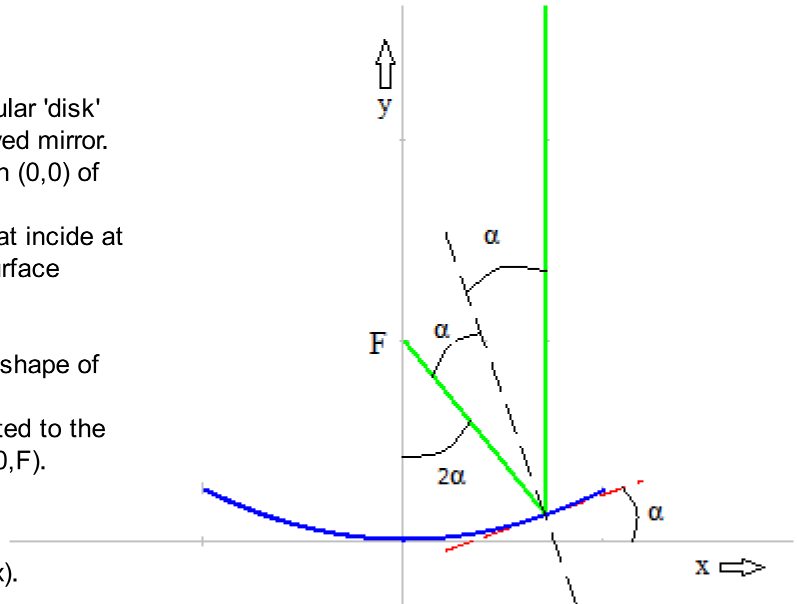
Optics

Mirror reflector

Consider an optical system of a circular 'disk' concave Mirror; that is, a hollow curved mirror. The 'top' of the curve lies at the origin (0,0) of the coordinate system.
The mirror is ideal, it reflects rays that incide at an angle α w.r.t. the normal on its surface back at an angle of $-\alpha$ with no loss.

Find the expression that defines the shape of the curve such that all rays inciding perpendicular to the x-axis are reflected to the focal point of the mirror which is at (0,F).



Let the mirror curve be defined by y(x).

The (green) ray falls straight onto the mirror, and hits the curve y(x) at an angle α with the normal (black line) on the curve y(x). This angle α is the same angle as the angle between the tangential of curve y(x) with the x-axis. Note that α is a function of x, thus α(x)

That tangential is $\frac{d}{dx}y(x)$ and is $\tan(\alpha(x))$ thus $\frac{d}{dx}y(x) = \tan(\alpha(x))$

The ray is reflected again at an angle $-\alpha$ w.r.t. the normal towards the point (0,F) on the y-axis.
The ray hits the y-axis at an angle equal to 2α; this gives:

$$\tan(2\cdot\alpha(x)) = \frac{x}{F - y(x)}$$

Then this is the system of 2 equations for the mirror, with boundary conditions and 2 unknowns:

$$EQ_M := \begin{pmatrix} \frac{d}{dx}y(x) - \tan(\alpha(x)) \\ \tan(2\cdot\alpha(x)) - \frac{x}{F - y(x)} \end{pmatrix} \quad BC_M := \begin{pmatrix} y(0) = 0 \\ \alpha(0) = 0 \end{pmatrix} \quad UK_M := \begin{pmatrix} y \\ \alpha \end{pmatrix} \quad UF_M := \begin{pmatrix} y(x) \\ \alpha(x) \end{pmatrix}$$

A symbolic solution is however not easily found:

$$EQ_M \text{ solve}, UF_M \rightarrow$$

Numerically it is possible, with: (r defines the radius of the mirror disk R as a fraction of F)

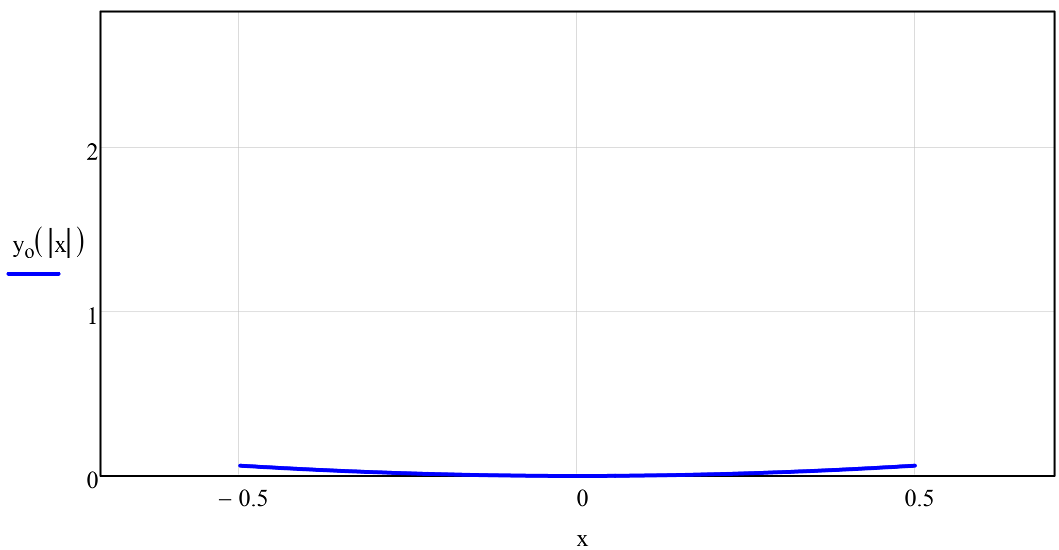
$F := 1 \qquad r := \frac{1}{2} \qquad R := r \cdot |F|$

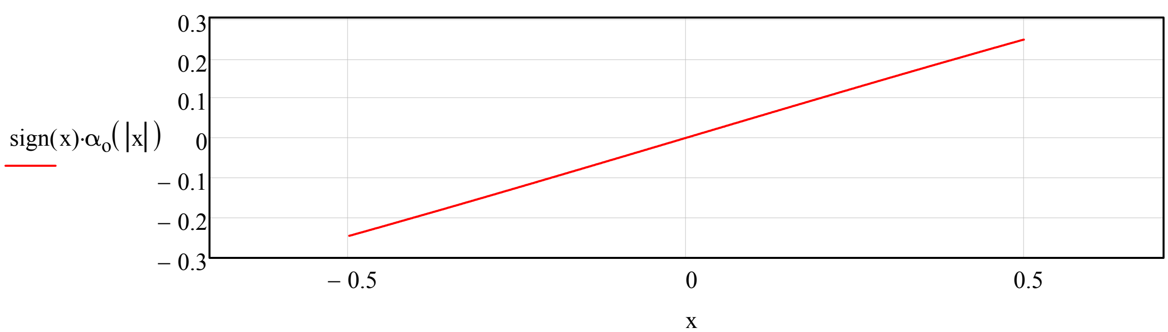
Given $\quad y(0) = 0 \quad \alpha(0) = 0$

$$\frac{d}{dx}y(x) - \tan(\alpha(x)) = 0 \qquad \tan(2\cdot\alpha(x)) - \frac{x}{F - y(x)} = 0$$

$$\begin{pmatrix} y_o \\ \alpha_o \end{pmatrix} := \text{Odesolve}\left[\begin{pmatrix} y \\ \alpha \end{pmatrix}, x, R\right]$$

$x := -|R|, -|R| + \frac{|R|}{360} .. |R| \quad z := \sqrt{2} \quad y_{top} := z\cdot 4\cdot R\cdot(F > 0) \qquad y_{bot} := -z\cdot 4\cdot R\cdot(F < 0)$





At such scales $\alpha(|x|)$ appears to be linear with x. What if the radius of the lens R is larger than F ?

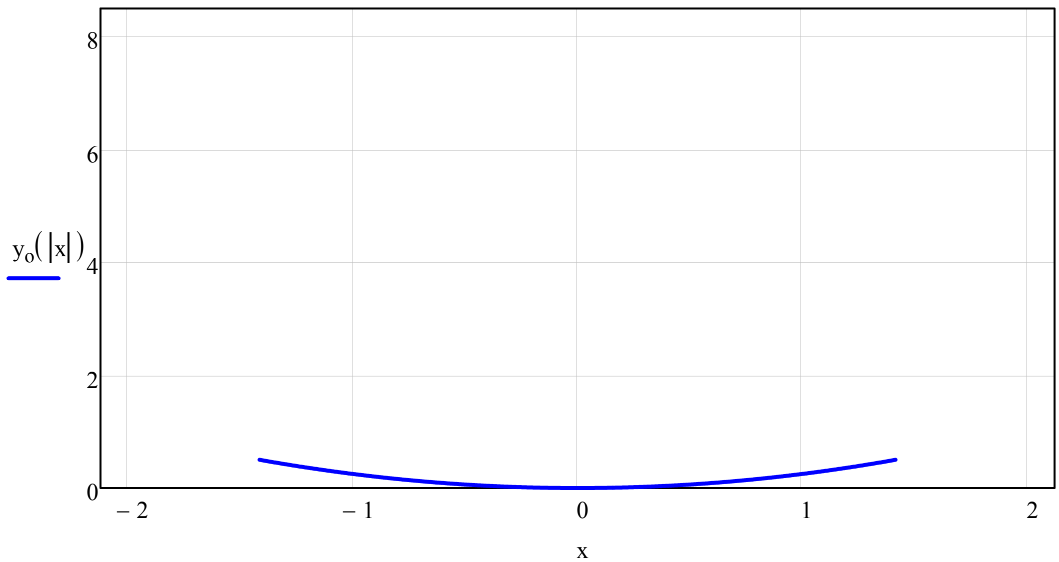
$F := 1 \qquad r := \sqrt{2} \qquad R := r \cdot |F|$

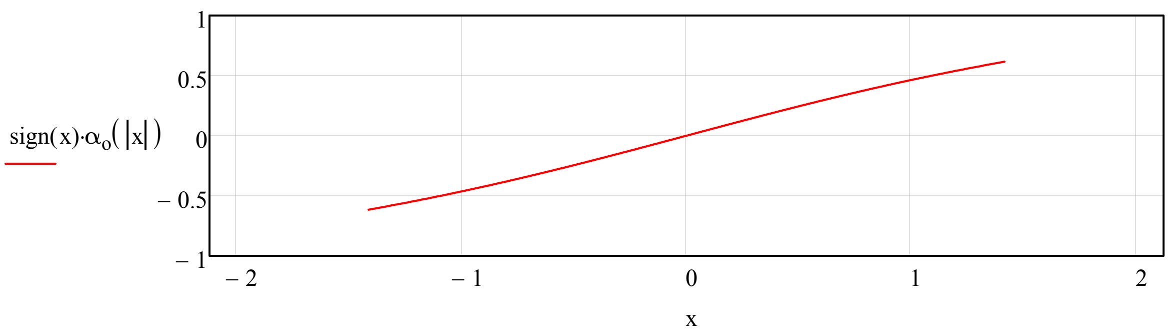
Given $\quad y(0) = 0 \quad \alpha(0) = 0$

$$\frac{d}{dx}y(x) - \tan(\alpha(x)) = 0 \qquad \tan(2\cdot\alpha(x)) - \frac{x}{F - y(x)} = 0$$

$$\begin{pmatrix} y_o \\ \alpha_o \end{pmatrix} := \text{Odesolve}\left[\begin{pmatrix} y \\ \alpha \end{pmatrix}, x, R\right]$$

$z := 1.5 \qquad x := -R, -R + \frac{R}{360} .. R \qquad y_{top} := z \cdot 4 \cdot R \cdot (F > 0) \qquad y_{bot} := -z \cdot 4 \cdot R \cdot (F < 0)$





$y_o(x)$ does look like a parabola.

suppose $\quad <y> := a \cdot x^2 \qquad <y(x)> := y(x) = <y> \rightarrow y(x) = a \cdot x^2$

then with: $\quad EQ_{M_0} \rightarrow \frac{d}{dx} y(x) - \tan(\alpha(x)) \qquad$ gives: $\quad <\tan(\alpha)> := EQ_{M_0} \left| \begin{array}{l} \text{substitute}, <y(x)> \\ \text{solve}, \tan(\alpha(x)) \end{array} \right. \rightarrow$

and: $\quad <\alpha> := EQ_{M_0} \left| \begin{array}{l} \text{substitute}, <y(x)> \\ \text{solve}, \alpha(x) \end{array} \right. \rightarrow$

and this gives substitutions for: $<\tan(\alpha(x))> := \tan(\alpha(x)) = <\tan(\alpha)> \rightarrow$

and: $<\alpha(x)> := \alpha(x) = <\alpha> \rightarrow$

knowing trigonometry rule: $\quad TR_9 \rightarrow \tan(2 \cdot a) = -\frac{2 \cdot \tan(a)}{\tan(a)^2 - 1}$

gives substitution: $\quad <\tan(2\alpha(x))> := TR_9 \text{ substitute}, a = \alpha(x) \rightarrow \tan(2 \cdot \alpha(x)) = -\frac{2 \cdot \tan(\alpha(x))}{\tan(\alpha(x))^2 - 1}$

then with $\quad EQ_{M_1} \rightarrow \tan(2 \cdot \alpha(x)) - \frac{x}{F - y(x)} \qquad$ find a: $\quad <a> := EQ_{M_1} \left| \begin{array}{l} \text{substitute}, <\tan(2\alpha(x))> \\ \text{substitute}, <\tan(\alpha(x))> \\ \text{substitute}, <y(x)> \\ \text{solve}, a \end{array} \right. \rightarrow$

which gives a substitution: $<a> := a = <a> \rightarrow$

so with $\quad y\_M(x, F) := <y> \text{ substitute}, <a> \rightarrow \qquad y_M(x, F) := \left| \begin{array}{l} \text{return } 0 \text{ if } (F = 0) \\ y\_M(x, F) \end{array} \right.$

and $\quad \alpha\_M(x, F) := <\alpha> \text{ substitute}, <a> \rightarrow \qquad \alpha_M(x, F) := \left| \begin{array}{l} \text{return } 0 \text{ if } (F = 0) \\ \alpha\_M(x, F) \end{array} \right.$

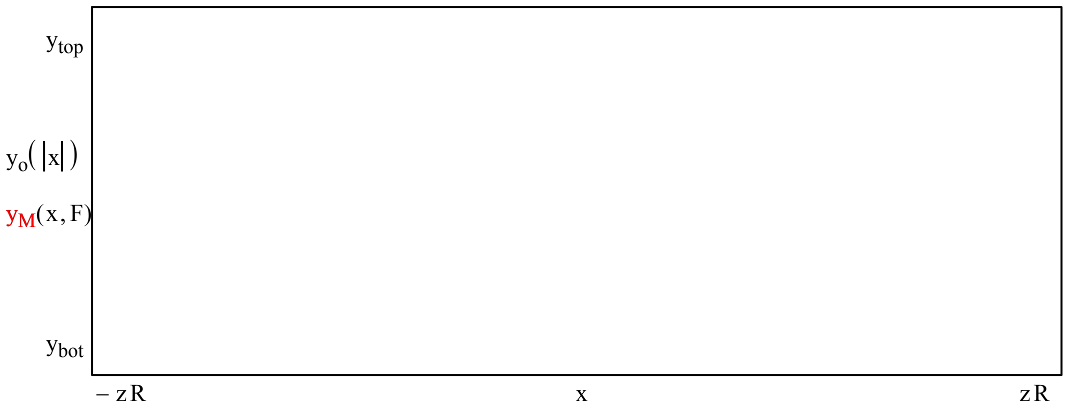
the set of differential equations is solved

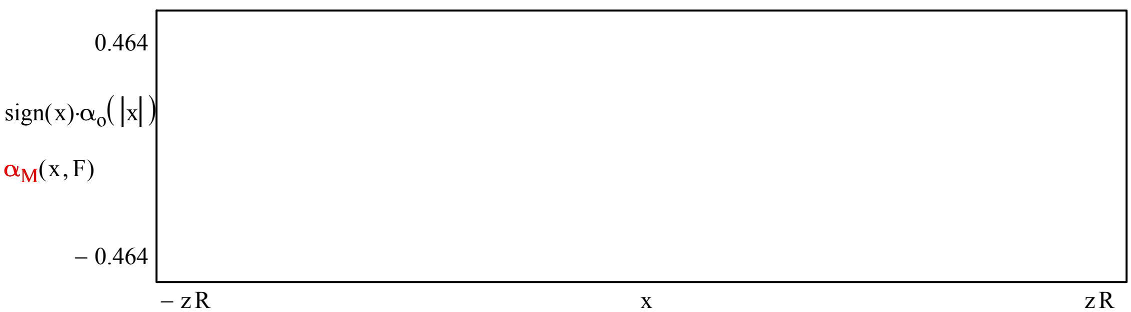
Now compare the symbolic with the numeric solution.

$R := |F| \quad z := 1.5 \quad x := -R, -R + \frac{R}{360} .. R$

Note that for |x|>R the Odesolve solutions are off.

$y_{top} := z \cdot |y_o(R)| \cdot (F > 0) \qquad y_{bot} := -z \cdot |y_o(R)| \cdot (F < 0)$





Draw the trajectory of a ray that falls on the mirror at a distance W from the centre, where w is a fraction of R .

$F := 1 \qquad X := 10 \qquad z := 2$

$i := ORIGIN, ORIGIN + 1 .. ORIGIN + r := \frac{1}{2} \qquad R := r \cdot |F| + X \cdot (F = 0) \qquad w := \sqrt{\frac{1}{2}} \qquad W := w \cdot R$

$d(W,F,X) := \tan\left(\frac{\pi}{2} + 2 \cdot \alpha_M(W,F)\right) \cdot (X - 1) \cdot W + y_M(W,F)$

$ray_d(w) := \left[\infty \;\; y_M(w,F) \;\; F \cdot (F > 0) + d(W,F,X) \cdot (F < 0)\right]^T \qquad ray_w(w) := [w \;\; w \;\; XW \cdot (F < 0)]^T$

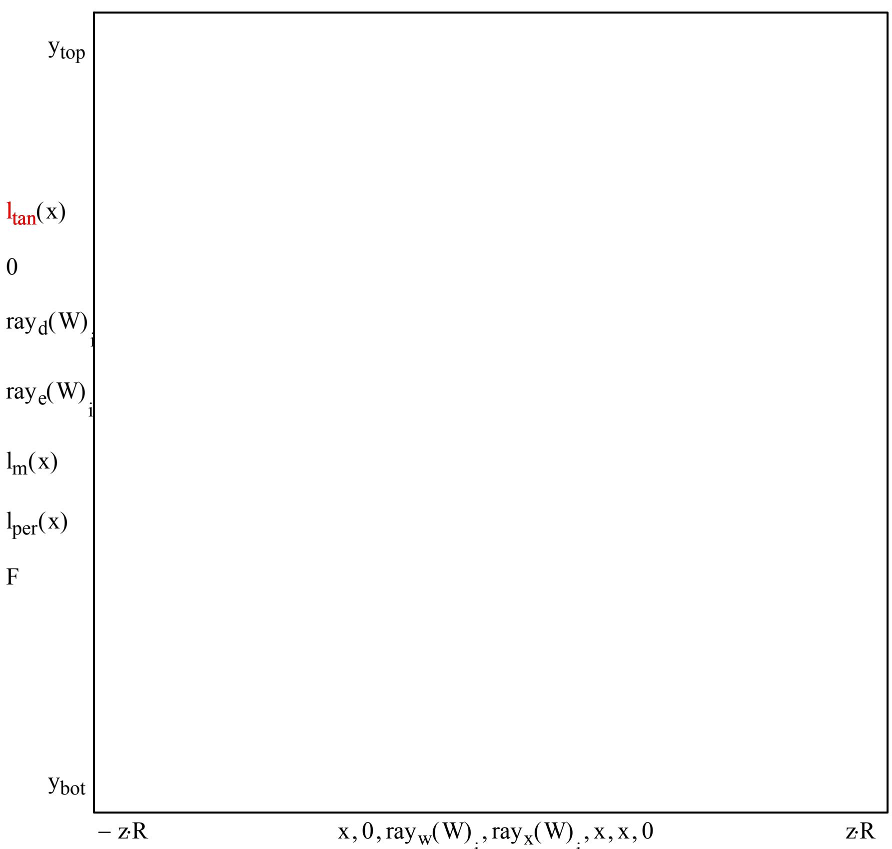
$ray_e(w) := \left[XF - (X - 1) \cdot y_M(w,F) \;\; y_M(w,F)\right]^T \qquad ray_x(w) := [-(X - 1)w \;\; w]^T$

mirror curve: $l_m(x) := \text{if}\left(|x| \le R, y_M(x,F), NaN\right)$

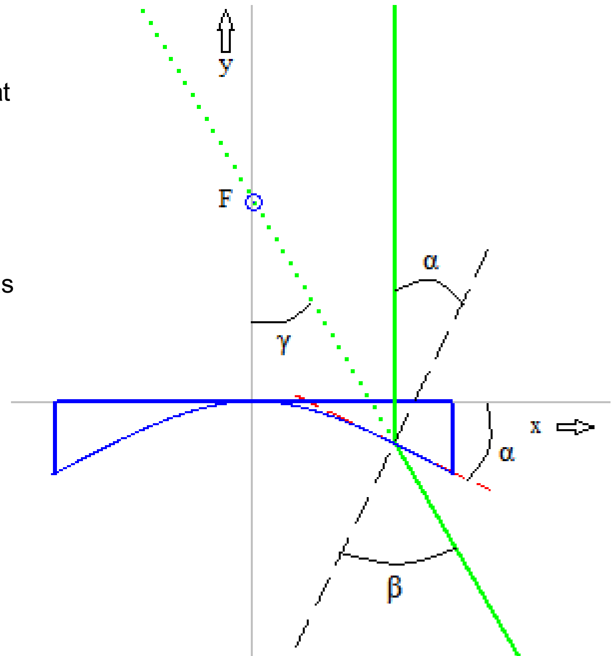
tangential: $l_{tan}(x) := \text{if}\left[|x - W| < \frac{r}{2}, (x - W) \cdot \tan\left(\alpha_M(W,F)\right) + y_M(W,F), NaN\right]$

perpendicular: $l_{per}(x) := \text{if}\left[|x - W| < \frac{r}{2}, (x - W) \cdot \tan\left(\frac{\pi}{2} + \alpha_M(W,F)\right) + y_M(W,F), NaN\right]$

$x := -z \cdot R, -z \cdot R + \frac{z \cdot R}{400} .. z \cdot R \qquad y_{top} := \frac{3}{2} \cdot z \cdot R \cdot (F > 0) + \frac{z}{2} \cdot R \cdot (F < 0)$



Plano-convex/concave lens



Consider an optical system of a circular 'disk' plano-convex/concave lens; that is, a lens that is flat on the side where light rays incide, and the other side, where light rays exit, is curved.
The 'top' of the curve lies at the origin (0,0) of the coordinate system, the flat surface of the lens is at y=c, where c is an arbitrary positive constant, or 0.
The lens is of material with a refractive index of n, its environment has refractive index 1.

Find the expression that defines the shape of the curve such that all rays that incide perpendicular to the flat surface, on exit via the curved surface are bent away from the central axis with respect to the focal point of the lens which is at (0,F).
If F is negative, rays are bent towards the y-axis.

Let the lens curve be defined by y(x)

The (green) ray falls straight through the plane side of the lens, and hits the curve y(x) at an angle α with the normal (black line) on the curve y(x). This angle α is the opposite angle from the angle between the tangential of curve y(x) with the x-axis. Note that α is a function of x, thus α(x)

That tangential is $\frac{d}{dx}y(x)$ and is $\tan(\alpha(x))$ thus $\frac{d}{dx}y(x) = \tan(\alpha(x))$

Snell's law describes that where a ray crosses a boundary from medium 1 to medium 2, the angle (to the normal of the boundary) $\alpha 1$ at which the ray hits the boundary, is related to the angle (to the normal) $\alpha 2$ at which the ray exits from the boundary by:

$\frac{\sin(\alpha 1)}{\sin(\alpha 2)} = \frac{n2}{n1}$ where n1 and n2 are the refractive indexes of the respective media.

From this law, and assuming the medium surrounding the lens has a refractive index of 1, β is derived as:

$\frac{\sin(\alpha(x))}{\sin(\beta(x))} = \frac{1}{n}$ solve, $\sin(\beta(x)) \rightarrow$ or $\sin(\beta(x)) = n \cdot \sin(\alpha(x))$

Finally observe that: $\tan(\gamma) = \frac{x}{y(x) - F}$ and together with $\gamma = \beta - \alpha$ gives: $\tan(\beta(x) - \alpha(x)) = \frac{x}{y(x) - F}$

Then the system of 3 equations for the plano-convex lens, with boundary conditions and 3 unknowns is:

$$EQ_{PC} := \begin{pmatrix} \frac{d}{dx}y(x) - \tan(\alpha(x)) \\ \sin(\beta(x)) - n \cdot \sin(\alpha(x)) \\ \tan(\beta(x) - \alpha(x)) - \frac{x}{y(x) - F} \end{pmatrix} \quad BC_{PC} := \begin{pmatrix} y(0) = 0 \\ \alpha(0) = 0 \\ \beta(0) = 0 \end{pmatrix} \quad UK_{PC} := \begin{pmatrix} y \\ \alpha \\ \beta \end{pmatrix} \quad UF_{PC} := \begin{pmatrix} y(x) \\ \alpha(x) \\ \beta(x) \end{pmatrix}$$

A symbolic solution is however not easily found: EQ_{PC} solve, $UF_{PC} \rightarrow$

Numerically it is possible, with: (r defines the radius of the lens disk R as a fraction of F)

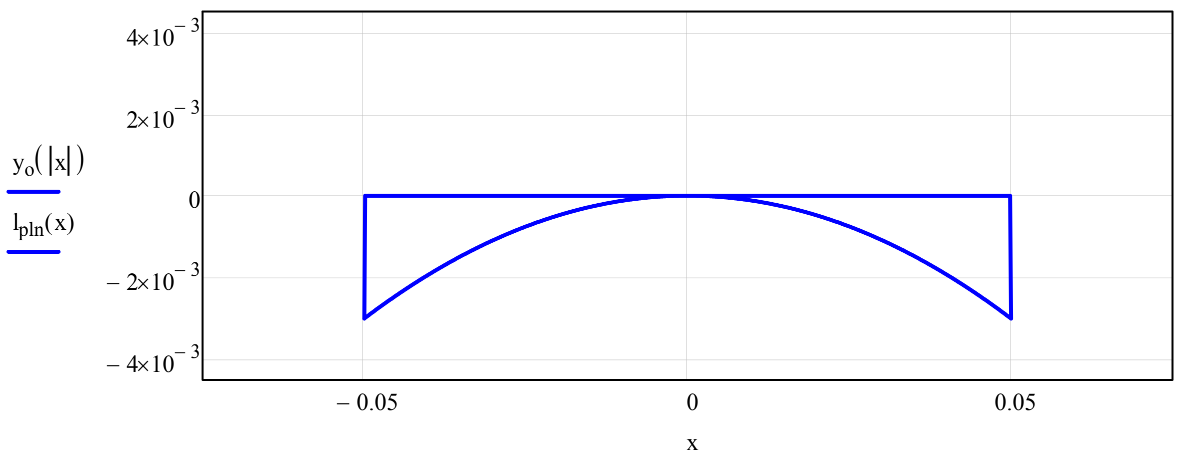
$n := \sqrt{2}$ $F := 1$ $r := \frac{1}{20}$ $R := r \cdot |F|$

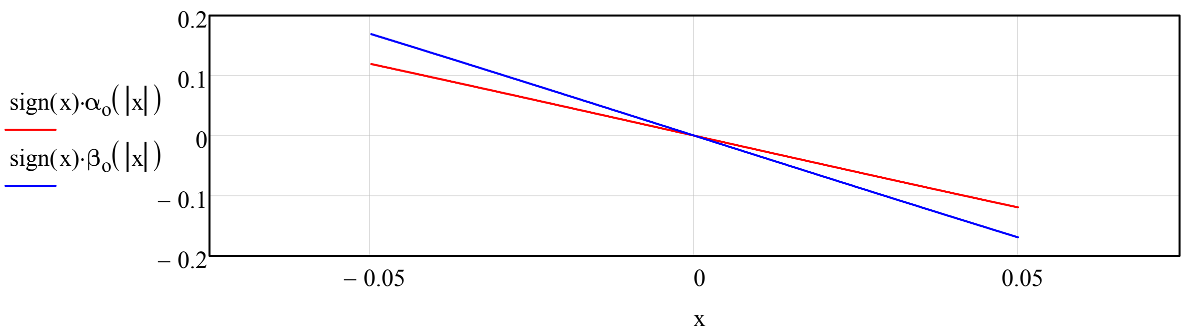
Given $y(0) = 0$ $\alpha(0) = 0$ $\beta(0) = 0$

$n \cdot \sin(\alpha(x)) - \sin(\beta(x)) = 0$ $\frac{d}{dx} y(x) - \tan(\alpha(x)) = 0$ $\tan(\beta(x) - \alpha(x)) - \frac{x}{y(x) - F} = 0$

$$\begin{pmatrix} y_o \\ \alpha_o \\ \beta_o \end{pmatrix} := \text{Odesolve}\left[\begin{pmatrix} y \\ \alpha \\ \beta \end{pmatrix}, x, R\right]$$

$z := 1.5$ $x := -R, -R + \frac{R}{360} .. R$ $l_{pln}(x) := \text{if}\left[|x| < R, \max\left(\begin{pmatrix} 0 & y_o(R) \end{pmatrix}\right), \text{if}\left(|x| = R, y_o(R), \text{NaN}\right)\right]$





At such scales α(|x|) and β(|x|) appear to be linear with x.
What if the radius of the lens R is larger than F ?

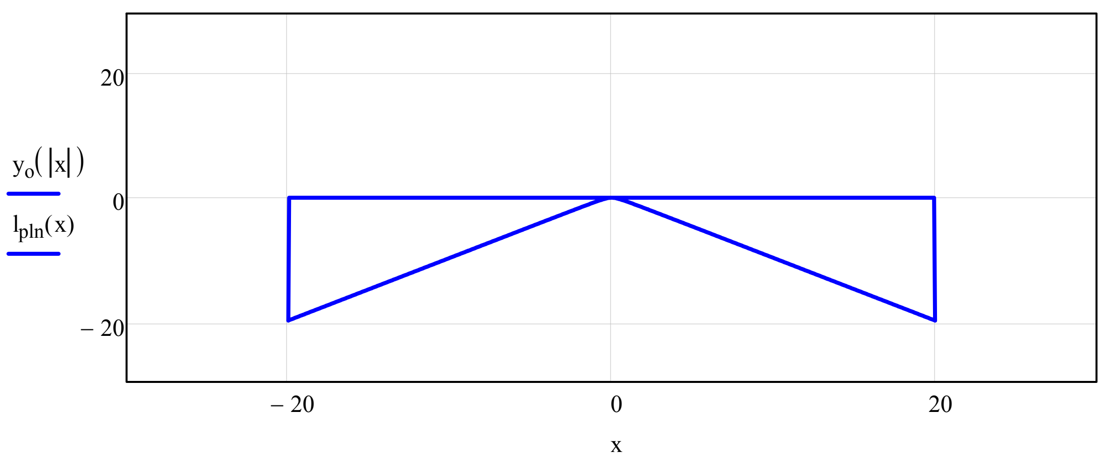
$n := \sqrt{2}$ $F := 1$ $r := 20$ $R := r \cdot |F|$

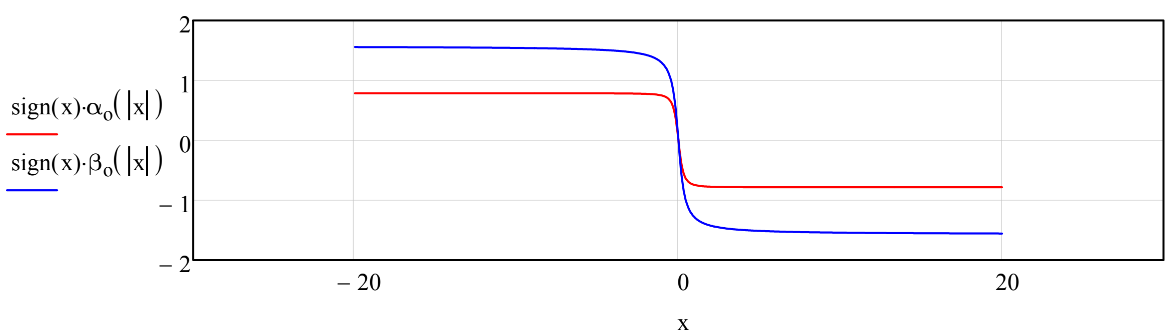
Given $y(0) = 0$ $\alpha(0) = 0$ $\beta(0) = 0$

$n \cdot \sin(\alpha(x)) - \sin(\beta(x)) = 0$ $\frac{d}{dx} y(x) - \tan(\alpha(x)) = 0$ $\tan(\beta(x) - \alpha(x)) - \frac{x}{y(x) - F} = 0$

$$\begin{pmatrix} y_o \\ \alpha_o \\ \beta_o \end{pmatrix} := \text{Odesolve}\left[\begin{pmatrix} y \\ \alpha \\ \beta \end{pmatrix}, x, R\right]$$

$x := -R, -R + \frac{R}{360} .. R$ $z := 1.5$ $l_{pln}(x) := \text{if}\left[|x| < R, \max\left(\begin{pmatrix} 0 & y_o(R) \end{pmatrix}\right), \text{if}\left(|x| = R, y_o(R), \text{NaN}\right)\right]$





Now observe that for large values of lens radius, y(x) goes almost linearly with |x|, α(x) resembles a tanh(x) and β(x) resembles an atan(x). In fact:

$\beta_o(R) = -0.495\,\pi$ almost $\pi/2$

Now suppose: $<\beta> := \mathrm{atan}(b \cdot x)$ $<\beta(x)> := \beta(x) = <\beta> \rightarrow \beta(x) = \mathrm{atan}(b \cdot x)$

then, with: $EQ_{PC_1} \rightarrow \sin(\beta(x)) - n \cdot \sin(\alpha(x))$

find α: $<\alpha> := EQ_{PC_1} \left| \begin{array}{l} \text{solve}, \alpha(x) \\ \text{substitute}, <\beta(x)> \end{array} \right. \rightarrow$

so: $<\alpha(x)> := \alpha(x) = <\alpha> \rightarrow$

now knowing trigonometry rule: $TR_{10} \rightarrow \tan(a - b) = \dfrac{\tan(a) - \tan(b)}{\tan(a) \cdot \tan(b) + 1}$

with the substitution: $<\tan(\alpha\text{-}\beta)> := TR_{10} \left| \begin{array}{l} \text{substitute}, a = \beta(x) \\ \text{substitute}, b = \alpha(x) \end{array} \right. \rightarrow -\tan(\alpha(x) - \beta(x)) = -\dfrac{\tan(\alpha(x)) - \tan(\beta(x))}{\tan(\alpha(x)) \cdot \tan(\beta(x)) + 1}$

and from: $EQ_{PC_2} \rightarrow \tan(\beta(x) - \alpha(x)) + \dfrac{x}{F - y(x)}$ it is possible to find y as:

$$<y> := EQ_{PC_2} \left| \begin{array}{l} \text{substitute}, <\tan(\alpha\text{-}\beta)> \\ \text{solve}, y(x) \\ \text{substitute}, <\beta(x)> \\ \text{simplify} \\ \text{collect}, \tan(\alpha(x)) \end{array} \right. \rightarrow$$

This gives the substitution: $<y(x)> := y(x) = <y>$

Finally check: $EQ_{PC_0} \rightarrow \left| \begin{array}{l} \text{substitute}, <y(x)> \\ \text{substitute}, <\alpha(x)> \\ \text{simplify} \end{array} \right. \rightarrow$

proves that the symbolic solution is found with y(x), α(x) and β(x).

Now find b with: $<b> := <y> \left| \begin{array}{l} \text{substitute}, <\alpha(x)> \\ \text{simplify} \\ \text{substitute}, x = 0 \\ \text{solve}, b \end{array} \right. \rightarrow$ $<b> \rightarrow$

which gives a substitution: $<b> := b = <b> \rightarrow$

And with that make the final definitions of y(x), α(x) and β(x):

$$y_0(x,F,n,b) := \langle y\rangle \;\Big|\; \begin{array}{l} \text{substitute}, \langle\alpha(x)\rangle \\ \text{simplify} \end{array} \rightarrow$$

This can be reworked to

$$y_1(x,F,n,b) := \frac{-F + b\cdot x^2 + \left[n^2 + b^2\cdot x^2\cdot\left(n^2-1\right)\right]^{\frac{1}{2}}\cdot\left(\frac{1}{b}+F\right)}{\left[\left[n^2 + b^2\cdot x^2\cdot\left(n^2-1\right)\right]^{\frac{1}{2}} - 1\right]}$$

$$y_2(x,F,n) := y_1(x,F,n,b)\ \text{substitute}, \langle b\,\rangle \rightarrow$$

which can be reworked to:

$$y\_PC(x,F,n) := F \cdot \frac{\left(\frac{x}{F}\right)^2 \cdot \frac{n}{(1-n)} + \sqrt{1 + \frac{(n+1)}{(n-1)} \cdot \left(\frac{x}{F}\right)^2} - 1}{n \cdot \sqrt{1 + \frac{(n+1)}{(n-1)} \cdot \left(\frac{x}{F}\right)^2} - 1}$$

$$y_{PC}(x,F,n) := \begin{array}{|l} \text{return } 0 \text{ if } F = 0 \\ y\_PC(x,F,n) \end{array}$$

Now we can build $\alpha(x)$:

$$\alpha_0(x,F,n) := \langle\alpha\rangle \begin{array}{|l} \text{substitute}, \langle a \rangle \\ \text{substitute}, \langle b \rangle \end{array} \rightarrow$$

$$\alpha\_PC(x,F,n) := \text{asin}\left[\frac{x}{F \cdot (1-n)} \cdot \frac{1}{\sqrt{1 + \frac{n^2}{(n-1)^2} \cdot \left(\frac{x}{F}\right)^2}}\right]$$

$$\alpha_{PC}(x,F,n) := \begin{array}{|l} \text{return } 0 \text{ if } F = 0 \\ \alpha\_PC(x,F,n) \end{array}$$

and $\beta(x)$:

$$\beta_0(x,F,n) := \langle\beta\rangle \text{ substitute}, \langle b \rangle \rightarrow$$

$$\beta\_PC(x,F,n) := \text{atan}\left(\frac{n}{1-n} \cdot \frac{x}{F}\right)$$

$$\beta_{PC}(x,F,n) := \begin{array}{|l} \text{return } 0 \text{ if } F = 0 \\ \beta\_PC(x,F,n) \end{array}$$

Let's find out which is the better approximation for y: a sphere or a parabola?

$n := \sqrt{2}$ $\quad F := -1$ $\quad r := 0.01$ $\quad R := r \cdot |F|$ $\quad TOL := 10^{-10}$ $\quad \delta := 1 \cdot 10^{-9}$

$x := -R, -R + \frac{R}{400} .. R$ $\quad r := 0.005 \cdot |F|$ $\quad s := 0.1$ $\quad p := 0.1$ $\quad YR := y_{PC}(R, F, n) \cdot (F < 0)$

Sphere: $\quad sy(x, s) := s \cdot \left[1 - \sqrt{1 - \left(\frac{x}{s}\right)^2}\right]$

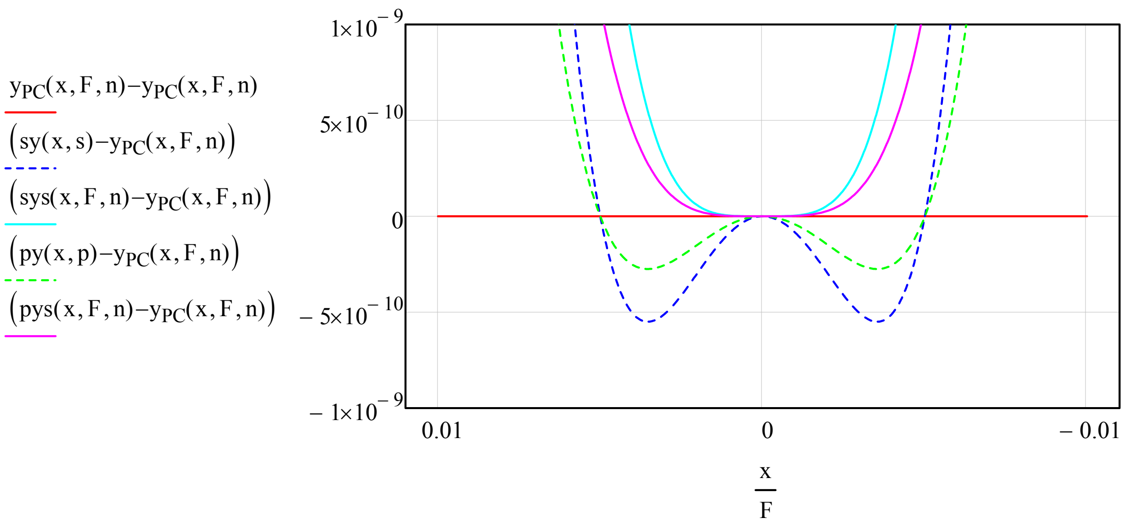
$s := root(sy(r, s) - y_{PC}(r, F, n), s)$ $\quad s = 0.414244$

approximate s with: $\quad (1 - n) \cdot F = 0.414$ $\quad sys(x, F, n) := (1 - n) \cdot F \cdot \left[1 - \sqrt{1 - \left[\frac{x}{(1 - n) \cdot F}\right]^2}\right]$

Parabola: $\quad py(x, p) := p \cdot x^2$

$p := root(py(r, p) - y_{PC}(r, F, n), p)$ $\quad p = 1.207063$

approximate p with: $\quad \frac{1}{2 \cdot (1 - n) \cdot F} = 1.207$ $\quad pys(x, F, n) := \frac{x^2}{2 \cdot (1 - n) \cdot F}$ $\quad z := 1.1$



The parabola wins, which should come as no surprise since, for a completely reflective lens, where n = -1 :

$y\_PC(x, F, n)$ substitute, $n = -1$ $\rightarrow \frac{x^2}{4 \cdot F}$ $\quad$ which is the same as for the mirror. $y\_M(x, F) \rightarrow$

And β = $-\alpha$:

$\beta\_PC(x, F, n)$ | substitute, $n = -1$ simplify $\rightarrow -\text{atan}\left(\frac{\frac{1}{2}}{F} \cdot x\right)$ $\quad -\alpha\_M(x, F)$ simplify $\rightarrow$

which was used in setting up the equations for the mirror.

Draw the trajectory of a ray that falls on the lens at a distance W from the centre, where w is a fraction of R .

$F := 1 \quad n := \sqrt{2} \quad X := 10 \quad z := 2$

$i := ORIGIN, ORIGIN + 1 .. ORIGIN + 2 \quad r := \frac{1}{2} \quad R := r \cdot |F| + X \cdot (F = 0) \quad w := \sqrt{\frac{1}{2}} \quad W := w \cdot R$

$d(W,F,n,X) := \tan\left[\frac{\pi}{2} - \left(\beta_{PC}(W,F,n) - \alpha_{PC}(W,F,n)\right)\right] \cdot (X - 1) \cdot W + y_{PC}(W,F,n)$

$ray_d(w) := \left[\infty \quad y_{PC}(w,F,n) \quad F \cdot (F < 0) + d(W,F,n,X) \cdot (F > 0)\right]^T \quad ray_w(w) := [w \quad w \quad XW \cdot (F > 0)]^T$

$ray_e(w) := \left[XF - (X - 1) \cdot y_{PC}(w,F,n) \quad y_{PC}(w,F,n)\right]^T \quad ray_x(w) := [-(X - 1) \cdot w \quad w]^T$

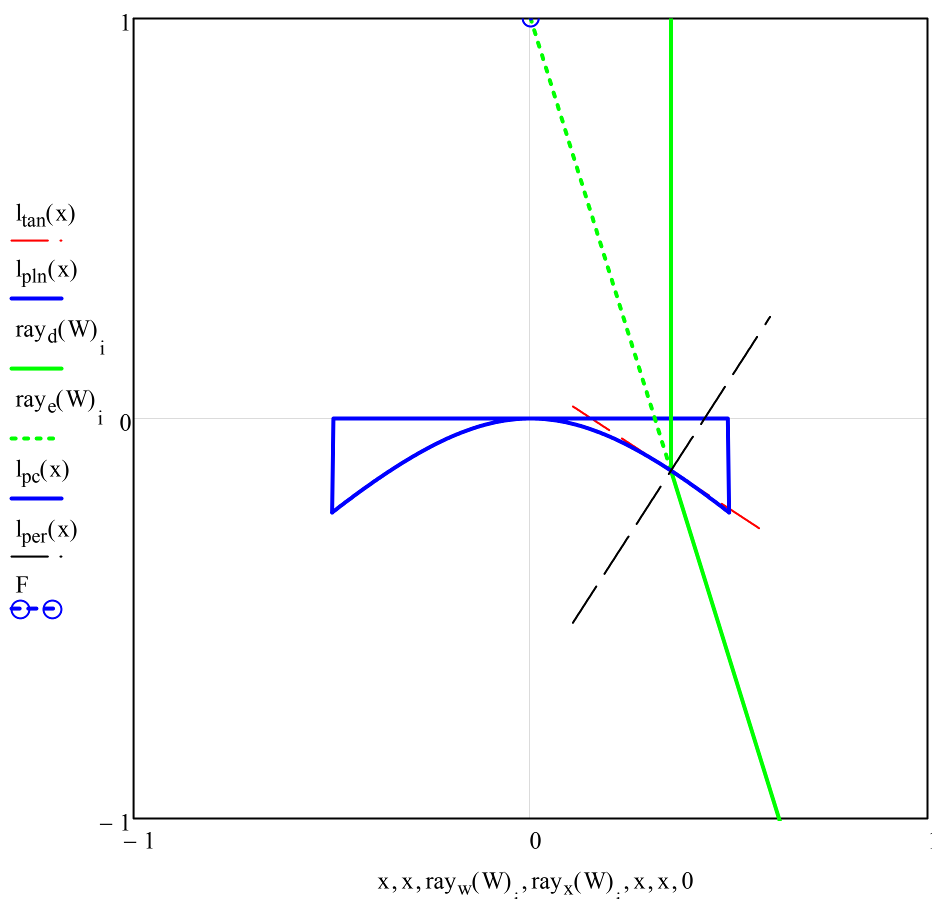
lens curve: $l_{pc}(x) := \text{if}\left(|x| \le R, y_{PC}(x,F,n), NaN\right)$

plano flat: $l_{pln}(x) := \text{if}\left[|x| < R, \max\left(\left(0 \quad y_{PC}(R,F,n)\right)\right), \text{if}\left(|x| = R, y_{PC}(R,F,n), NaN\right)\right]$

tangential: $l_{tan}(x) := \text{if}\left[|x - W| < \frac{r}{2}, (x - W) \cdot \tan\left(\alpha_{PC}(W,F,n)\right) + y_{PC}(W,F,n), NaN\right]$

perpendicular: $l_{per}(x) := \text{if}\left[|x - W| < \frac{r}{2}, (x - W) \cdot \tan\left(\frac{\pi}{2} + \alpha_{PC}(W,F,n)\right) + y_{PC}(W,F,n), NaN\right]$

$x := -z \cdot R, -z \cdot R + \frac{z \cdot R}{300} .. z \cdot R \quad y_{top} := \frac{z \cdot 2}{2} \cdot R \quad y_{bot} := \frac{-2}{2} \cdot z \cdot R$



Trigonometry Rules:

$$TR := \begin{pmatrix} \sin(a+b) = \sin(a)\cdot\cos(b)+\cos(a)\cdot\sin(b) \\ \sin(2\cdot a) = 2\cdot\sin(a)\cdot\cos(a) \\ \sin(a-b) = \sin(a)\cdot\cos(b)-\cos(a)\cdot\sin(b) \\ \sin(a-a) = 0 \\ \cos(a+b) = \cos(a)\cdot\cos(b)-\sin(a)\cdot\sin(b) \\ \cos(2\cdot a) = \cos(a)^2-\sin(a)^2 \\ \cos(a-b) = \cos(a)\cdot\cos(b)+\sin(a)\cdot\sin(b) \\ \cos(a-a) = 1 = \cos(a)^2+\sin(a)^2 \\ \tan(a+b) = \dfrac{\tan(a)+\tan(b)}{1-\tan(a)\cdot\tan(b)} \\ \tan(2a) = \dfrac{2\cdot\tan(a)}{1-\tan(a)^2} \\ \tan(a-b) = \dfrac{\tan(a)-\tan(b)}{1+\tan(a)\cdot\tan(b)} \\ \tan(a-a) = 0 \end{pmatrix}$$

$$HY := \begin{pmatrix} \sinh(x) = \dfrac{e^x-e^{-x}}{2} \\ \cosh(x) = \dfrac{e^x+e^{-x}}{2} \\ \tanh(x) = \dfrac{e^x-e^{-x}}{e^x+e^{-x}} \end{pmatrix}$$

$F := 1$ $\quad F := F$

$x := 0$ $\quad x := x$

$y := 0$ $\quad y(x) := 0$ $\quad y(x) := y(x)$ $\quad y := y$

$\alpha := 0$ $\quad \alpha(x) := 0$ $\quad \alpha(x) := \alpha(x)$ $\quad \alpha := \alpha$

$a := 1$ $a := a$

$F := F$

$R := R$

$x := 0$ $x := x$

$y := 0$ $y(x) := 0$ $y(x) := y(x)$ $y := y$

$\alpha := 0$ $\alpha(x) := 0$ $\alpha(x) := \alpha(x)$ $\alpha := \alpha$

$r := r$ $R := R$ $w := w$ $W := W$

$$y_{bot} := \frac{-3}{2} \cdot z \cdot R \cdot (F < 0) + \frac{-z}{2} \cdot R \cdot (F > 0)$$

$F := -1$			$F := F$
$n := 1$			$n := n$
$x := 0$			$x := x$
$y := 0$	$y(x) := 0$	$y(x) := y(x)$	$y := y$
$\alpha := 0$	$\alpha(x) := 0$	$\alpha(x) := \alpha(x)$	$\alpha := \alpha$
$\beta := 0$	$\beta(x) := 0$	$\beta(x) := \beta(x)$	$\beta := \beta$

$b := 1$ $\quad b := b$

$F := -1$ $\quad F := F$

$n := \sqrt{2}$ $\quad n := n$

$x := 0$ $\quad x := x$

$y := 0 \quad y(x) := 0 \quad y(x) := y(x) \quad y := y$

$\alpha := 0 \quad \alpha(x) := 0 \quad \alpha(x) := \alpha(x) \quad \alpha := \alpha$

$\beta := 0 \quad \beta(x) := 0 \quad \beta(x) := \beta(x) \quad \beta := \beta$

$y\_M(x,F) \rightarrow$

$$a := \lim_{x \to \infty} \frac{y\_PC(x,F,n)}{x} \rightarrow -\frac{1}{F \cdot (n-1) \cdot \sqrt{\frac{n+1}{F^2 \cdot (n-1)}}}$$

$$b := \lim_{x \to \infty} (y\_PC(x,F,n) - a \cdot x) \rightarrow \frac{F}{n+1}$$

$$as(x,F,n) := \frac{-1}{F \cdot (n-1) \cdot \sqrt{\frac{n+1}{(n-1) \cdot F^2}}} \cdot \sqrt{x^2} + \frac{F}{n+1}$$

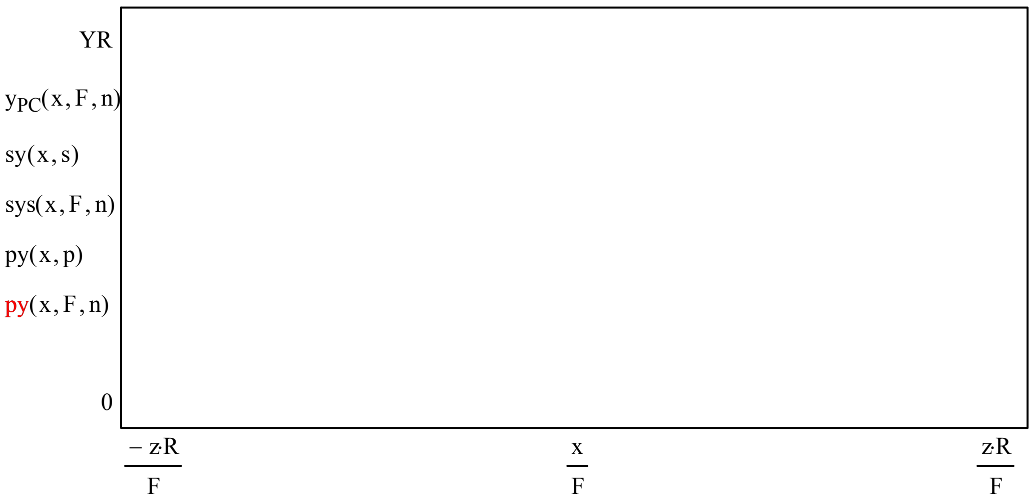
$\alpha_{PC}(x,F,n) = 0$

$\beta_{PC}(x,F,n) = 0$

$n := n \quad F := F \quad R := R$

$x := x \quad r := r$

$s := s \quad p := p$



$$\alpha\_PC(x,F,n) \text{ substitute}, n = -1 \rightarrow \operatorname{asin}\left(\frac{x}{F\cdot\sqrt{\frac{4\cdot F^2+x^2}{F^2}}}\right)$$

$r := r \quad R := R \quad w := w \quad W := W$