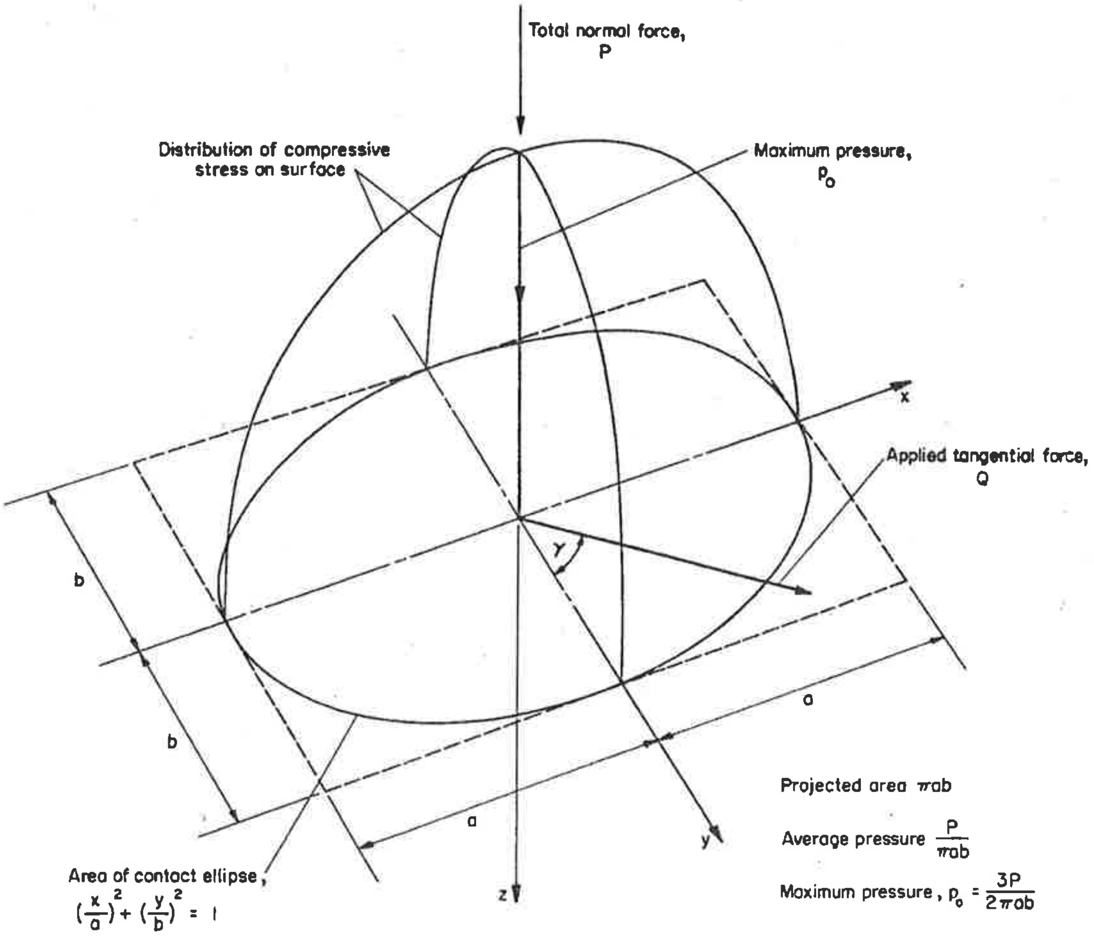


Sketch 3.1 Area of contact, forces and stresses

4. GENERAL DESCRIPTION OF THE CALCULATIONS

These calculations were obtained from Derivations 10 to 13 inclusive, but some modifications and simplications have been made.

For the purposes of the calculations, spatial coordinates are rendered non-dimensional by dividing them by the major semi-axis of the ellipse of contact:

$$x_1 = x/a; \qquad y_1 = y/a; \qquad z_1 = z/a.$$

For the special case of line contacts, when , $a \to \infty$ the dimensional coordinates are divided by the half-width b:

$$x_2 = x/b = x_1/\beta; \qquad y_2 = y/b = y_1/\beta; \qquad z_2 = z/b = z_1/\beta;$$

where $\beta = b/a$ is the axis ratio of the ellipse, and $\beta \to 0$ for a line contact. It will normally be convenient to express the results for the general case in terms of (x_2, y_2, z_2) rather than of (x_1, y_1, z_1).

The stress components are rendered non-dimensional by dividing them by the maximum compressive stress p_0,

$$\bar{f}_i = f_i/p_0; \quad \bar{q}_{ik} = q_{ik}/p_0; \qquad etc, \qquad i, k = x, y, z \quad .$$

Three sets of components are given, corresponding with the effects of the three loading cases assumed:

1. Normal loading: $$f_z = -p_0\left(1-(x/a)^2-(y/b)^2\right)^{1/2} \tag{4.1}$$
2. Tangential loading parallel to short dimension: $$q_{zy} = -\mu p_0\left(1-(x/a)^2-(y/b)^2\right)^{1/2}\cos\gamma \tag{4.2}$$
3. Tangential loading parallel to long dimension: $$q_{zx} = -\mu p_0\left(1-(x/a)^2-(y/b)^2\right)^{1/2}\sin\gamma \tag{4.3}$$

The forces of friction are in the positive directions of the *x*- and *y*-axes. The three sets are distinguished by numerical subscripts, *e.g.* $(f_x)_1$, $(q_{xy})_2$, *etc.* Sets 1 and 3 have been obtained directly from the Derivations, with μ replaced by μ $\sin\gamma$, while set 2 has been obtained from set 3 by interchanging *x* and *y*, *a* and *b*, β and $1/\beta$, $\sin\gamma$ and $\cos\gamma$ and by reversing the direction of *x*, *i.e.* reversing the sign of $(q_{xy})_2$ and $(q_{zx})_2$. The justification for this procedure is shown in Sketch 4.1.

The general elliptical case is considered first, and is followed by the simpler cases of circular and line contacts. The general case requires the evaluation of three numerical integrals, either directly or by conversion to the standard forms of the elliptic integrals, for which a recursive method of evaluation is provided.

Simplified forms are given for use with points which are either on the surface $(z = 0)$ or on the axis $(x = 0, y = 0)$. Unless these special forms are used, it is recommended that any zeros in the dimensionless space coordinates be replaced by small numbers, *e.g.* 10^{-6}. These should be small compared with β, but their squares should be large compared with the smallest digit retained by the calculator or computer. In this way, trouble with infinities and indeterminate numbers may be avoided. The signs of square roots may be taken as positive unless otherwise indicated.

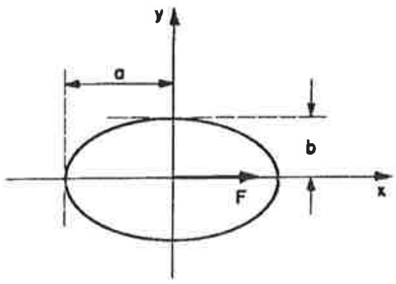
Many of the quantities are defined in different ways in the different sections, *e.g.* s_1^2 is defined in Section 5.2 as the root of the equation

$$\frac{x_1^2}{1+s_1^2} + \frac{y_1^2}{\beta^2+s_1^2} + \frac{z_1^2}{s_1^2} = 1,$$

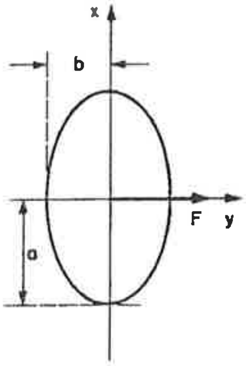
but in Section 9.2 as the root of

$$\frac{\rho_1^2}{1+s_1^2} + \frac{z_1^2}{s_1^2} = 1.$$

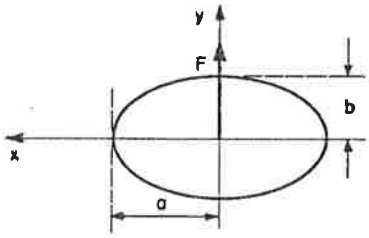
The second version is derived from the first as the special case $\beta = 1$, but because of the large number of symbols needed, the same symbol s_1 has been employed in both cases.



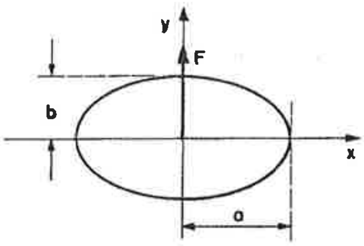
(a) Loading Case 3. Configuration used in Derivation 12



(b) Interchange x and y, a and b



(c) Rotate 90° anticlockwise



(d) Reverse direction of x. This is now the configuration required for loading Case 2

Sketch 4.1 Transformation from loading Case 3 to loading Case 2

This example is an obvious one, but in the case of the function J and its partial derivatives, the definitions for the circular case in Sections 9.3, 9.6 and 9.7 are apparently unrelated to those for the general elliptical case in Sections 5.18 to 5.20. However, the special case may be shown to be derived from the general case in the limit $B \to 1$, $n \to 0$.

In many cases, some checks on the working have been provided. These involve the sums of the three direct stresses for each of the three loading stresses given by Equations (4.1) to (4.3).

5. THE GENERAL ELLIPTICAL CONTACT: POINTS WITHIN THE CONTACT OR ON THE SURFACE AND OUTSIDE THE CONTACT ELLIPSE

5.1 Establish dimensionless coordinates based on the major semi-axis of the ellipse of contact:

$$x_1 = x/a; \quad y_1 = y/a; \quad z_1 = z/a .$$

5.2 Find s_1^2 as the highest root of the equation:

$$\left(\frac{x_1^2}{1+s_1^2}\right) + \left(\frac{y_1^2}{\beta^2+s_1^2}\right) + \left(\frac{z_1^2}{s_1^2}\right) = 1$$

where $\beta = b/a$, the axis ratio of the ellipse of contact $(\beta \le 1)$.

Notes on the procedure for solving this equation are given in Section 6. If the point (x, y, z) is both on the surface and on or within the ellipse of contact, $s_1 = 0$ and the special forms provided for this case should be used (Section 7). If $\beta = 1$ the forms provided for the special case of circular contact should be used (Sections 9 to 12).

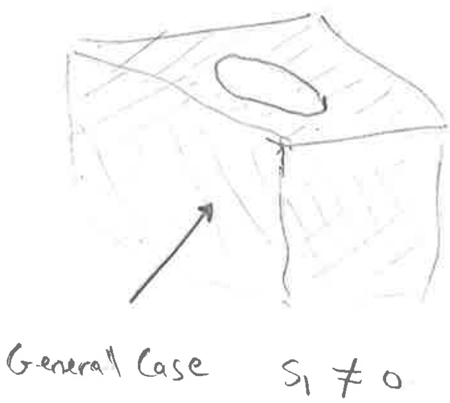
5.3 $$r_1 = \left(s_1^2+1\right)^{1/2}\left(s_1^2+\beta^2\right)^{-1/2}$$

5.4 $$G = \left(s_1^2+1\right)^{1/2}\left(s_1^2+\beta^2\right)^{1/2}$$

5.5 $$H = \left(\frac{x_1}{1+s_1^2}\right)^2 + \left(\frac{y_1}{\beta^2+s_1^2}\right)^2 + \left(\frac{z_1}{s_1^2}\right)^2$$

5.6 $$L = z_1/(s_1^3 HG)$$

5.7 $$n = \left(1-\beta^2\right)^{1/2}$$



5.8 $$I_1 = \int_{s_1}^{\infty}\left(\frac{\mathrm{d}w_1}{\left(1+w_1^2\right)^{3/2}\left(\beta^2+w_1^2\right)^{1/2}}\right)$$

5.9 $$I_2 = \int_{s_1}^{\infty}\left(\frac{\mathrm{d}w_1}{\left(1+w_1^2\right)^{1/2}\left(\beta^2+w_1^2\right)^{3/2}}\right)$$

5.10 $$I_3 = \int_{s_1}^{\infty} \left(\frac{dw_1}{w_1^2\left(1 + w_1^2\right)^{1/2} \left(\beta^2 + w_1^2\right)^{1/2}} \right), \quad s_1 \neq 0 .$$

A note on the evaluation of the integrals I_1, I_2 and I_3 is given in Section 8.

5.11 $$N = Gz_1/s_1$$

5.12 $$C = s_1^2 + 1 + N$$

5.13 $$D = s_1^2 + \beta^2 + N$$

5.14 $$E = \left(2Ns_1^4 + z_1^2(s_1^4 - \beta^2)\right)/\left(NHs_1^4G^2\right)$$, for the case of $z_1 = 0$, $E = 2/(HG^2)$

5.15 $$F = C + D - x_1^2 - y_1^2$$

5.16 $$\theta_1 = \tan^{-1}\left(ny_1/D\right), \quad -\pi/2 \le \theta_1 \le \pi/2$$

5.17 $$\theta_2 = \tanh^{-1}\left(nx_1/C\right)$$

Notes: (a) These definitions of θ_1 and θ_2 may be shown to be equivalent to those given in the Derivations, with a change of sign.

(b) If the inverse tangent is expressed in degrees, it must be multiplied by $(\pi/180)$ to convert it to radians.

(c) $\tanh^{-1}Z = \tfrac{1}{2}\ln\left(\frac{1+Z}{1-Z}\right)$.

5.18 $$J = \frac{1}{n^3}\left(x_1\theta_1 - y_1\theta_2\right)$$

5.19 $$\frac{\partial J}{\partial x_1} = \frac{1}{n^3}\left[\theta_1 - \frac{ny_1C}{(s_1^2 + 1)F} - \frac{n^3x_1^2\,y_1E}{(s_1^2 + 1)F}\right]$$

5.20 $$\frac{\partial J}{\partial y_1} = \frac{1}{n^3}\left[-\theta_2 + \frac{nx_1D}{(s_1^2 + \beta^2)F} - \frac{n^3x_1y_1^2E}{(s_1^2 + \beta^2)F}\right]$$

5.21 $$K = \frac{1}{n^3}\left(y_1\theta_1 + x_1\theta_2 - n\right)$$

5.22 $$(\bar{f}_x)_1 = \beta\left[2(1-\sigma)z_1I_1 - 2\sigma z_1I_3 - \left(\frac{x_1s_1}{Gr_1}\right)^2 L + (1-2\sigma)[K + z_1/(n^2r_1s_1)]\right]$$

5.23 $$(\bar{f}_y)_1 = \beta\left[2(1-\sigma)z_1I_2 - 2\sigma z_1I_3 - \left(\frac{y_1s_1r_1}{G}\right)^2 L - (1-2\sigma)[K + z_1r_1/(n^2s_1)]\right]$$

5.24 $$(\bar{f}_z)_1 = -\beta z_1^2\, L/s_1^2$$

5.25 $$(\bar{q}_{xy})_1 = (1-2\sigma)\beta J - (\beta x_1y_1s_1^2\, L/G^2)$$

5.26 $$(\bar{q}_{yz})_1 = -\beta y_1z_1Lr_1/G$$

5.27 $$(\bar{q}_{zx})_1 = -\beta z_1x_1L/(Gr_1)$$

5.28 $$(\bar{f}_x)_2 = \mu\beta\cos\gamma\left[\left(\frac{1}{n^2}(I_1 - I_2) - \frac{x_1^2s_1}{r_1HG^4}\right)2\sigma y_1 + (1-2\sigma)z_1\frac{\partial J}{\partial x_1}\right]$$

5.29 $$(\bar{f}_y)_2 = \mu\beta\cos\gamma\left[-\left((n^2 - \sigma\beta^2)I_2 + \sigma I_1\right)\frac{2y_1}{n^2} - (1-2\sigma)\, z_1\frac{\partial J}{\partial x_1} + \frac{y_1z_1^2\, r_1}{HG^2s_1^3} + \frac{2\sigma x_1^2y_1s_1}{r_1HG^4}\right]$$

5.30 $$(\bar{f}_z)_2 = -\mu\beta\cos\gamma\left(\frac{y_1z_1^2\, r_1}{s_1^3\, G^2H}\right)$$

5.31 $$(\bar{q}_{xy})_2 = -\mu\beta\cos\gamma\left[\left(2\sigma(I_1 - \beta^2I_2) - n^2I_1\right)\frac{x_1}{n^2} + (1-2\sigma)z_1\frac{\partial J}{\partial y_1} - \frac{2\sigma x_1y_1^2\, r_1s_1}{HG^4}\right]$$

5.32 $$(\bar{q}_{yz})_2 = \mu\beta\cos\gamma\left(z_1(I_2 - I_3) - \frac{y_1^2\, z_1r_1^2}{s_1HG^3}\right)$$

5.33 $$(\bar{q}_{zx})_2 = \mu\beta\cos\gamma\left(\frac{x_1y_1z_1}{s_1HG^3}\right)$$

5.34 $$(\bar{f}_x)_3 = \mu\beta\sin\gamma\left[-\left((n^2+\sigma)I_1 - \sigma\beta^2 I_2\right)\frac{2x_1}{n^2} - (1-2\sigma)z_1\frac{\partial J}{\partial y_1} + \frac{x_1 z_1^2}{HG^2 r_1 s_1^3} + \frac{2\sigma x_1 y_1^2\, s_1 r_1}{HG^4}\right]$$

5.35 $$(\bar{f}_y)_3 = \mu\beta\sin\gamma\left[\left(\frac{\beta^2}{n^2}(I_1 - I_2) - \frac{y_1^2\, s_1 r_1}{HG^4}\right)2\sigma x_1 + (1-2\sigma)z_1\frac{\partial J}{\partial y_1}\right]$$

5.36 $$(\bar{f}_z)_3 = -\mu\beta\sin\gamma\left(\frac{x_1 z_1^2}{s_1^3 G^2 H r_1}\right)$$

5.37 $$(\bar{q}_{xy})_3 = \mu\beta\sin\gamma\left[\left(2\sigma(I_1 - \beta^2 I_2) - n^2 I_2\right)\frac{y_1}{n^2} + (1-2\sigma)z_1\frac{\partial J}{\partial x_1} - \frac{2\sigma x_1^2\, y_1 s_1}{HG^4 r_1}\right]$$

5.38 $$(\bar{q}_{yz})_3 = -\mu\beta\sin\gamma\left(\frac{x_1 y_1 z_1}{s_1 HG^3}\right)$$

5.39 $$(\bar{q}_{zx})_3 = \mu\beta\sin\gamma\left(z_1(I_1 - I_3) - \frac{x_1^2\, z_1}{s_1 HG^3 r_1^2}\right)$$

5.40 $$\begin{aligned}
\bar{f}_x &= (\bar{f}_x)_1 + (\bar{f}_x)_2 + (\bar{f}_x)_3 \\
\bar{f}_y &= (\bar{f}_y)_1 + (\bar{f}_y)_2 + (\bar{f}_y)_3 \\
\bar{f}_z &= (\bar{f}_z)_1 + (\bar{f}_z)_2 + (\bar{f}_z)_3 \\
\bar{q}_{xy} &= (\bar{q}_{xy})_1 + (\bar{q}_{xy})_2 + (\bar{q}_{xy})_3 \\
\bar{q}_{yz} &= (\bar{q}_{yz})_1 + (\bar{q}_{yz})_2 + (\bar{q}_{yz})_3 \\
\bar{q}_{zx} &= (\bar{q}_{zx})_1 + (\bar{q}_{zx})_2 + (\bar{q}_{zx})_3
\end{aligned}$$

Note that a useful check on the working is obtained by taking the sums of the three direct stresses for each of the three loading stresses:

$$\begin{aligned}
(\bar{f}_x)_1 + (\bar{f}_y)_1 + (\bar{f}_z)_1 &= -2\beta(1+\sigma)z_1 I_3 \\
(\bar{f}_x)_2 + (\bar{f}_y)_2 + (\bar{f}_z)_2 &= -2\mu\beta\cos\gamma\,(1+\sigma)y_1 I_2 \\
(\bar{f}_x)_3 + (\bar{f}_y)_3 + (\bar{f}_z)_3 &= -2\mu\beta\sin\gamma\,(1+\sigma)x_1 I_1 .
\end{aligned}$$

6. SOLVING THE EQUATION FOR s_1^2

6.1 The equation is

$$f(s_1) = \frac{x_1^2}{1 + s_1^2} + \frac{y_1^2}{\beta^2 + s_1^2} + \frac{z_1^2}{s_1^2} - 1 = 0 .$$

In general, this last equation will have three real roots, of which only one, the required largest root, is positive. The equation may be re-written as

$$s_1^6 + 3b_2 s_1^4 + 3b_1 s_1^2 - 3b_0 = 0 ,$$

where

$$b_2 = \frac{1}{3}\left(1 + \beta^2 - x_1^2 - y_1^2 - z_1^2\right)$$

$$b_1 = \frac{1}{3}\left(\beta^2 - \beta^2 x_1^2 - y_1^2 - (1 + \beta^2) z_1^2\right)$$

$$b_0 = \frac{1}{3}\beta^2 z_1^2 .$$

⇒ $sis^3 + (3b_2) sis^2 + (3b_1) sis - 3b_0 = 0$

Coefficients in Mathcad.

$b0_3 = 1$

$b0_2 = 3b_2$

$b0_1 = 3b_1$

$b0_0 = -3b_0$

6.2 The following expressions are now calculated

$$q = b_2^2 - b_1$$

$$r = \frac{3}{2}\left(b_1 b_2 + b_0\right) - b_2^3$$

$$\psi = \frac{1}{3}\cos^{-1}\left(r q^{-3/2}\right)$$

In general q will be positive and its square root may also be taken as positive, while r may be positive, zero or negative. The inverse cosine should be placed in the first quadrant $(0 \le \psi \le \pi/2)$ if r is positive and in the second quadrant $(\pi/2 \le \psi \le \pi)$ if r is negative.

6.3 The required largest root is

$$s_1^2 = 2q^{1/2}\cos\psi - b_2$$

The other roots may be obtained by increasing ψ by $2\pi/3$ (120 deg) and $4\pi/3$ (240 deg) respectively.

6.4 If $\beta < 10^{-2}$ the above procedure may give inaccurate results. The value of the function $f(s_1)$ should always be checked, and if it differs significantly from zero, corrections may be made by Newton's method. The value of s_1 found by the method of Section 6.3 may be used as a starting value of $(s_1)_0$, subsequent corrections being made by

$$(s_1)_{i+1} = (s_1)_i + \frac{f[(s_1)_i]}{2(s_1)_i H[(s_1)_i]} ,$$

18. WORKED EXAMPLES

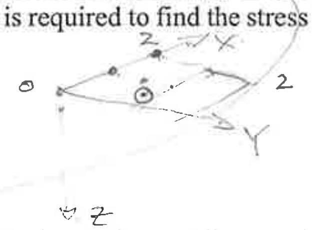
18.1 Example 1

Example 1 of Data Item No. 78035 (Reference 1) describes two cylinders, of diameters 50 mm (2 in) and 150 mm (6 in) crossed at an angle of 40 degrees and subject to a normal load of 1.25 kN (281 lbf). The properties of the contact, estimated by the methods given in Reference 1, are reproduced in Table 18.1.

TABLE 18.1 Contact stresses and dimensions – Example 1

Maximum compressive stress	P_0	0.783 GPa	113600 lbf in^{-2}
Major semi-axis	a	1.896 mm	0.0746 in
Minor semi-axis	b	0.404 mm	0.0159 in
Axis ratio	β	0.213	

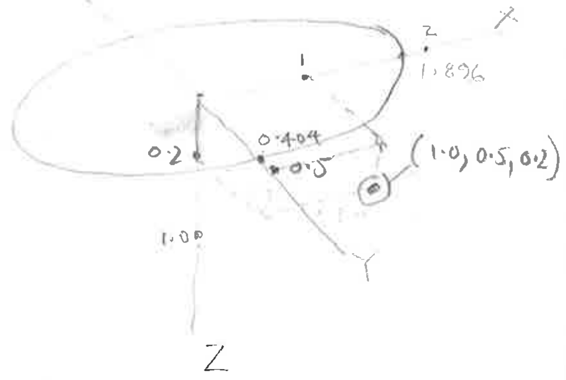
The material has a Poisson's ratio of 0.3, and shear stresses with a coefficient of friction, μ , of 0.45 will be imposed at an angle, γ, of 30 deg. with the minor axis of the ellipse. It is required to find the stress field at the point given by the dimensional coordinates:



$$x = 1.0 \text{ mm},$$
$$y = 0.5 \text{ mm},$$
$$z = 0.2 \text{ mm}.$$

The number at the beginning of each line in the following working refers to the section and line number in the main text.

18.1.1 Preliminaries



5.1 $$x_1 = 1.0/1.896 = 0.5274$$

$$y_1 = 0.5/1.896 = 0.2637$$

$$z_1 = 0.2/1.896 = 0.1055$$

5.7 $$n^2 = \left(1 - 0.213^2\right) = 0.954631$$

6.1 $$b_2 = \frac{1}{3}\left(1 + 0.213^2 - 0.5274^2 - 0.2637^2 - 0.1055^2\right) = 0.2288501$$

$$b_1 = \frac{1}{3}\left(0.213^2 - \left(0.213 \times 0.5274\right)^2 - 0.2637^2 - 0.1055^2(1 + 0.213^2)\right)$$

$$= -0.01614111$$

$$b_0 = \frac{1}{3}\left(0.213 \times 0.1055\right)^2 = 1.683228 \times 10^{-4}$$

6.2 $q = 0.2288501^2 + 0.01614111 = 0.06851348$

$$r = \frac{3}{2}\left((-0.01614111 \times 0.2288501) + 1.683228 \times 10^{-4}\right) - 0.2288501^3$$

$$= -0.01727378$$

$$\psi = \frac{1}{3}\cos^{-1}\left(-\frac{0.1727378}{0.06851348^{3/2}}\right) = 54.803742 \text{ deg.}$$

6.3 $s_1^2 = 2 \times (0.06851348)^{1/2}\cos(54.803742°) - 0.2288501 = 0.0728852$

$$s_1 = 0.26997$$

6.4 $$f(s_1) = \frac{0.5274^2}{1.0728852} + \frac{0.2637^2}{0.213^2 + 0.0728852} + \frac{0.1055^2}{0.0728852} - 1 = -6.78 \times 10^{-8}$$

The value of s_1 will therefore be taken as 0.2700.

18.1.2 Calculation of the integrals I_1, I_2 and I_3

Three alternative methods will be given.

Method A. Direct numerical integration.

This method is recommended when the number of points to be calculated is small, and when a sub-routine for numerical integration is available. The substitution $u = w_1^{-1}$ puts the integrals into the forms:

$$I_1 = \int_0^{s_1^{-1}} \left(u^2\left(1 + u^2\right)^{-3/2}\left(1 + \beta^2 u^2\right)^{-1/2}\right) du$$

$$I_2 = \int_0^{s_1^{-1}} \left(u^2\left(1 + u^2\right)^{-1/2}\left(1 + \beta^2 u^2\right)^{-3/2}\right) du$$

$$I_3 = \int_0^{s_1^{-1}} \left(u^2\left(1 + u^2\right)^{-1/2}\left(1 + \beta^2 u^2\right)^{-1/2}\right) du$$

Numerical integration by Simpson's rule gives the results shown in Table 18.2.

TABLE 18.2 Results of numerical integration – Example 1

N_i †	I_1	I_2	I_3
10	0.963458	4.112769	5.321722
20	0.962983	4.112683	5.321650
50	0.962983	4.112682	5.321650

† N_i = number of divisions into which the integration interval is subdivided