Hermite–gaussian functions of complex argument as optical-beam eigenfunctions

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Optical-resonator modes and optical-beam-propagation problems have been conventionally analyzed using as the basis set the hermite–gaussian eigenfunctions $\psi_n(x,z)$ consisting of a hermite polynomial of real argument $H_n[\sqrt{2}x/w(z)]$ times the complex gaussian function $\exp[-jkx^2/2q(z)]$, in which $q(z)$ is a complex quantity. This note shows that an alternative and in some ways more-elegant set of eigensolutions to the same basic wave equation is a hermite-gaussian set $\hat{\psi}_n(x,z)$ of the form $H_n[\sqrt{c}x]\exp[-cx^2]$, in which the hermite polynomial and the gaussian function now have the same complex argument $\sqrt{c}x \equiv (jk/2q)^{1/2}x$. The conventional functions ψ_n are orthogonal in x in the usual fashion. The new eigenfunctions $\hat{\psi}_n$, however, are not solutions of a hermitian operator in x and hence form a biorthogonal set with a conjugate set of functions $\hat{\phi}_n(\sqrt{c}x)$. The new eigenfunctions $\hat{\psi}_n$ are not by themselves eigenfunctions of conventional spherical-mirror optical resonators, because the wave fronts of the $\hat{\psi}_n$ functions are not spherical for $n > 1$. However, they may still be useful as a basis set for other optical resonator and beam-propagation problems.

Index Heading: Resonant modes.

An optical beam traveling in the z direction may be written in the scalar approximation in the form

$$u(x,y,z)=\psi(x,y,z)e^{-jkz}. \qquad (1)$$

Under the usual assumption that $\psi(x,y,z)$ is slowly varying compared to a wavelength, the paraxial wave equation as given by Kogelnik and Li[1] reduces to the form

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}-2k\frac{\partial\psi}{\partial z}=0, \qquad (2)$$

where a $\partial^2\psi/\partial z^2$ term has been ignored. The eigensolutions to this equation are most commonly given as the hermite–gaussian set $\psi_{nm}(x,y,z)=\psi_n(x,z)\psi_m(y,z)$ where

$$\psi_n(x,z)=\left(\frac{1}{w(z)}\right)^{\frac{1}{2}}H_n\left(\frac{\sqrt{2}x}{w(z)}\right)e^{-[jk/2q(z)]x^2}e^{j(n+\frac{1}{2})\phi(z)}, \qquad (3)$$

and similarly for the y coordinate. The z dependence is given by $dq(z)/dz=1$, $w^2(z)=-\lambda/\pi\,\mathrm{Im}[1/q(z)]$, and $\phi(z)$ is the Guoy phase-shift factor.[1] A similar Laguerre-gaussian expansion in cylindrical coordinates is also possible.

These conventional solutions show a somewhat inelegant lack of symmetry between the complex argument of the gaussian function and the purely real argument of the hermite function. There is a similar lack of elegance in the way in which the complex parameter $q(z)$ appears in one part of the solution, whereas only the real spot size $w(z)$ appears in other parts. This note points out that it is possible to obtain an alternative set of eigensolutions $\hat{\psi}_{nm}(x,y,z)=\hat{\psi}_n(x,z)\times\hat{\psi}_m(y,z)$ that satisfy exactly the same differential Eq. (2), but that have the more-symmetric form

$$\hat{\psi}_n(x,z)=A(q)H_n\{\sqrt{(jk/2q)}x\}e^{-(jk/2q)x^2}. \qquad (4)$$

The z dependence is now entirely contained in the complex parameter $q(z)$, which has the same form as before. The complex amplitude $A(q)$, which is the analog of the $[1/w(z)]^{\frac{1}{2}}\exp[j(n+\frac{1}{2})\phi(z)]$ factor in the usual expansion, also has a particularly simple form.

ANALYSIS

Since we expect the usual gaussian-mode factor $\exp[-(jk/2q)x^2]$ to be a basic part of the eigenfunction in any case, we write our proposed alternative solutions in the form

$$\hat{\psi}_n(x,z)=A(q)H_n(\sqrt{c}x)e^{-cx^2}, \qquad (5)$$

where c is the complex parameter $c=c(z)=jk/2q(z)$ and the functional form of $H_n(\sqrt{c}x)$ is initially undetermined. Putting Eq. (5) into the wave Eq. (2) and making use of the usual condition $dq(z)/dz=1$ immediately leads to the two separated equations

$$H_n''(\sqrt{c}x)-2\sqrt{c}xH_n'(\sqrt{c}x)+2nH_n(\sqrt{c}x)=0, \qquad (6a)$$

$$\frac{q}{A}\frac{dA}{dq}=-\frac{n+1}{2}. \qquad (6b)$$

The function $H_n(\sqrt{c}x)$ is evidently a Hermite polynomial of complex argument $\sqrt{c}x$, whereas the amplitude factor $A(q)$ becomes a simple function of $q(z)$ only. From Eqs. (6a) and (6b), the resulting complex-argument hermite–gaussian normal modes are

$$\hat{\psi}_n(x,z)=(q_0/q)^{(n+1)/2}H_n(\sqrt{c}x)e^{-cx^2}, \qquad (7)$$

where $c\equiv jk/2q$. This is certainly an at least superficially neater alternative set of hermite–gaussian eigensolutions to the basic wave Eq. (2).

DISCUSSION

The new eigensolutions Eqs. (4) or (7) are not the same as the conventional eigensolutions Eq. (3) on any one-to-one basis. For example, at a waist where $q=j\pi w^2/\lambda$, the conventional solutions are

$$\psi_n(x)=H_n\left(\frac{\sqrt{2}x}{w}\right)e^{-x^2/w^2}, \tag{8}$$

while the new solutions reduce to

$$\hat{\psi}_n(x)=H_n\left(\frac{x}{w}\right)e^{-x^2/w^2}. \tag{9}$$

As functions of x, only the conventional hermite–gaussian functions satisfy the differential equation

$$\frac{d^2\psi_n}{dx^2}+(2n+1-x^2)\psi_n=0, \tag{10}$$

whereas the new complex-argument hermite–gaussian functions are solutions of the equation

$$\frac{d^2\hat{\psi}_n}{dx^2}+2cx\frac{d\hat{\psi}_n}{dx}+2(n+1)c\hat{\psi}_n=0. \tag{11}$$

This equation may written in operator form as

$$\mathfrak{L}\hat{\psi}_n=\lambda_n\hat{\psi}_n, \tag{12}$$

where the differential operator $\mathfrak{L}$ and its eigenvalues λ_n are

$$\mathfrak{L}\equiv\left[\frac{d^2}{dx^2}+2cx\frac{d}{dx}\right],\quad \lambda_n=-2(n+1)c. \tag{13}$$

This operator is not a hermitian operator, and its eigenfunctions $\hat{\psi}_n$ do not form an orthonormal set. The hermitian adjoint operator $\mathfrak{L}^+$ conjugate to this operator is

$$\mathfrak{L}^+\equiv\left[\frac{d^2}{dx^2}-\frac{d}{dx}(2c^*x)\right]. \tag{14}$$

The eigenfunctions $\hat{\phi}_n$ of the adjoint operator are the solutions of the adjoint equation $\mathfrak{L}^+\hat{\phi}_n=\mu_n\hat{\phi}_n$, which reduces to

$$\frac{d^2\hat{\phi}_n}{dx^2}-2c^*x\frac{d\hat{\phi}_n}{dx}-(2c^*+\mu_n)\hat{\phi}_n=0. \tag{15}$$

The eigensolutions to the adjoint equation are

$$\hat{\phi}_n(x)=H_n(\sqrt{c^*}x), \tag{16a}$$

$$\mu_n=-2(n+1)c^*. \tag{16b}$$

As expected, $\mu_n=\lambda_n^*$. There is, however, no gaussian factor associated with the adjoint functions $\hat{\phi}_n$. The original solutions $\hat{\psi}_n$ and the adjoint solutions $\hat{\phi}_n$ form a biorthogonal set, with the orthogonality relationship

$$\int_{\infty}^{\infty}\hat{\phi}_n^*(x)\hat{\psi}_m(x)dx=\int_{\infty}^{\infty}H_n(\sqrt{c}x)H_m(\sqrt{c}x)e^{-cx^2}dx =K_n\delta_{nm}. \tag{17}$$

This orthogonality relation checks for the case c purely real, which also provides a convenient method for evaluating the normalization coefficient K_n. If a given wave function $u(x)$ is to be expanded in the new complex-argument eigenfunctions $\hat{\psi}_n$, in the form

$$u(x)=\sum_n a_n\hat{\psi}_n(x), \tag{18}$$

then the coefficients a_n, assuming proper normalization of $\hat{\psi}_n$, will be given by

$$a_n=\int_{\infty}^{\infty}\hat{\phi}_n^*(x)u(x)dx. \tag{19}$$

Note, however, that the adjoint functions $\hat{\phi}_n$ cannot be normalized, because, without any gaussian factor, their areas diverge.

These more-elegant eigenfunctions $\hat{\psi}_n$ are not the appropriate basis set for analyzing stable spherical-mirror optical resonators because the wave fronts of the higher-order modes ($n>1$) are not spherical. The complex argument $\sqrt{c}x$ in the hermite polynomials gives an additional phase variation in $H_n(\sqrt{c}x)$ that modifies the usual gaussian spherical-phase variation. However, the new eigenfunctions with their greater simplicity may be useful in other sorts of optical-beam propagation and optical-resonator problems, in free space or where more-general phase variations are involved.

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REFERENCES

[1]H. Kogelnik and T. Li, Proc. IEEE **54**, 1312 (1966).