\right)\right)$$

simplifies to

$$cos\left(k \cdot \left(n - \frac{1}{2}\right) \cdot \frac{\pi}{N+1}\right)$$

$$k := 0 .. N$$

$$c_k := \frac{2}{N+1} \cdot \sum_n \left(f\left(h\left(\cos\left(\left(n - \frac{1}{2}\right) \cdot \frac{\pi}{N+1}\right)\right)\right) \cdot \cos\left(k \cdot \left(n - \frac{1}{2}\right) \cdot \frac{\pi}{N+1}\right)\right)$$

$$c_0 := \frac{1}{2} \cdot c_0$$

So the expression for the approximation to **f** in terms of Chebyshev polynomials of order up to **N** is

$$A(x) := \sum_k \left(c_k \cdot T(k, g(x))\right)$$

Since the function **h** scales [-1, 1] to the actual domain of approximation, we have to scale the arguments to the **T**s by **g**, which is the inverse of **h**.

Let's compare some values of **f** and the Chebyshev approximation:

$A(1) = 0.000296$	$f(1) = 0$
$A(1.5) = 0.404583$	$f(1.5) = 0.404449$
$A(4) = -1.049481$	$f(4) = -1.049151$

The approximation is in fact exact at the points which correspond to the roots **p** defined above:

$$A\left(h\left(p_5\right)\right) = 0.640937486048965$$

$$f\left(h\left(p_5\right)\right) = 0.640937486048965$$

In the **N**th degree approximation to **f(h(x))**, the coefficients of the polynomial in **y** are contained in the vector

$$w := C \cdot c$$

The approximating polynomial is

$$P(y) := \sum_k \left(w_k \cdot y^k \right)$$

The coefficients of the powers of **x** in the approximation to **f(x)** over the interval [**a**, **b**] are given by

$$j := 0 .. N \qquad d_k := \sum_j \textbf{if}\left((k<j) \cdot (k>0), w_j \cdot r^k \cdot s^{j-k} \cdot \frac{j!}{k! \cdot j-k!}, 0 \right)$$

The approximating polynomial, and explicit coefficients are shown below.

$$k = \begin{bmatrix} 0 \\ \vdots \end{bmatrix} \quad d_k = \begin{bmatrix} 0 \\ \vdots \end{bmatrix}$$

$$q(x) := \sum_k \left(d_k \cdot x^k \right)$$

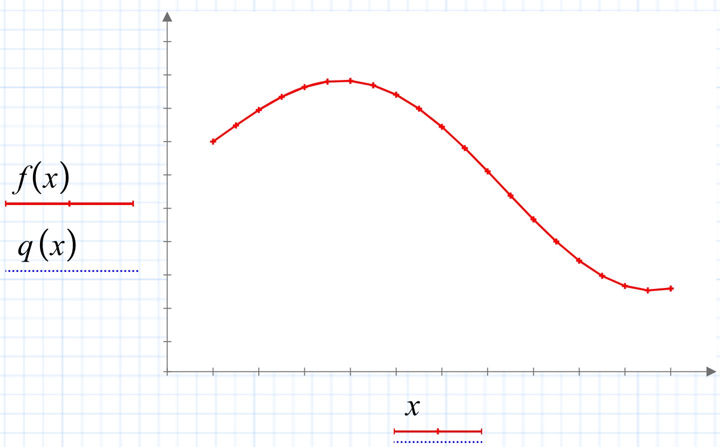

Fig. 14.2 Comparison of the approximation with f(x)

Compare the values of **f** and **q** at some points in [**a**, **b**]:

$f(2) = 0.63028$ $\qquad q(2) = 33.76522$

$f(2.3) = 0.6211$ $\qquad q(2.3) = 37.07042$

Expressing Powers of x As Chebyshev Sum

In some applications it is useful to express a power of **x** as a sum of Chebyshev polynomials. The coefficients in this sum are given by the appropriate column of the inverse of **C**.

$$E := C^{-1} \qquad E^{\langle 4 \rangle} = \begin{bmatrix} 0.375 \\ 0 \\ 0.5 \\ 0 \\ 0.125 \\ 0 \\ 0 \end{bmatrix}$$

To check this expression for the fourth power of **x**, define

$$X4(x) := \sum_k \left(\left(E^{\langle 4 \rangle} \right)_k \cdot T(k, x) \right)$$

and calculate a few values:

$$X4(1) = 1 \qquad X4(-2) = 16 \qquad X4(10) = 1 \cdot 10^4$$

Approximating an Mth Degree Polynomial with One of Lower Degree

To illustrate the application of these formulas, the following equations show how to approximate the sixth degree polynomial defined by

$$I := 0..6 \qquad u_I := \frac{1}{I!} \qquad F(x) := \sum_I \left(u_I \cdot x^I \right)$$

with a polynomial of lower degree, so that the maximum error over the interval **[-1,1]** will be less than .001. The strategy is to calculate the Chebyshev coefficients for the polynomial, discard the coefficients smaller than the allowable error, and then reexpress this result as a polynomial of lower degree.

The coefficients **v** in the expression for **F** as a sum of Chebyshev polynomials of degree less than or equal to 6 are given by

$$v := E \cdot u \qquad v = \begin{bmatrix} 1.266 \\ 1.13 \\ 0.271 \\ 0.044 \\ 0.005 \\ 5.208 \cdot 10^{-4} \\ 4.34 \cdot 10^{-5} \end{bmatrix}$$

Since **|T(n, x)|** is less than or equal to 1 for **x** in [-1, 1], the last two coefficients can be omitted without exceeding the maximum allowable error. Set them to zero explicitly:

$$v_5 := 0 \qquad v_6 := 0$$

The coefficients in the fourth degree approximation are

$$t := C \cdot v$$

The polynomial is

$$G(x) := \sum_I \left(t_I \cdot x^I \right)$$

The coefficients are

$$I = \begin{bmatrix} 0 \\ \vdots \end{bmatrix} \quad t_I = \begin{bmatrix} 1 \\ \vdots \end{bmatrix}$$

$$y := -1, -.9 .. 1$$

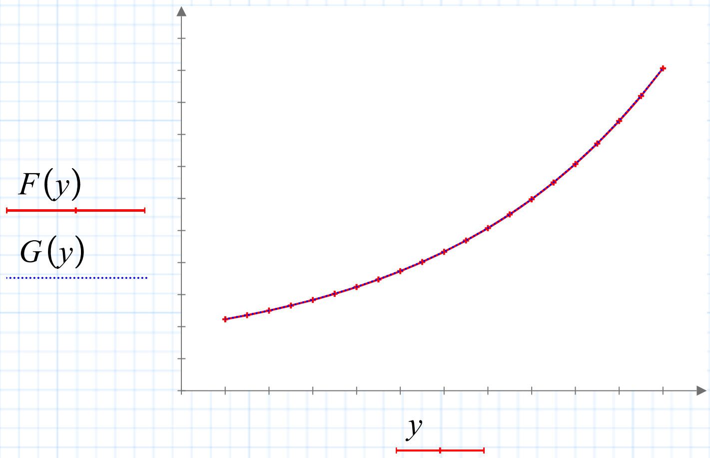

Fig. 14.3 Compare F with approximation G

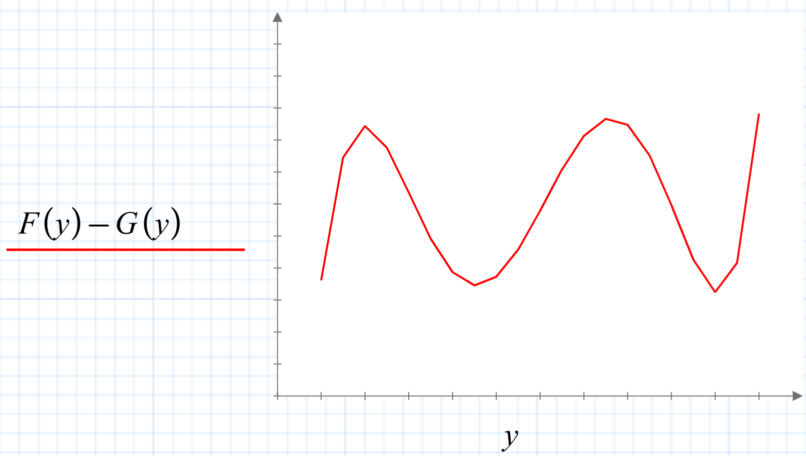

Fig. 14.4 Error as a function of y (F-G)