Regular Article

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Is the cubic parabola really the best railway transition curve?

https://doi.org/10.1515/eng-2021-0123
received June 27, 2021; accepted August 05, 2021

Abstract: In the current study, the authors tested two railway transition curves in terms of their usefulness for railway practice. The first curve was a cubic parabola – the curve most popular in railway engineering. The second curve was a polynomial of ninth degree, and this curve was chosen due to the fact that this curve satisfies advanced geometrical demands. In this study, the model of a two-axle vehicle was applied. The study relied on the passage of the vehicle model through the route consisted of straight track, railway transition, and circular arc. Three different circular arc radii were used. The value – 0.6 m/s^2 – of maximum lateral acceleration in the circular arc was applied. Three different vehicle velocities were also used. In this study, the authors simulated vehicle dynamics for different curves and ten lengths of the curves. As the main criterion, lateral acceleration of the vehicle body was applied. The results for these two mentioned curves were compared.

Keywords: railway transition curves, rail vehicle, dynamical assessment, computer simulation

1 Introduction

The aim of this article is the assessment of two railway transition curves. The inspiration for this article was an earlier study [1]. In the mentioned study, its authors used a model of railway vehicle to assess the practical usefulness of popular railway transition curves – first of all, the third-degree (cubic) parabola – the most popular railway transition, but also polynomial and trigonometric curves. Among these criteria of the assessment four the most important criteria were as follows:

– vehicle body lateral and vertical acceleration,
– wheel/rail lateral and vertical forces,
– derailment coefficient, and
– reduction rate of wheel load.

In an earlier study [1], the authors simulated the movement of rail vehicle negotiating six curves. As a result, also six (above-mentioned) parameters were simulated by simulation software. Different curves resulted in different courses of dynamical characteristics, such as the vehicle body lateral and vertical acceleration. The results of the simulations were also compared by the authors.

The authors of the current article also possess the models of the rail vehicle. These are a two-axle rail vehicle of average values of parameters [2,3] and a four-axle vehicle model [2,3]. In this article, due to computations times, the two-axle vehicle model was used. Its structure was fully described in an earlier study, for example, Zboinski [2]. In Section 4 of this article, this model is also shortly presented.

As the transition curves, two transition curves were chosen for the assessment. The first curve was the cubic parabola, whereas the second curve was a polynomial curve of ninth degree. In an earlier study [1], the ninth degree is the highest degree polynomial used in polynomials, which could be used as the transition curve. The mentioned curve has the third derivative of the curvature with respect to curve length equal to zero in the beginning and the endpoint of the curve. In their middle part, it also has the inflection point.

2 Literature review

Nowadays, a large number of studies, which deal with the problem of railway transition curve formation and the assessment, can be observed. It is especially visible in the context of the high-speed train line construction. The examples of such studies are Klauder and Chrismer [4] and Li et al. [5].

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From the point of view of the current article, the most important are the studies, where their authors use the rail vehicle model to find new better shapes of the transitions. In this area, in the authors' opinion, the crucial are earlier studies [6–13]. The aim of the mentioned studies is the study of the influence of railway transition curve shape on vehicle dynamics through the use of computer vehicle model. As an effect, the vehicle dynamics represented by vehicle body lateral displacements and accelerations can be obtained. In many studies, however, the relatively simple vehicle model is used. Sometimes, it is just a mathematical point, which is, in the lights of high-speed trains, not sufficient. In such an approach, transition curves are assessed on the basis that maximum values of the vehicle body lateral acceleration and its change acting on the passengers should not be exceeded.

The second group of the studies are those where the curve is treated as a mathematical object, and its properties are examined by the authors [14–20]. In general, these works are aimed to find the curves better than the third-degree parabola.

The current study uses the two-axle advanced rail vehicle for the railway transition curve assessment. This model was used due to the fact that the times of simulations are significantly shorter than the times of simulations obtained using the four-axle models. The advantage of this study is the fact that the mentioned two-axle model allows to obtain the simulations that are not available when the rail vehicle is treated as a mathematical point. Moreover, this fact should be highlighted, as is it the discrepancy with the mentioned studies [14–20].

3 Theory

In road engineering, the most popular transition curve is the clothoid, being an example of a spiral. This curve is characterized by the fact that its curvature linearly changes versus the curve length. Coordinates x and y of the curve in the function of curve length l are presented, for example, in an earlier study [14], and are as follows:

$$x(l) = l[1 - l^4/(40C^2) + l^8/(3,456C^4) - \ldots], \quad (1)$$

$$y(l) = l^3/(6C)[1 - l^4/(56C^2) + l^8/(7,040C^4) - \ldots], \quad (2)$$

where constant C is a product of curve radius R and the total curve length L.

In railway engineering, a certain simplification is used. This mentioned simplification is a cubic parabola. In this curve, two simplifications are applied. First of all, x coordinate is approximatively equal to curve length. Second – only the first term from power series expansion (2) is used. So the final formulae for y coordinate has the following form:

$$y(x) = x^3/(6RL). \quad (3)$$

It is also worthy of mentioning that this curve does not possess the linear curvature of the function of x coordinate if we apply exact formulae for the curvature of the curve in $y(x)$ form. In such a case, the curve (3) loses linearity at the end of the curve. If we applied simplified formulae for curvature $k(x) = \mathrm{d}^2y(x)/\mathrm{d}x^2$ [21], such a linearity exists.

In the current study, curve (3) was used for testing. The second curve, as mentioned, is the curve in the following form:

$$y(l) = 1/R[-5l^9/(18L^7) - 5l^8/(4L^6) - 2l^7/(L^5) + 7l^6/(6L^4)], \quad (4)$$

The curvature of this curve is presented in an earlier study [1].

Each transition curve has a minimum length. This minimum length arises from the fact that two quantities cannot be exceeded. These kinematic parameters are maximum unbalanced lateral acceleration and vertical wheel rise along the curve. So, two formulae for the minimum length for each curve always was as follows:

$$L_{\min} = \max(AvH/f; Ava/d), \quad (5)$$

where A – constant, v – vehicle velocity (m/s), H – cant (mm), f – velocity of vertical wheel rise along superelevation ramp (abrupt change of cant; for the third-degree parabola – 50 mm/s [22] and for the polynomial curve of ninth degree – 70 mm/s [23]), a – unbalanced lateral acceleration (cant deficiency; 0.6 m/s^2 [24]), and d – change of the lateral acceleration (abrupt change of cant deficiency; 0.45 m/s^3 [22,25]).

The constant A is equal to 1 [1] for a cubic parabola and 2.1785 for the curve of ninth degree [1]. In the current study for a given curve radius, the third-degree parabola always had a smaller length than the curve of ninth, so as to make the objective comparison of the curves, the length of the third-degree parabola was elongated to this length.

Moreover, the quantities – vehicle velocity v, radius R, cant H, and unbalanced lateral acceleration a have the following engineering relation:

$$v^2/R = (g/s)H + a, \quad (6)$$

where g – gravity and s – track gauge (here s = 1,500 mm).

Using formula (6), in this study, the vehicle velocities were calculated.

Table 1: Parameters assumed in the simulations

R (m)	*a* (m/s^2)	*H* (mm)	*v* (m/s)	*L* (m)
600	0.6	150	30.79	118.86
900	0.6	65	30.37	88.58
1,200	0.6	75	36.17	105.50

4 Method

The simulations made in the needs of the current study relied on a passage of the rail vehicle through the route consisted of straight track, transition curve, and circular arc. In the current study, three values of circular arc were assumed. These three values were 600, 1,200, and 2,000 m. In each case, the relatively large value of unbalanced lateral acceleration was assumed. Also, three values of cant were assumed. Considering the relationship (6), three vehicle velocities were calculated. The full set of assumed parameters is presented in Table 1.

In the study, one model of rail freight vehicle was applied. The mentioned model has a two-axle structure. In the past, the authors called the model a "two-axle (freight) wagon." The model has a body connected with two wheelsets with spring-damping elements. The structure of the model and its parameters correspond to a typical real wagon. The nominal model of this vehicle is shown in Figure 1c. The vehicle model is supplemented with the model of the track. Here, it is presented in Figure 1a and b. The entire track-vehicle system is discussed in detail, for example, in an earlier study [13]. The model parameters of this system are also presented in Zboinski and Woznica [13]. In Table 2, all parameters of the model are presented.

The model applied has all elements of vehicle dynamical models named in railway vehicle dynamics as key mass elements, that is, wheelsets and vehicle body, suspension elements – stiffness and damping elements, and wheel and rail geometry. In the model, tangential contact forces calculated applied the nonlinear simplified contact theory invented by Kalker. During the calculation of all inertia components arising from the negotiation of the curved track, the authors made no simplification. As mentioned earlier, the whole dynamical system also includes the track model. So, the dynamical vehicle-track system is also analyzed. The model also possesses a typical pair of wheel-rail (S1002/60E1) profiles. This pair is introduced in the model as the table of contact parameters. This table is built with the use of the software presented in [6].

In the article, the authors applied one criterion of the assessment of the usefulness of the railway transition curve. It is the integral of vehicle body mass center lateral acceleration along the route. This mentioned criterion is as follows:

$$\mathrm{QF} = L_C^{-1}\int_0^{L_C} |\mathrm{d}^2y/\mathrm{d}^2t|\mathrm{d}l, \tag{7}$$

where L_C is the total route, and $\mathrm{d}^2y/\mathrm{d}^2t$ is the lateral acceleration of the vehicle.

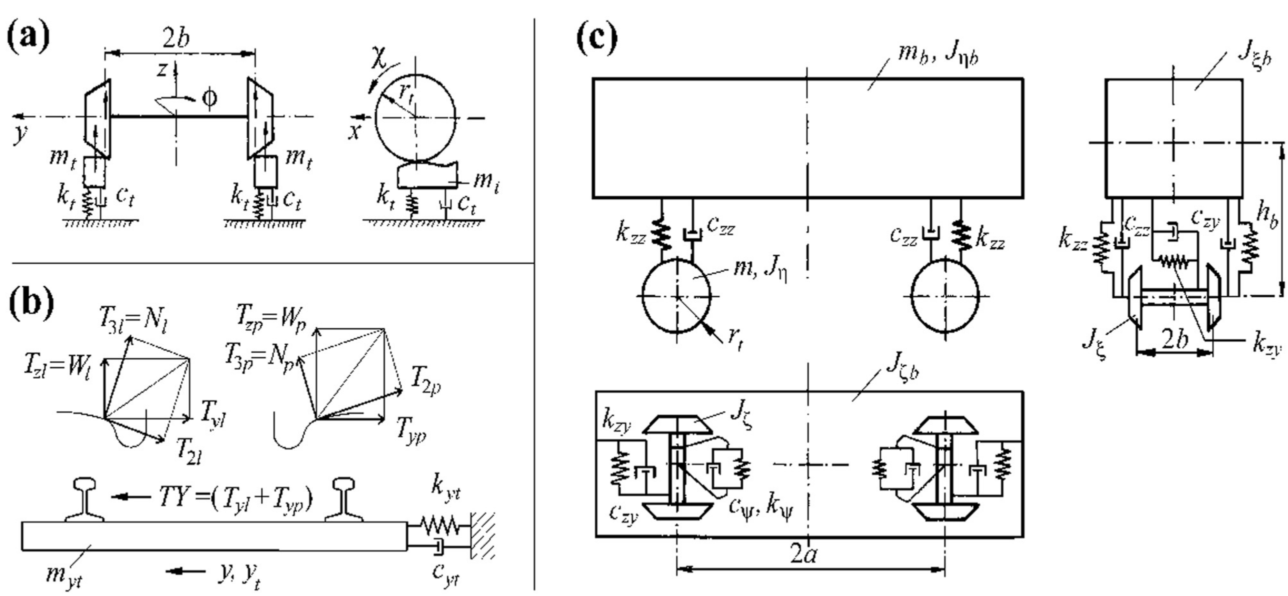

Figure 1: Vehicle model: a) track vertically, b) track laterally, c) vehicle model.

Table 2: Parameters of the adopted vehicle-track system

	Description	Unit	Value
m_b	Vehicle body mass	kg	30,000
m	Wheelset mass	kg	2,400
$I_{\xi b}$	Vehicle body moment of inertia, longitudinal axis	kg m^2	17,500
$I_{\eta b}/I_{\zeta b}$	Vehicle body moment of inertia, lateral/vertical axis	kg m^2	185,000
I_{ξ}	Wheelset moment of inertia, longitudinal axis	kg m^2	1,700
I_{η}	Wheelset moment of inertia, lateral axis	kg m^2	200
I_{ξ}	Wheelset moment of inertia, vertical axis	kg m^2	1,700
k_{zz}	Longitudinal stiffness of the first level of suspension	kN/m	1,000
k_{zy}/k_{zx}	Lateral/vertical stiffness of the first level of suspension	kN/m	800
c_{zz}	Longitudinal damping of the first level of suspension	kN s/m	60
c_{zy}	Lateral damping of the first level of suspension	kN s/m	47
c_{zx}	Vertical damping of the first level of suspension	kN s/m	42
a	Semi-wheel base	m	3.16
h_b	Vertical distance between mass centers of wheelset and vehicle body	m	1.04
r_t	Wheelset rolling radius	m	0.46
m_t	Vertical mass of the rail	kg	200
k_t	Vertical stiffness of the rail	kN/m	70,000
c_t	Vertical damping of the rail	kN s/m	200
m_{ty}	Lateral mass of the track	kg	500
k_{ty}	Lateral stiffness of the track	kN/m	25,000
c_{ty}	Lateral damping of the track	kN s/m	500

5 The results

The results of this study are the simulations in the form of vehicle body mass center lateral, vertical, and angular displacements and accelerations. The fundamental results are, however, the values of the quality function (7) for three different circular arc values and two transition curves in the function of transition curve length. In general, the curve length ranged from 40 to 200 m. The length of the curve was discrete, with the step of 20 m. In Table 3, the authors presented the mentioned obtained values of quality functions.

Making the analysis of Table 3, we see that the greater length of the transition curve, the smaller value of the quality function. This fact is valid both for the third-degree parabola and the polynomial of ninth degree. It is visible that for small values of the lengths in all three cases, the third-degree parabola dominates over the polynomial of ninth degree. For the greatest lengths, the situation is the opposite. Here, the polynomial curve has significantly better properties. In the middle part of the curve length range, the limit value of the length below which the third-degree parabola dominates exists.

Figures 2–4 present a graphical representation of quality values for circular arc 600, 900, and 1,200 m for the third-degree parabola (dashed line) and the polynomial of ninth degree versus the curve length. Intersection points of the curves define limit values, which divide the whole range of the lengths into two parts. These limit values are 160, 140, and 170 m, respectively, for each case, and these are the values above which the curve of ninth degree is always better. Below this value, we see that, as mentioned, the third-degree parabola is better. It is worthy of mention

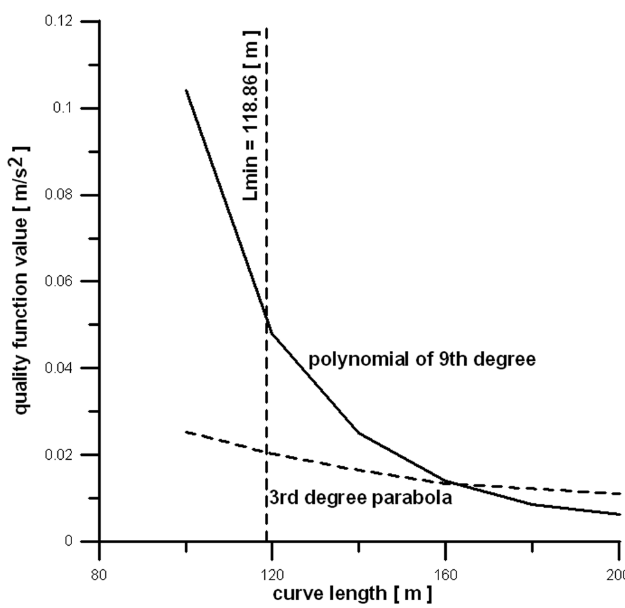

Figure 2: Values of quality function for R = 600 m (min value of the length = 118.86 m).

Table 3: Numerical values of the quality function (m/s^2)

Length (m)	R = 600 m		R = 900 m		R = 1,200 m	
	Third degree	Ninth degree	Third degree	Ninth degree	Third degree	Ninth degree
40	6.01×10^{-02}	$2.95 \times 10^{+00}$	7.37×10^{-02}	$2.84 \times 10^{+00}$	1.15×10^{-01}	$4.23 \times 10^{+00}$
60	5.24×10^{-02}	$1.01 \times 10^{+00}$	6.22×10^{-02}	9.35×10^{-01}	8.91×10^{-02}	$1.79 \times 10^{+00}$
80	2.95×10^{-02}	2.64×10^{-01}	3.95×10^{-02}	2.36×10^{-01}	6.07×10^{-02}	6.26×10^{-01}
100	2.53×10^{-02}	1.04×10^{-01}	3.18×10^{-02}	9.43×10^{-02}	4.44×10^{-02}	1.96×10^{-01}
120	2.02×10^{-02}	4.80×10^{-02}	2.53×10^{-02}	4.31×10^{-02}	3.71×10^{-02}	9.59×10^{-02}
140	1.64×10^{-02}	2.51×10^{-02}	2.11×10^{-02}	2.01×10^{-02}	2.75×10^{-02}	4.81×10^{-02}
160	1.34×10^{-02}	1.39×10^{-02}	1.63×10^{-02}	1.18×10^{-02}	2.28×10^{-02}	2.47×10^{-02}
180	1.22×10^{-02}	8.49×10^{-03}	1.33×10^{-02}	8.25×10^{-03}	1.86×10^{-02}	1.49×10^{-02}
200	1.10×10^{-02}	6.32×10^{-03}	1.14×10^{-02}	6.55×10^{-03}	1.66×10^{-02}	1.03×10^{-02}

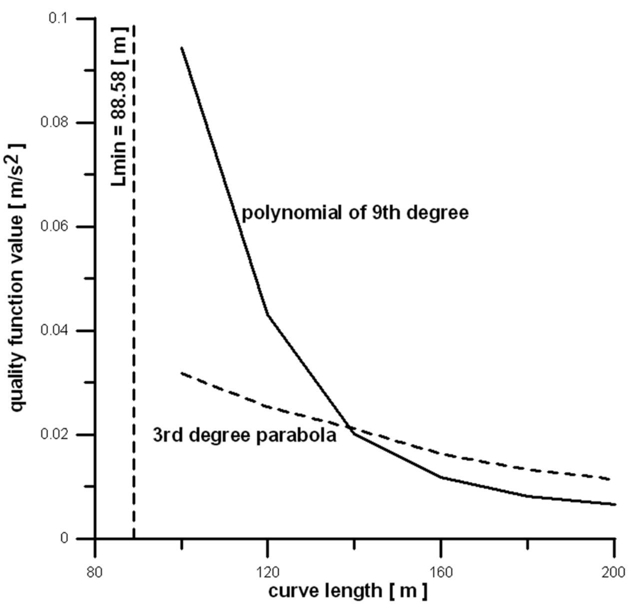

Figure 3: Values of quality function for R = 600 m (min value of the length = 88.58 m).

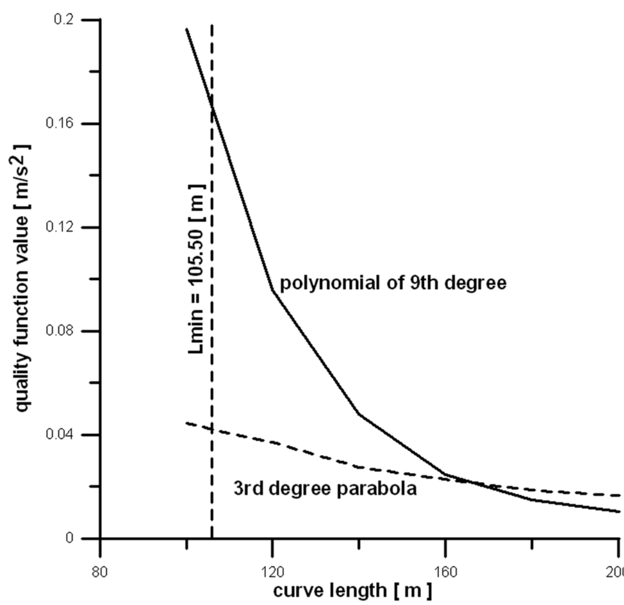

Figure 4: Values of quality function for R = 600 m (min value of the length = 105.50 m).

that the theoretical minimum values of the lengths (Table 1) must be considered.

In the current study also, the graphical results of the simulation are presented for two relatively extreme cases – for the curve length of 100 and 200 m and the curve radius of 600 m. The rest of the dynamical results have a qualitatively similar character. Figures 5 and 6 present the mentioned accelerations for the lengths of 100 and 200 m and the two transition curves – the third-degree parabola and the polynomial of ninth degree. Mentioned results are the courses of lateral acceleration for vehicle body mass centers. Considering the lateral accelerations and the length of 100 m, we may observe that a significantly better situation is for the third-degree parabola. Here, no tangency of the curvature in the beginning and the last point of the curve seem to have no matter. When we consider the length of 200 m, the polynomial of ninth degree dominates. It is especially visible at the beginning of the circular arc (indicated in Figure 6 by an arrow). Accelerations for the cubic parabola start to have larger values.

6 Conclusion

In the study, the effective method of the assessment of two railway transition curves was presented and used, which allows to assess the railway transition curves better than the methods using only a mathematical point as the rail vehicle model.

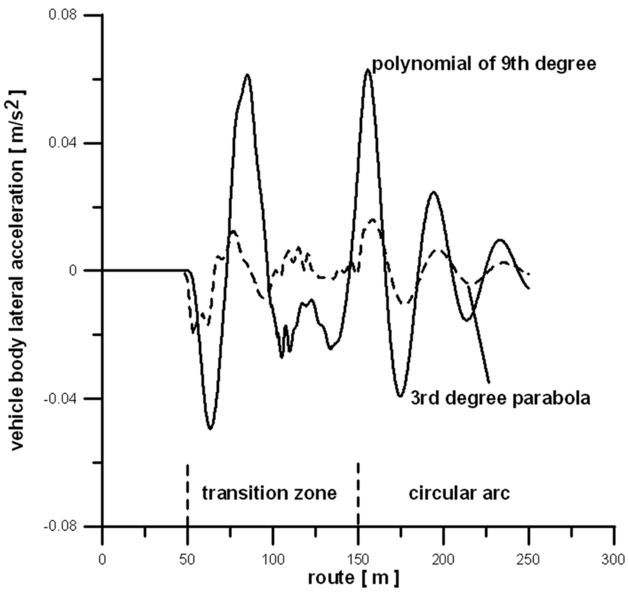

Figure 5: Vehicle body lateral accelerations for the third-degree parabola and the curve of ninth degree (length = 100 m).

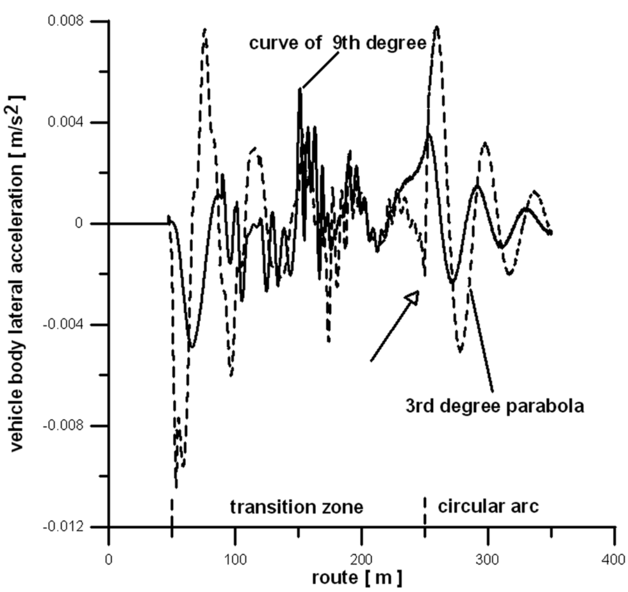

Figure 6: Vehicle body lateral accelerations for the third-degree parabola and the curve of ninth degree (length = 200 m).

The authors used the two-axle rail vehicle model pioneered by the leading author of the article. Using the model, properties of both the third-degree parabola and the polynomial curve of ninth degree for different lengths of the curves were examined. The analysis presented the ranges of lengths of transition curves for which the transition of ninth degree is more favorable than the most used parabola, the third-degree parabola.

The study showed the most important practical conclusion that for very long transition curves (above 150 m), the most popular transition in railway engineering does not have the chance to be the best transition curve. Such curve can be considered only for relatively short transitions. This conclusion is a key element in the lights of the high-speed train line construction.

The study shows a need for further research on transition curves, especially the long ones, which are used on high-speed railway lines. As the transition curve suitable for the railway practice, the authors propose polynomial of tenth and eleventh degrees, and splines.

Conflict of interest: Authors state no conflict of interest.

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