Effects of TOL and Method on Definite Integration

This QuickSheet illustrates the impact of the TOL parameter and the integration method on definite integral evaluations. Set TOL from **Worksheet Options** under the **Tools** menu or directly in the worksheet, as shown.

The symbolic result of this integral is exact:

$$\int_{1}^{e^5} \frac{1}{t1} \, dt1 \rightarrow 5$$

Default tolerance: TOL = 0.001

$$\int_{1}^{e^5} \frac{1}{t1} \, dt1 = 5.000000000068143$$

Looser tolerance

$$TOL := 0.1$$

$$\int_{1}^{e^5} \frac{1}{t1} \, dt1 = 5.00000005437786$$

Tighter tolerance

TOL := 0.00001

$$\int_{1}^{e^5} \frac{1}{t1}\, dt1 = 5.000000000000009$$

For an improper integral, plot the integrand and determine the interval of integration that really contains the bulk of the area. Experiment with various values of TOL. There is no reason, in the second case, to raise the upper limit of integration beyond

90 (for example), since the integrand $\dfrac{\sin(x)^6}{x^5}$ is so rapidly

decreasing.

Also, setting TOL to its minimum value of of 10e-15 is, in general, too small for highly discontinuous functions, as more often than not the algorithm cannot compute a good estimate in these cases. Consider the following discontinuous sawtooth function. The combination of the large scale of the amplitudes and sharp discontinuities makes this function unstable at some values of the integral.

$$f(x) := 2 \cdot 10^5 \cdot \Phi(x - 12) + 2 \cdot 10^5 \cdot \Phi(x - 2) - 2 \cdot 10^4 \cdot x - 3 \cdot 10^4$$
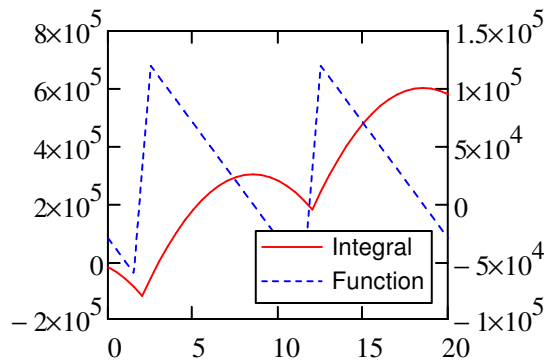
TOL := 10^{-5} $\quad$ ISaw(x) := $\int_{-1}^{x} f(x)\, dx$ $\qquad$ x := 0, .5 .. 20
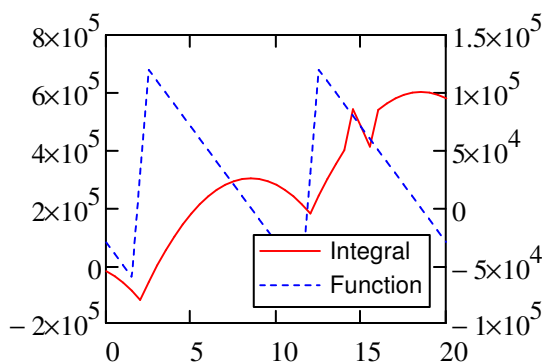
Note the spike in the integral around x = 15. This sort of problem can occur when integrating discontinuous functions. Notice that decreasing TOL below 10^{-10} actually makes this worse.

Break this integral into pieces corresponding to the function discontinuities to get a valid answer:

$$\mathrm{ISaw}(x) := \mathrm{if}\left(x < 12, \int_{-1}^{x} f(x)\, dx, \int_{-1}^{12} f(x)\, dx + \int_{12}^{x} f(x)\, dx\right)$$

It's easy to arrive at functions with these kinds of discontinuities and scale factors when dealing with dimensioned quantities.

Integration Methods

In addition to the required tolerance on the solution, the selected method of integration changes the results from numerical integrals. Some methods are more appropriate than others. In general, Mathcad selects an appropriate method based on the shape of the function to be integrated. This is called Autoselect. You can tell which method has been chosen by right-clicking on the integral symbol.

$$\int_{0}^{10} e^{-x^2}\, dx = 0.886226925446677$$

The Romberg method is chosen to solve this integral, which involves creating trapezoids under the curve and bisecting the width of the trapezoids until the approximation stabilizes. By comparison, the following integral uses an adaptive method, since the function is highly oscillatory.

$$\int_{0}^{2\pi} \sin(16x)^2\, dx = 3.141592653589792$$

In the adaptive method, two approximations with different numbers of approximation points are compared to see if the behavior in any particular chunk of the integration region is being well-modeled. Integration subregions are sized adaptively to get the best picture of the behavior of the function at any particular point.

Other types of problem integrals are handled numerically, such as

$$\int_0^4 \frac{1}{\sqrt{x}} \, dx = 4.000000000000001$$

which uses the Singular Endpoint method to deal with the case of dividing by 0. This method is useful for integrating asymptotic functions, but essentially you are throwing away information at the endpoints of the integration region. It's not recommended to use this method unless an endpoint is actually infinite. Finally, there is a method for integrating functions with infinite limits:

$$\int_0^\infty \frac{1}{1 + u^2} \, du = 1.570796326794897$$

which is again useful for integration of asymptotic functions.

One of the limitations of numerical integration is that very narrow pulses in a mostly zero-valued function often integrate to zero. Typically, if the integrand is zero in greater than 95% of the region of integration, then the algorithm may not evaluate the integrand at one of the non-zero points.

$$f(t1) := \begin{vmatrix} 1 & \text{if } 0.45 < t1 < 0.5 \\ 0 & \text{otherwise} \end{vmatrix}$$

$$\int_0^1 f(t1) \, dt1 = 0 \qquad \int_{0.1}^1 f(t1) \, dt1 = 0.049999691341824$$

You can correct this problem by integrating over a smaller region containing the non-zero part of the integrand. You might also try a technique like **<u>Monte Carlo Integration</u>**.