

# Theory of modulated LII (MLII) version 2

## Reference sheets

a necessary reference worksheet which contains data extraction functions. This version has been modified so that a value for Cp.soot is returned for any temperature

- Reference:D:\Mathcad-collab\TMP\Useful\_2001.xmcdz(R)
- Reference:D:\Mathcad-collab\TMP\constant3\_DRs.xmcdz
- Reference:D:\Mathcad-collab\TMP\Useful\_2001.xmcdz(R)

Units & Fundtl. constants

## Problem

We have a laser source modulated light source at a frequency  $f$  that is inducing periodic changes in soot temperature. In general both the light source and the resultant signal can have a DC component as well as the AC term at frequency  $f$ . In the derivation below I will not consider any DC component to the driving signal since our laser is purely AC. In general one can write the source term as:  $I_{mod} = Cnst \cdot (1 + Mod_I \cdot \sin(\omega \cdot t))$  where, in our case  $Mod_I=1$

$$I_{mod} = Cnst \cdot (1 + Mod_I \cdot \sin(\omega \cdot t)) \quad \text{For modulation at angular frequency } \omega=2\pi f$$

For definition of modulation depth see figure below

See "R:\Optical Diagnostics\Spencer\_AnnNYAcaSci158\_69.pdf"

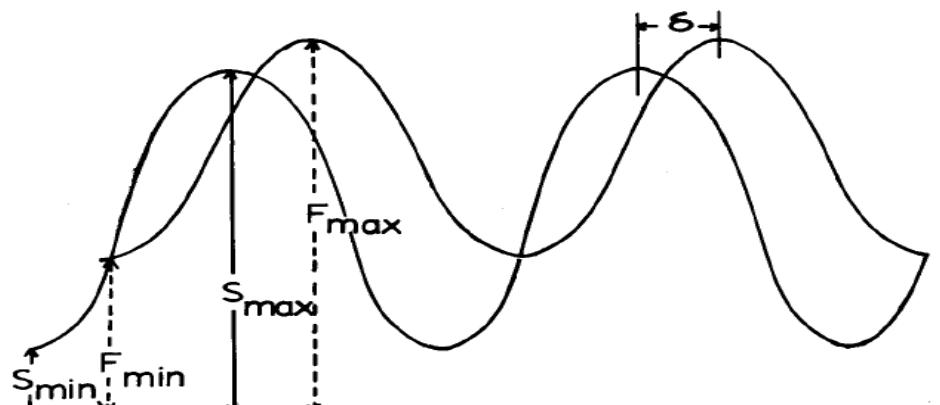
364

Annals New York Academy of Sciences

### ALTERNATE METHODS OF MEASUREMENT OF $I$

#### 1. PHASE DIFFERENCE WITH SCATTERER.

#### 2. RELATIVE MODULATION.



MODULATION FREQUENCY =  $f$

$\delta$  = PHASE DIFFERENCE;  $\tan \delta = 2\pi f \tau$

$$D_F = F_{\max} - F_{\min} / F_{\max} + F_{\min}$$

$$D_S = S_{\max} - S_{\min} / S_{\max} + S_{\min}$$

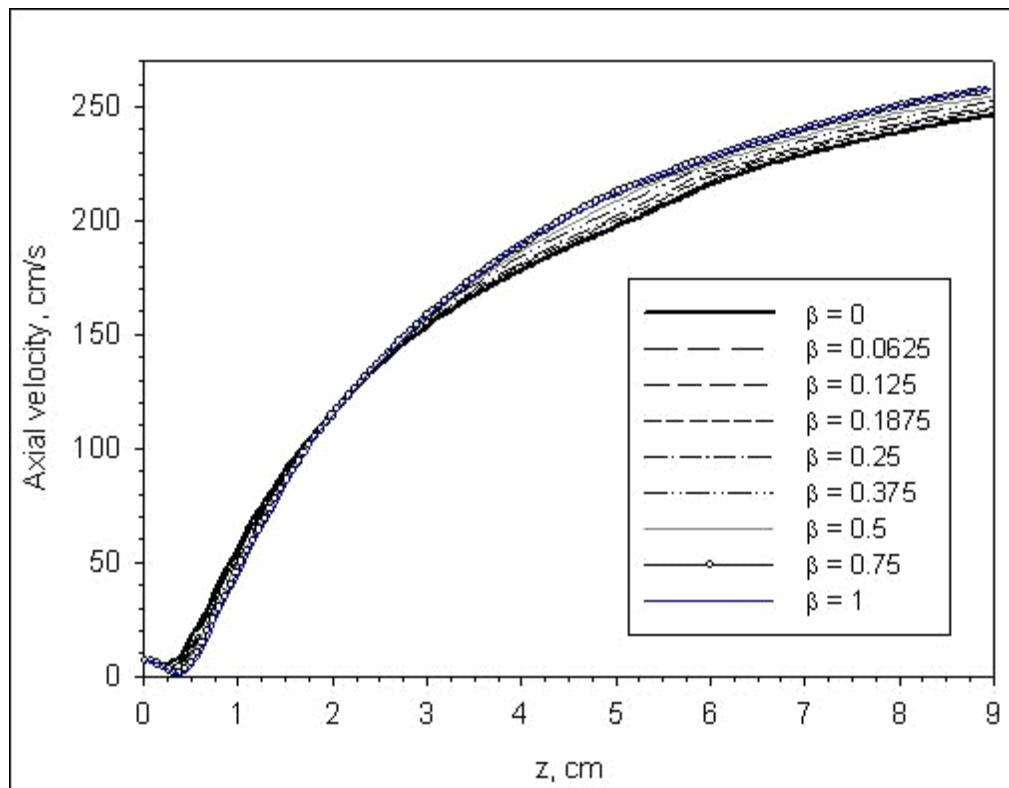
D = RELATIVE MODULATION =

$$D_F / D_S = 1 / \sqrt{1 + 4\pi^2 f^2 \tau^2} = \cos \delta$$

FIGURE 2. Schematic diagram showing the phase lag and demodulation of the photocurrents due to scatter,  $S$ , and a fluorescent solution,  $F$ .

In the file "A:\Analysis\Modulated L of modulated LII (MLLI). It became approach here

Source gas velocity



This is Fengshan's model data for centre line gas velocity with height above burner z

Axial velocity increases from ~ 25 to - 200 cm.sec

Therefore the residence time for an excitation region of approximately 1 mm is

$$\frac{1 \cdot \text{mm}}{50 \cdot \frac{\text{cm}}{\text{sec}}} = 2 \cdot \text{ms} \quad \text{to} \quad \frac{1 \cdot \text{mm}}{200 \cdot \frac{\text{cm}}{\text{sec}}} = 0.5 \cdot \text{ms}$$

For soot volume fraction of 2.5 PPM

$$\text{svf} := 2.5 \cdot 10^{-6}$$

At 42 mm height

$$\frac{1 \cdot \text{mm}}{175 \cdot \frac{\text{cm}}{\text{sec}}} = 0.571 \cdot \text{ms}$$

what is diffusion distance in this time for air at 1730 K?

$$D_{\text{air}} := 3.6 \cdot 10^{-4} \cdot \frac{\text{m}^2}{\text{sec}} \quad x_{\text{dif}} = (2 \cdot D \cdot t)^{0.5}$$

$$\left( 2 \cdot D_{\text{air}} \cdot \frac{1 \cdot \text{mm}}{175 \cdot \frac{\text{cm}}{\text{sec}}} \right)^{0.5} = 0.641427 \cdot \text{mm}$$

$$\left( 2 \cdot D_{\text{air}} \cdot \frac{1 \cdot \text{mm}}{25 \cdot \frac{\text{cm}}{\text{sec}}} \right)^{0.5} = 1.697056 \cdot \text{mm}$$

Distance diffused in time gas flow moves 1 mm

$$\left( 2 \cdot D_{\text{air}} \cdot \frac{1 \cdot \text{mm}}{175 \cdot \frac{\text{cm}}{\text{sec}}} \right)^{0.5} = 0.641427 \cdot \text{mm}$$

Note: that this was previously 1\*sec/25; the units of mm and cm were dropped and the answer was consequently off (above 5 mm). I don't believe this influences any calculations below since you use the 42 mm condition, which was done correctly.

## Soot and gas heat capacity and modulated CW laser heating

### Heat capacity of soot

Heat capacity soot  $f_{\text{cp}}(3000) = 2274.801287 \frac{1}{\text{kg} \cdot \text{K}} \cdot \text{J}$

volumetric heat capacity is:

$$f_{\text{cp}}(3000) \cdot \text{RHOS} = 4.310748 \times 10^6 \frac{1}{\text{m}^3 \cdot \text{K}} \cdot \text{J}$$

$$\text{HC}_{\text{soot}}(T, \text{svf}) := \text{RHOS} \cdot \text{svf} \cdot f_{\text{cp}}\left(\frac{T}{K}\right)$$

heat capacity of soot per unit volume of soot/air mix

### Heat capacity of air

$$C_{\text{p,air}}(293) = \text{■} \cdot \text{J}$$

$$C_{\text{v,air}}(1700) = \text{■} \cdot \text{J}$$

$$C_{\text{v,air}}(293) = \text{■} \cdot \text{J}$$

$$C_{\text{p,air}}(293) = \text{■} \cdot \text{J}$$

c. f. handbook of Chemistry and Physics to "our value

$$C_{\text{p,air,300}} := .2401 \cdot \frac{\text{cal}}{\text{gm} \cdot \text{K}} = 1005.25068 \frac{1}{\text{kg} \cdot \text{K}} \cdot \text{J}$$

$$C_{\text{p,air}}(300) = \text{■} \cdot \text{J}$$

very close

the gas equation gives

$$P \cdot V = n \cdot R \cdot T$$

where n is in moles

$$\frac{n}{V} = \frac{P}{R \cdot T}$$

So density of air  $\rho_{\text{air}}(T)$  is

$$\rho_{\text{air}}(T) := \frac{1 \cdot \text{atm}}{T \cdot \text{K} \cdot \text{R}} \cdot \left( 28.96 \cdot \frac{\text{gm}}{\text{mole}} \right)$$

$$\text{From handbook } \rho_{air293} := .001205 \cdot \frac{\text{kg}}{\text{L}} = 1.205 \frac{\text{kg}}{\text{m}^3} \quad \text{compare} \quad \rho_{air(293)} = 1.20444 \frac{\text{kg}}{\text{m}^3}$$

Therefore the volumetric heat capacity of air at 1 atm. and temperature T is

$$HC_{air}(T) := C_{p,air} \left( \frac{T}{K} \right) \cdot \frac{1 \cdot \text{atm}}{T \cdot R} \cdot \left( 28.96 \cdot \frac{\text{gm}}{\text{mole}} \right)$$

ignoring any dependence of Mol. Weight of air with T

$$HC_{air}(1730 \cdot K) = \text{■} \cdot J$$

Define ratio

$$fr(T, svf) := \frac{HC_{soot}(T, svf)}{HC_{air}(T)}$$

$$fr(1730 \cdot K, 2.5 \cdot 10^{-6}) = \text{■}$$

$$\text{Note } 1 + fr(T, svf) = \frac{HC_{soot}(T, svf) + HC_{air}(T)}{HC_{air}(T)}$$

$$\frac{1}{fr(T, svf)} + 1 = \frac{HC_{air}(T) + HC_{soot}(T, svf)}{HC_{soot}(T, svf)} = \frac{1 + fr(T, svf)}{fr(T, svf)}$$

## Laser heating of soot

If  $P_d$  is the unmodulated laser power density.e.g. W/mm<sup>2</sup>

The formula for the absorption of radiation a single primary particle is the product of the absorption cross-section and the laser power density. Where the absorption

cross-section is a product of the absorption efficiency  $\frac{4 \cdot \pi \cdot dp \cdot Em}{\lambda_{laser}}$  and the cross physical cross section  $\frac{\pi \cdot dp^2}{4}$

$$\text{Power}_{abs} = P_d \cdot \frac{\pi^2 \cdot dp^3 \cdot Em}{\lambda_{laser}}$$

for a single primary particle

$$N_{pp} = \frac{svf}{\left( \frac{\pi \cdot dp^3}{6} \right)}$$

The heating (power absorbed) per unit volume of soot laden gas will be

$$P_{vol} = P_d \cdot \frac{\pi^2 \cdot dp^3 \cdot Em}{\lambda_{laser}} \cdot \frac{svf}{\left( \frac{\pi \cdot dp^3}{6} \right)} = P_d \cdot \frac{6 \cdot \pi \cdot Em}{\lambda_{laser}} \cdot svf$$

The heat capacity of the soot per unit volume of soot laden gas is:

$$HC_{soot}(T, svf) = RHOS \cdot svf \cdot fCp \left( \frac{T}{K} \right)$$

The rate of soot temperature increase by the laser will then be

$$\frac{dT}{dt} = P_d \cdot \frac{\pi^2 \cdot dp^3 \cdot Em}{\lambda_{laser}} \cdot \frac{svf}{\left( fCp(T) \cdot \frac{\pi \cdot dp^3}{6} \cdot RHOS \cdot svf \right)} = \frac{6 \cdot \pi \cdot P_d \cdot Em}{\lambda_{laser} \cdot fCp(T) \cdot RHOS}$$

RHS Is the rate of soot heating i.e. the rate of absorption of energy divided by the soot particle heat capacity

So the power absorbed per unit volume of soot laden gas is

$$P_{volabs} = Powe$$

The heat capacity of the soot/unit volume of gas is

$$N_{pp} \cdot \frac{\pi \cdot dp^3}{6} \cdot fc$$

define

$$\text{Lasheat}(P_d, T) := \frac{6 \cdot P_d \cdot \pi \cdot E_m}{\lambda_{\text{laser}} \cdot fC_p\left(\frac{T}{K}\right) \cdot RHOS}$$

$$P_d = 75000 \frac{\text{kg}}{\text{s}^3}$$

$$\text{Lasheat}(P_d, 1730 \cdot K) = 1.824556 \times 10^5 \frac{\text{K}}{\text{s}}$$

What is  $\text{Lasheat}(P_d)$  for our experimental conditions?

What value for  $P_d$ ?

$$T_{0\text{gas}} := 1730 \cdot K$$

$$E_m := 0.4$$

$$\lambda_{\text{laser}} := 805 \cdot \text{nm}$$

$$svf := 2.5 \cdot 10^{-6}$$

805 nm laser has a maximum CW output of 170 mW. If we modulate it with a sine wave then the maximum peak to peak of the fundamental of the sine wave should be  $\sim 150$  mW assuming we don't lose too much in harmonics. The laser cross section is  $\sim 1 \text{ mm}^2$  so  $P_d$  is  $\sim 150 \text{ mW mm}^{-2}$  maximum peak to peak and the average value will be  $75 \text{ mW mm}^{-2}$

$$P_d := 75 \cdot 10^{-3} \cdot \frac{W}{\text{mm}^2}$$

So  $\text{Lasheat}(P_d)$  is

$$\text{Lasheat} := \frac{6 \cdot P_d \cdot \pi \cdot E_m}{\lambda_{\text{laser}} \cdot fC_p(1730) \cdot RHOS} = 1.824556 \times 10^5 \frac{\text{K}}{\text{s}}$$

$$P_d \cdot \frac{6 \cdot \pi \cdot E_m}{\lambda_{\text{laser}}} \cdot svf = 1.75617 \times 10^6 \frac{1}{\text{m}^3 \cdot \text{s}} \cdot J$$

$$HC_{\text{soot}}(T, svf) := RHOS \cdot svf \cdot fC_p\left(\frac{T}{K}\right)$$

$$HC_{\text{soot}}(1730 \cdot K, 2.5 \cdot 10^{-6}) = 9.625188 \frac{1}{\text{m}^3 \cdot \text{K}} \cdot J$$

$$fC_p(1730) = 2031.702048 \frac{1}{\text{kg} \cdot \text{K}} \cdot J$$

Consider soot and gas at some undisturbed temperature  $T_{0\text{gas}}$  and neglecting gas flow then the differential equations describing temperature change with laser illumination follow

To make this problem tractable I will make the following assumptions. The undisturbed soot temperature is  $T_{0\text{gas}}$  ad the excitation regions at a uniform temperature  $T_{\text{soot}}$  and  $T_{\text{gas}}$  where the latter area function of time

Neglecting gas flow the differential equations describing soot heating are

The differential equation describing gas and soot heating are

$$\frac{d}{dt} T_{\text{gas}} = fr(T, svf) \cdot \frac{(T_{\text{soot}} - T_{\text{gas}})}{\tau_{\text{cool}}} \quad fr(T, svf) = \frac{HC_{\text{soot}}(T, svf)}{HC_{\text{air}}(T)}$$

$$\frac{d}{dt} T_{\text{soot}} = \frac{6 \cdot \pi \cdot E_m \cdot P_d \cdot (1 + \sin(\omega \cdot t))}{RHOS \cdot \lambda_{\text{laser}} \cdot fC_p(T)} - \frac{(T_{\text{soot}} - T_{\text{gas}})}{\tau_{\text{cool}}}$$

$$\frac{d}{dt} (T_{\text{soot}} - T_{\text{gas}}) = \frac{d}{dt} T_{\text{soot}} - \frac{d}{dt} T_{\text{gas}} = \frac{6 \cdot \pi \cdot E_m \cdot P_d \cdot svf \cdot (1 + \sin(\omega \cdot t))}{RHOS \cdot \lambda_{\text{laser}} \cdot fC_p(T)} - \frac{(T_{\text{soot}} - T_{\text{gas}})}{\tau_{\text{cool}}} - fr(T, svf) \cdot \frac{(T_{\text{soot}} - T_{\text{gas}})}{\tau_{\text{cool}}}$$

$$\text{Let } \text{Lasheat} = \frac{6 \cdot P_d \cdot \pi \cdot E_m \cdot svf}{\lambda_{\text{laser}} \cdot fC_p(T) \cdot RHOS}$$

$$\frac{d}{dt}(T_{soot} - T_{gas}) = \text{Lasheat} \cdot (\sin(\omega \cdot t) + 1) - \frac{(T_{soot} - T_{gas})}{\tau_{cool}} \cdot (1 + fr(T, svf))$$

What is solution of the  $\frac{d}{dt}(T_{soot} - T_{gas})$  equation?

Multiply left hand side by integrating factor  $\exp\left[\frac{t \cdot (1 + fr(T, svf))}{\tau_{cool}}\right]$

$$svf := svf$$

$$\exp\left[\frac{t \cdot (1 + fr(T, svf))}{\tau_{cool}}\right] \cdot \left[ \frac{d}{dt}(T_{soot} - T_{gas}) + \frac{(T_{soot} - T_{gas}) \cdot (1 + fr(T, svf))}{\tau_{cool}} \right] = \frac{d}{dt} \left[ \exp\left[\frac{t \cdot (1 + fr(T, svf))}{\tau_{cool}}\right] \cdot (T_{soot} - T_{gas}) \right] = \exp\left[\frac{t \cdot (1 + fr(T, svf))}{\tau_{cool}}\right] \cdot [\text{Lasheat}(P_d) \cdot (\sin(\omega \cdot t) + 1)]$$

Integrating we get  $\text{Lasheat} := \text{Lasheat}$   $fr := fr$   $\tau_{cool} := \tau_{cool}$

$$\int \exp\left[\frac{t \cdot (1 + fr)}{\tau_{cool}}\right] \cdot [\text{Lasheat} \cdot (\sin(\omega \cdot t) + 1)] dt \rightarrow \frac{\text{Lasheat} \cdot \tau_{cool} \cdot e^{\frac{t}{\tau_{cool}} + \frac{fr \cdot t}{\tau_{cool}}} \cdot (2 \cdot fr + \sin(\omega \cdot t) + fr^2 \cdot \sin(\omega \cdot t) + fr^2 + 2 \cdot fr \cdot \sin(\omega \cdot t) + \omega^2 \cdot \tau_{cool}^2 - \omega \cdot \tau_{cool} \cdot \cos(\omega \cdot t) - \omega \cdot fr \cdot \tau_{cool} \cdot \cos(\omega \cdot t) + 1)}{(fr + 1) \cdot (\omega^2 \cdot \tau_{cool}^2 + fr^2 + 2 \cdot fr + 1)}$$

I get error pattern match exception for this evaluation

It calculated in a blank worksheet to give:

$$\int \exp\left[\frac{t \cdot (1 + fr)}{\tau_{cool}}\right] \cdot [\text{Lasheat} \cdot (\sin(\omega \cdot t) + 1)] dt \rightarrow \frac{\text{Lasheat} \cdot \tau_{cool} \cdot e^{\frac{t}{\tau_{cool}} + \frac{fr \cdot t}{\tau_{cool}}} \cdot (2 \cdot fr + \sin(\omega \cdot t) + fr^2 \cdot \sin(\omega \cdot t) + fr^2 + 2 \cdot fr \cdot \sin(\omega \cdot t) + \omega^2 \cdot \tau_{cool}^2 - \omega \cdot \tau_{cool} \cdot \cos(\omega \cdot t) - \omega \cdot fr \cdot \tau_{cool} \cdot \cos(\omega \cdot t) + 1)}{(fr + 1) \cdot (\omega^2 \cdot \tau_{cool}^2 + fr^2 + 2 \cdot fr + 1)}$$

Therefore

$$T_{soot} - T_{gas} = \frac{\text{Lasheat} \cdot \tau_{cool} \cdot (2 \cdot fr(T, svf) + \sin(\omega \cdot t) + fr(T, svf)^2 \cdot \sin(\omega \cdot t) + fr(T, svf)^2 + 2 \cdot fr(T, svf) \cdot \sin(\omega \cdot t) + \omega^2 \cdot \tau_{cool}^2 - \omega \cdot \tau_{cool} \cdot \cos(\omega \cdot t) - \omega \cdot fr(T, svf) \cdot \tau_{cool} \cdot \cos(\omega \cdot t) + 1)}{(fr(T, svf) + 1) \cdot (\omega^2 \cdot \tau_{cool}^2 + fr(T, svf)^2 + 2 \cdot fr(T, svf) + 1)}$$

$$T_{soot} - T_{gas} = \frac{\text{Lasheat} \cdot \tau_{cool} \cdot \left[ (fr(T, svf) + 1)^2 + \sin(\omega \cdot t) \left[ (fr(T, svf) + 1)^2 + \omega^2 \cdot \tau_{cool}^2 - \omega \cdot \tau_{cool} \cdot \cos(\omega \cdot t) \cdot (1 + fr(T, svf)) \right] \right]}{(fr(T, svf) + 1) \cdot \left[ \omega^2 \cdot \tau_{cool}^2 + (fr(T, svf) + 1)^2 \right]}$$

$$T_{soot} - T_{gas} = \frac{\text{Lasheat} \cdot \tau_{cool} \cdot \left[ 1 + \sin(\omega \cdot t) + \frac{\omega^2 \cdot \tau_{cool}^2}{(fr(T, svf) + 1)^2} - \frac{\omega \cdot \tau_{cool} \cdot \cos(\omega \cdot t)}{1 + fr(T, svf)} \right]}{(fr(T, svf) + 1) \cdot \left[ \frac{\omega^2 \cdot \tau_{cool}^2}{(fr(T, svf) + 1)^2} + 1 \right]}$$

We can recast the equation in terms of a phase angle

Let

$$\frac{\omega \cdot \tau_{cool}}{1 + fr(T,svf)} = \tan(\phi) \quad \text{Note} \quad \tan(\phi)^2 + 1 = \frac{1}{\cos(\phi)^2}$$

$$T_{soot} - T_{gas} = \frac{\text{Lasheat} \cdot \tau_{cool} \cdot \left[ \sin(\omega \cdot t) + \left( \frac{1}{\cos(\phi)^2} \right) - \tan(\phi) \cdot \cos(\omega \cdot t) \right]}{(fr(T,svf) + 1) \cdot \left( \frac{1}{\cos(\phi)^2} \right)}$$

$$T_{soot} - T_{gas} = \frac{\text{Lasheat} \cdot \tau_{cool} \cdot \left( \sin(\omega \cdot t) \cdot \cos(\phi)^2 + 1 - \cos(\phi)^2 \cdot \tan(\phi) \cdot \cos(\omega \cdot t) \right)}{(fr(T,svf) + 1)}$$

$$T_{soot} - T_{gas} = \frac{\text{Lasheat} \cdot \tau_{cool} \cdot \cos(\phi) \cdot (\sin(\omega \cdot t) \cdot \cos(\phi) + 1 - \sin(\phi) \cdot \cos(\omega \cdot t))}{(fr(T,svf) + 1)}$$

$$T_{soot} - T_{gas} = \frac{\text{Lasheat} \cdot \tau_{cool} \cdot (1 + \cos(\phi) \cdot \sin(\omega \cdot t - \phi))}{(fr(T,svf) + 1)}$$

The integrals above do not include the integration constant, but, since at  $t=0 \phi=0$  and  $T_{soot} - T_{gas}=0$ , the integration constant is zero

Check that original Mathcad symbolic solution and my rearrangements of it give the same results

Check that original Mathcad symbolic solution and my rearrangements of it give identical results

$$fr(T,svf) := \frac{HC_{soot}(T,svf)}{HC_{air}(T)} *$$

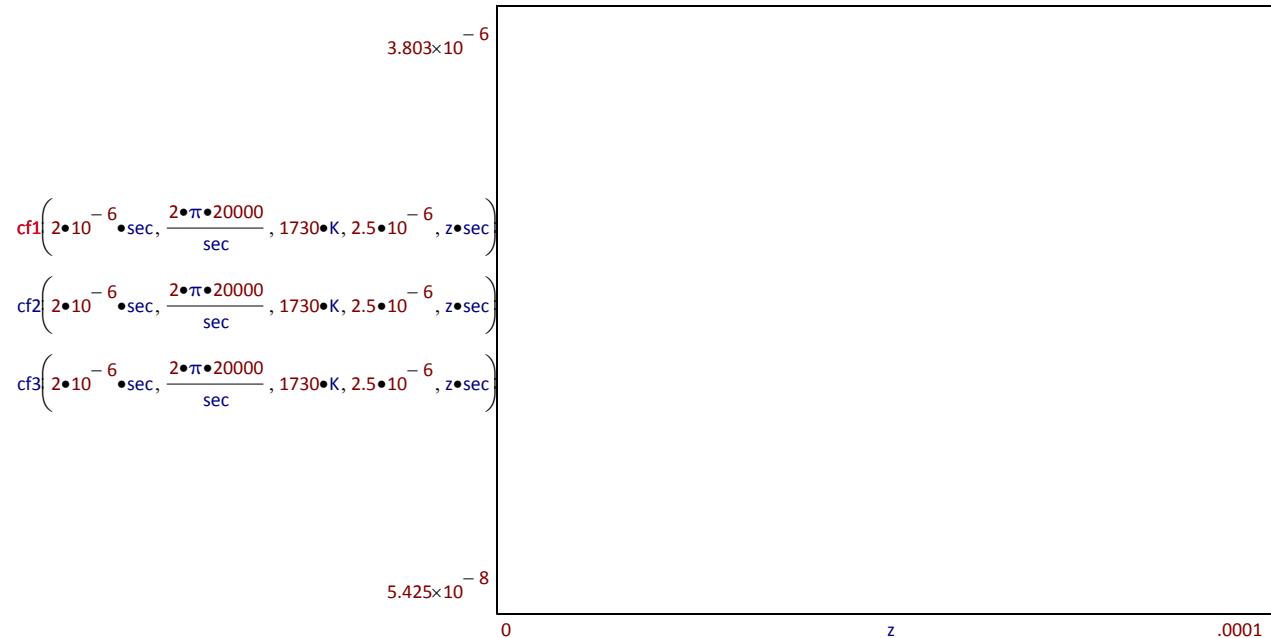
$$fr(1730 \cdot K, 2 \cdot 10^{-6}) = ■$$

$$\phi(\tau_{cool}, \omega, T, svf) := \text{atan}\left(\frac{\omega \cdot \tau_{cool}}{1 + fr(T, svf)}\right) *$$

$$cf1(\tau_{cool}, \omega, T, svf, t) := \frac{\tau_{cool} \cdot \left[ 1 + \sin(\omega \cdot t) + \frac{\omega^2 \cdot \tau_{cool}^2}{(fr(T, svf) + 1)^2} - \frac{\omega \cdot \tau_{cool} \cdot \cos(\omega \cdot t)}{1 + fr(T, svf)} \right]}{(fr(T, svf) + 1) \cdot \left[ \frac{\omega^2 \cdot \tau_{cool}^2}{(fr(T, svf) + 1)^2} + 1 \right]}$$

$$cf2(\tau_{cool}, \omega, T, svf, t) := \frac{\tau_{cool} \cdot \left( 2 \cdot fr(T, svf) + \sin(\omega \cdot t) + fr(T, svf)^2 \cdot \sin(\omega \cdot t) + fr(T, svf)^2 + 2 \cdot fr(T, svf) \cdot \sin(\omega \cdot t) + \omega^2 \cdot \tau_{cool}^2 - \omega \cdot \tau_{cool} \cdot \cos(\omega \cdot t) - \omega \cdot fr(T, svf) \cdot \tau_{cool} \cdot \cos(\omega \cdot t) + 1 \right)}{(fr(T, svf) + 1) \cdot \left( \omega^2 \cdot \tau_{cool}^2 + fr(T, svf)^2 + 2 \cdot fr(T, svf) + 1 \right)}$$

$$cf3(\tau_{cool}, \omega, T, svf, t) := \frac{\tau_{cool} \cdot \left( 1 + \cos\left(\operatorname{atan}\left(\frac{\omega \cdot \tau_{cool}}{1 + fr(T, svf)}\right)\right) \cdot \sin\left(\omega \cdot t - \operatorname{atan}\left(\frac{\omega \cdot \tau_{cool}}{1 + fr(T, svf)}\right)\right) \right)}{(fr(T, svf) + 1)}$$



Check that original Mathcad symbolic solution and my rearrangements of it give the same results

$fr(1730 \cdot K)$

Solve the equation for  $T_{gas}$

The differential equation describing gas heating is

$$\frac{d}{dt} T_{gas} = fr(T, svf) \cdot \frac{(T_{soot} - T_{gas})}{\tau_{cool}}$$

Using solution above for  $(T_{soot} - T_{gas})$  we get:

$$\frac{d}{dt} T_{gas} = \frac{Lasheat \cdot (1 + \cos(\phi) \cdot \sin(\omega \cdot t - \phi)) \cdot fr(T, svf)}{(1 + fr(T, svf))}$$

Substitute  $fr$  for  $fr(T, svf)$

Integrating    Lasheat := Lasheat        fr := fr         $\phi := \phi$

$$T_{\text{gas}} = \int \frac{\text{Lasheat} \cdot (1 + \cos(\phi) \cdot \sin(\omega \cdot t - \phi)) \cdot fr}{(1 + fr)} dt \rightarrow T_{\text{gas}} = -\frac{\text{Lasheat} \cdot fr \cdot (\cos(2 \cdot \phi - \omega \cdot t) + \cos(\omega \cdot t) - 2 \cdot \omega \cdot t)}{2 \cdot \omega + 2 \cdot \omega \cdot fr}$$

I get error pattern match exception for this evaluation

It calculated in a blank worksheet to give:

$$T_{\text{gas}} = \int \frac{\text{Lasheat} \cdot (1 + \cos(\phi) \cdot \sin(\omega \cdot t - \phi)) \cdot fr}{(1 + fr)} dt \rightarrow T_{\text{gas}} = -\frac{\text{Lasheat} \cdot fr \cdot (\cos(2 \cdot \phi - \omega \cdot t) + \cos(\omega \cdot t) - 2 \cdot \omega \cdot t)}{2 \cdot \omega + 2 \cdot \omega \cdot fr}$$

$$T_{\text{gas}} = -\frac{\text{Lasheat} \cdot fr \cdot (\cos(2 \cdot \phi - \omega \cdot t) + \cos(\omega \cdot t) - 2 \cdot \omega \cdot t)}{2 \cdot \omega + 2 \cdot \omega \cdot fr} + \text{Intcon}$$

$$\text{Lasheat}(P_d, 1730 \cdot K) = 1.824556 \times 10^5 \frac{K}{s}$$

What is the integration constant?

$$\text{When } t=0 \quad T_{\text{gas}} = T_0 \text{gas} = -\frac{\text{Lasheat} \cdot fr \cdot (\cos(2 \cdot \phi) + \cos(0))}{2 \cdot \omega + 2 \cdot \omega \cdot fr} + \text{Intcon}$$

$$\text{Intcon} = T_0 \text{gas} + \frac{\text{Lasheat} \cdot fr \cdot (\cos(2 \cdot \phi) + 1)}{2 \cdot \omega + 2 \cdot \omega \cdot fr}$$

So final equation is:

$$T_{\text{gas}} = T_0 \text{gas} + \frac{\text{Lasheat} \cdot fr}{2 \cdot \omega + 2 \cdot \omega \cdot fr} \cdot [[\cos(2 \cdot \phi) + 1 - (\cos(2 \cdot \phi - \omega \cdot t) + \cos(\omega \cdot t) - 2 \cdot \omega \cdot t)]]$$

Calculate some numerical examples

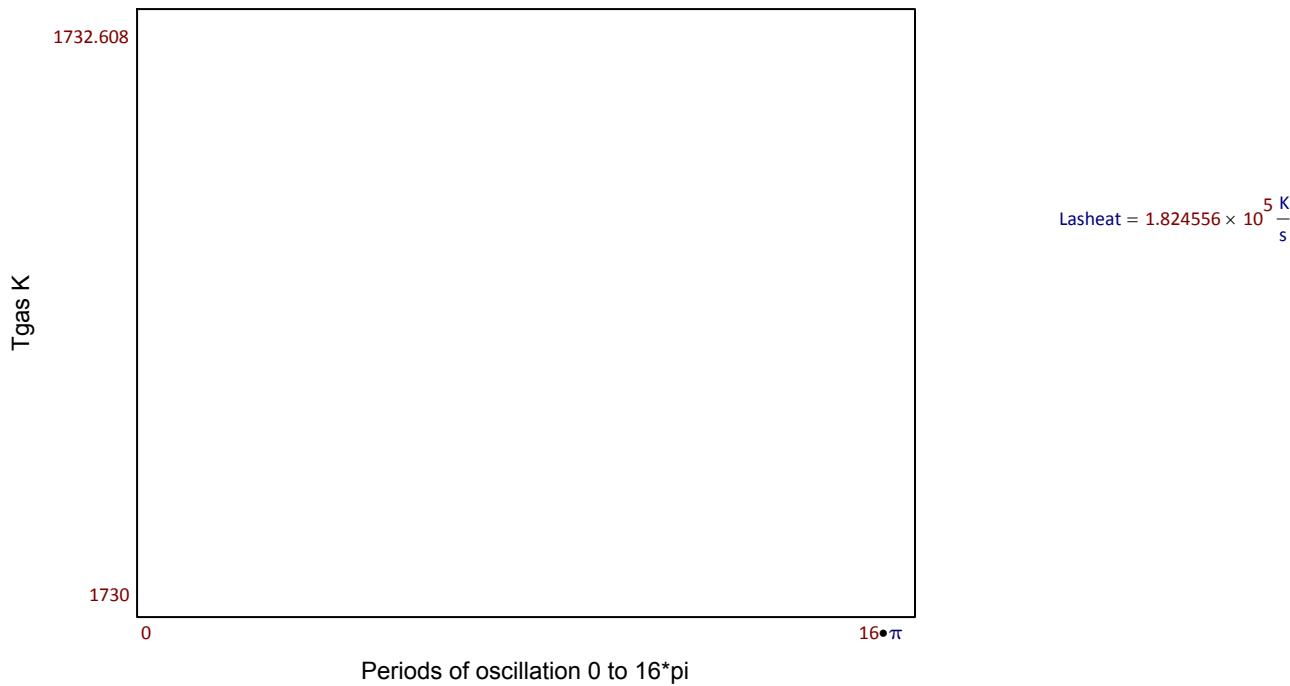
$$P_d := 75 \cdot 10^{-3} \cdot \frac{W}{mm^2} \quad \text{Lasheat}(P_d, T) := \frac{6 \cdot P_d \cdot \pi \cdot E_m}{\lambda_{\text{laser}} \cdot fC_p \left( \frac{T}{K} \right) \cdot RHOS}$$

$$fr(T, svf) := \frac{HC_{\text{soot}}(T, svf)}{HC_{\text{air}}(T)}$$

$$\phi(\tau_{\text{cool}}, \omega, T, svf) := \text{atan} \left( \frac{\omega \cdot \tau_{\text{cool}}}{1 + fr(T, svf)} \right)$$

$$T_{\text{gas}}(\tau_{\text{cool}}, \omega, T, svf, t) := T_0 \text{gas} + \frac{\text{Lasheat}}{2 \cdot (\omega + \omega \cdot fr(T, svf))} \cdot fr(T, svf) \cdot [\cos(2 \cdot \phi(\tau_{\text{cool}}, \omega, T, svf)) + 1 - (\cos(2 \cdot \phi(\tau_{\text{cool}}, \omega, T, svf) - \omega \cdot t) + \cos(\omega \cdot t) - 2 \cdot \omega \cdot t)]$$

20 kHz excitation, 2.5 ppm soot, cooling  $\tau=2 \cdot 10^{-6}$  sec



Solve the equation for  $T_{soot}$

$$\text{We have } T_{\text{gas}} = T_{0\text{gas}} + \frac{\text{Lasheat} \cdot fr}{2 \cdot \omega \cdot (1 + fr)} \cdot [[\cos(2 \cdot \phi) + 1 - (\cos(2 \cdot \phi - \omega \cdot t) + \cos(\omega \cdot t) - 2 \cdot \omega \cdot t)]]$$

$$\text{and } T_{\text{soot}} - T_{\text{gas}} = \frac{\text{Lasheat} \cdot \tau_{\text{cool}} \cdot (1 + \cos(\phi) \cdot \sin(\omega \cdot t - \phi))}{(fr + 1)}$$

Therefore

$$T_{\text{soot}} = T_{0\text{gas}} + \frac{\text{Lasheat} \cdot fr}{2 \cdot \omega \cdot (1 + fr)} \cdot [\cos(2 \cdot \phi) + 1 - (\cos(2 \cdot \phi - \omega \cdot t) + \cos(\omega \cdot t) - 2 \cdot \omega \cdot t)] + \frac{\text{Lasheat} \cdot \tau_{\text{cool}} \cdot (1 + \cos(\phi) \cdot \sin(\omega \cdot t - \phi))}{(fr + 1)}$$

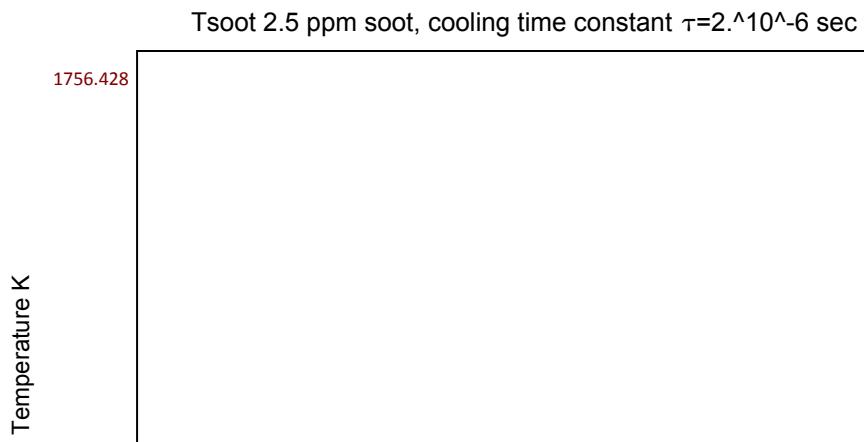
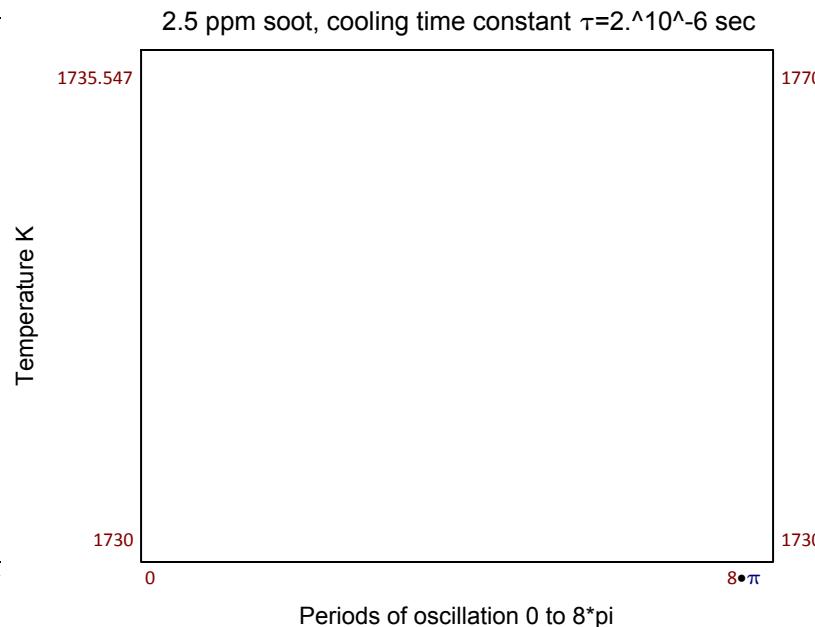
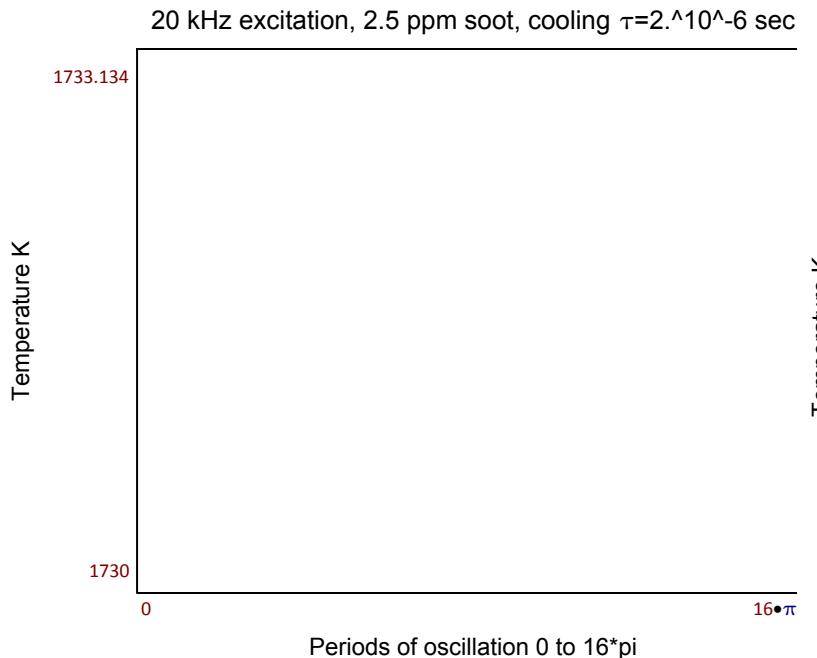
$$T_{\text{soot}} = T_{0\text{gas}} + \text{Lasheat}$$

$$T_{\text{soot}} = T_{0\text{gas}} + \frac{\text{Lasheat}}{(1 + fr)} \cdot \left[ \frac{fr \cdot [\cos(2 \cdot \phi) + 1 - (\cos(2 \cdot \phi - \omega \cdot t) + \cos(\omega \cdot t) - 2 \cdot \omega \cdot t)]}{2 \cdot \omega} + \tau_{\text{cool}} \cdot (1 + \cos(\phi) \cdot \sin(\omega \cdot t - \phi)) \right]$$

Calculate some numerical examples

$$T_{\text{soot}}(\tau_{\text{cool}}, \omega, T, svf, t) := T_{0\text{gas}} + \frac{\text{Lasheat}}{(1 + fr(T, svf))} \cdot \left[ \frac{fr(T, svf) \cdot [\cos(2 \cdot \phi(\tau_{\text{cool}}, \omega, T, svf)) + 1 - (\cos(2 \cdot \phi(\tau_{\text{cool}}, \omega, T, svf) - \omega \cdot t) + \cos(\omega \cdot t) - 2 \cdot \omega \cdot t)]}{2 \cdot \omega} + \tau_{\text{cool}} \cdot (1 + \cos(\phi(\tau_{\text{cool}}, \omega, T, svf))) \cdot \sin(\omega \cdot t - \phi(\tau_{\text{cool}}, \omega, T, svf)) \right]$$

The term  $\sin(\omega \cdot t - \phi(\tau_{\text{cool}}, \omega, T, svf))$  in the above expression shows error "this value must be a function but has the form R unitless"





Periods of oscillation 0 to  $16\pi$

At high frequencies there are many cycles of excitation during the gas replacement time and the heating tends towards a constant value. At lower frequencies there is only a fraction of a cycle within the gas replacement time and the heating rate is modulated depending on the instantaneous value of the modulated heating

Consider the effect of gas flow assuming cooling is very fast i.e.  $T_{soot} + T_{gas}$

The original equation for soot heating is:

$$\frac{dT_{soot}}{dt} = \frac{6 \cdot \pi \cdot E_m \cdot P_d}{RHOS \cdot \lambda_{laser} \cdot fCp(T)} \cdot (1 + \sin(\omega \cdot t)) - \frac{(T_{soot} - T_{0gas})}{\tau_{flow}}$$

$$Lasheat = \frac{6 \cdot \pi \cdot E_m \cdot P_d}{RHOS \cdot \lambda_{laser} \cdot fCp(T)}$$

In the above expression utilizing soot temperature is calculated by dividing the energy absorbed by the heat capacity of the soot. However, under the assumption that cooling is very fast the energy absorbed must teach both the soot and the gas resulting in a large reduction in the temperature rise. Above it was shown that:

$$\frac{1}{fr(T, svf)} + 1 = \frac{HC_{air}(T) + HC_{soot}(T, svf)}{HC_{soot}(T, svf)} = \frac{1 + fr(T, svf)}{fr(T, svf)}$$

we can correct for the effect of gas heating by multiplying the original expression by the ratio of the heat capacity of the soot divided by the total heat capacity of both soot and air.

$$\frac{dT_{soot}}{dt} = \frac{6 \cdot \pi \cdot E_m \cdot P_d}{RHOS \cdot \lambda_{laser} \cdot fCp(T)} \cdot \frac{fr(T, svf)}{(1 + fr(T, svf))} \cdot (1 + \sin(\omega \cdot t)) - \frac{(T_{soot} - T_{0gas})}{\tau_{flow}}$$

the above equation contains the following assumptions: at low frequencies relaxation of the soot temperature is very fast and the soot temperature is effectively the gas temperature. The second term on RHS of the above equation assumes that fresh gas replacement with a time constant  $\tau_{flow}$

setting  $fr = fr(T, svf)$

$$\text{Rearranging } \frac{dT_{soot}}{dt} + \frac{T_{soot}}{\tau_{flow}} = \frac{fr}{1 + fr} \cdot Lasheat \cdot (1 + \sin(\omega \cdot t)) + \frac{T_{0gas}}{\tau_{flow}} \quad \frac{fr}{1 + fr}$$

Multiply left hand side by integrating factor  $\exp\left(\frac{t}{\tau_{flow}}\right)$

$$\exp\left(\frac{t}{\tau_{flow}}\right) \cdot \left( \frac{dT_{soot}}{dt} + \frac{T_{soot}}{\tau_{flow}} \right) = \exp\left(\frac{t}{\tau_{flow}}\right) \cdot \left[ \frac{fr}{1 + fr} \cdot Lasheat \cdot (1 + \sin(\omega \cdot t)) + \frac{T_{0gas}}{\tau_{flow}} \right]$$

LHS is

$$\frac{d}{dt} \exp\left(\frac{t}{\tau_{flow}}\right) \cdot T_{soot}$$

integrating we get

$$\exp\left(\frac{t}{\tau_{flow}}\right) \cdot T_{soot}$$

Integrating RHS we get

$$fr := fr$$

$$T_{0gas} := T_{0gas} \quad \tau_{flow} := \tau_{flow}$$

$$\int \exp\left(\frac{t}{\tau_{flow}}\right) \cdot \left[ \frac{fr}{1+fr} \cdot Lasheat \cdot (1 + \sin(\omega \cdot t)) + \frac{T_{0gas}}{\tau_{flow}} \right] dt \rightarrow \frac{e^{\frac{t}{\tau_{flow}}} \cdot \left( T_{0gas} + T_{0gas} \cdot fr + T_{0gas} \cdot \omega^2 \cdot \tau_{flow}^2 + Lasheat \cdot fr \cdot \tau_{flow} + Lasheat \cdot fr \cdot \tau_{flow} \cdot \sin(\omega \cdot t) + Lasheat \cdot \omega^2 \cdot fr \cdot \tau_{flow}^3 + T_{0gas} \cdot \omega^2 \cdot fr \right)}{fr + \omega^2 \cdot \tau_{flow}^2 + \omega^2 \cdot fr \cdot \tau_{flow}^2 + 1}$$

I get error pattern match exception for this evaluation

$$\omega := \omega \quad t := t \quad \phi := \phi \quad \tau_{flow} := \tau_{flow} \quad \tau_{cool} := \tau_{cool} \quad Lasheat := Lasheat \quad T_{0gas} := T_{0gas} \quad T_{soot} := T_{soot}$$

The

$$\frac{T_{0gas} + T_{0gas} \cdot fr + T_{0gas} \cdot \omega^2 \cdot \tau_{flow}^2 + Lasheat \cdot fr \cdot \tau_{flow} + Lasheat \cdot fr \cdot \tau_{flow} \cdot \sin(\omega \cdot t) + Lasheat \cdot \omega^2 \cdot fr \cdot \tau_{flow}^3 + T_{0gas} \cdot \omega^2 \cdot fr \cdot \tau_{flow}^2 - Lasheat \cdot \omega \cdot fr \cdot \tau_{flow}^2 \cdot \cos(\omega \cdot t)}{fr + \omega^2 \cdot \tau_{flow}^2 + \omega^2 \cdot fr \cdot \tau_{flow}^2 + 1}$$

simplify  
 collect,  $T_{0gas}$  →  $\frac{fr \cdot \tau_{flow} - }{ }$   
 collect,  $Lasheat$

I get error emission failure for this evaluation

$$T_{soot} = \frac{fr \cdot \tau_{flow} + fr \cdot \tau_{flow} \cdot \sin(\omega \cdot t) + \omega^2 \cdot fr \cdot \tau_{flow}^3 - \omega \cdot fr \cdot \tau_{flow}^2 \cdot \cos(\omega \cdot t)}{fr + \omega^2 \cdot \tau_{flow}^2 + \omega^2 \cdot fr \cdot \tau_{flow}^2 + 1} \cdot Lasheat + T_{0gas} = \left[ \frac{1 + \sin(\omega \cdot t) + \omega^2 \cdot \tau_{flow}^2 - \omega \cdot \tau_{flow} \cdot \cos(\omega \cdot t)}{\left(1 + \frac{1}{fr}\right) \cdot \left(\omega^2 \cdot \tau_{flow}^2 + 1\right)} \cdot \tau_{flow} \cdot Lasheat + T_{0gas} \right]$$

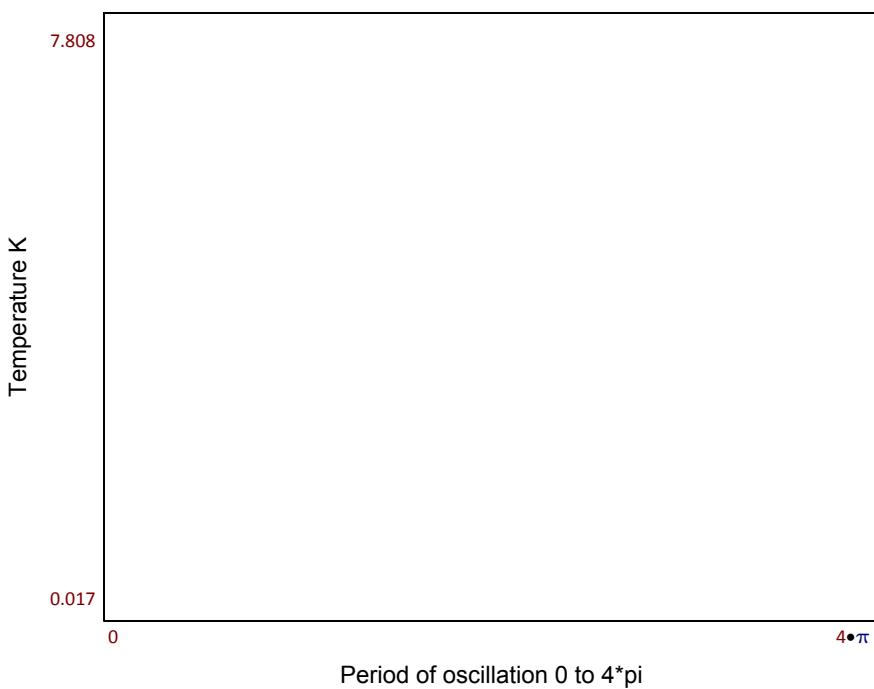
Calculate some numerical examples

$$fr(T, svf) := \frac{HC_{soot}(T, svf)}{HC_{air}(T)}$$

$$Trise(\omega, \tau_{flow}, T_{0gas}, t) := Lasheat(P_d, T_{0gas}) \cdot \tau_{flow} \cdot \left[ \frac{1 + \sin(\omega \cdot t) + \omega^2 \cdot \tau_{flow}^2 - \omega \cdot \tau_{flow} \cdot \cos(\omega \cdot t)}{\left(1 + \frac{1}{fr(T_{0gas}, svf)}\right) \cdot \left(\omega^2 \cdot \tau_{flow}^2 + 1\right)} \right]$$

For  $\sin(\omega \cdot t) + \omega^2 \cdot \tau_{flow}^2 - \omega \cdot \tau_{flow} \cdot \cos(\omega \cdot t)$  in the above expression shows error "this value must be a function but has the form R unless st"

Soot temperature modulation  $\tau_{\text{flow}} = 0.6 \text{ msec}$



Period of oscillation 0 to  $4\pi$

► consider the effect of gas flow by setting up an energy balance. This approach gives strange results and is almost certainly incorrect.

## Consider the effect of gas flow without any assumptions

$$\frac{dT_{soot}}{dt} = \text{Lasheat} \cdot (1 + \sin(\omega \cdot t)) - \frac{(T_{soot} - T_{0gas})}{\tau_{flow}} - \frac{(T_{soot} - T_{gas})}{\tau_{cool}}$$

$\tau_{flow} := \tau_{flow}$

$\phi_{col} := \phi_{col}$

$$fr(T, svf) = \frac{HC_{soot}(T, svf)}{HC_{air}(T)}$$

The starting differential equation describing soot & gas heating are:

$$\frac{dT_{gas}}{dt} = fr(T, svf) \cdot \frac{(T_{soot} - T_{gas})}{\tau_{cool}} - \frac{(T_{gas} - T_{0gas})}{\tau_{flow}}$$

does  $T_{0gas}$  in  $T_{soot} - T_{0gas}/\tau_{flow}$  represent the soot temperature of the soot outside of the heated volume (which should be equal to the gas temperature outside the volume)?

$$\frac{d}{dt}(T_{soot} - T_{gas}) = \frac{d}{dt}T_{soot} - \frac{d}{dt}T_{gas} = \text{Lasheat} \cdot (1 + \sin(\omega \cdot t)) - \frac{(T_{soot} - T_{gas})}{\tau_{cool}} - \frac{(T_{soot} - T_{0gas})}{\tau_{flow}} - fr \cdot \frac{(T_{soot} - T_{gas})}{\tau_{cool}} + \frac{(T_{gas} - T_{0gas})}{\tau_{flow}}$$

$$\frac{d}{dt}(T_{soot} - T_{gas}) = \text{Lasheat} \cdot (\sin(\omega \cdot t) + 1) - \left[ \frac{(T_{soot} - T_{gas})}{\tau_{cool}} \cdot (1 + fr) + \frac{T_{soot} - T_{gas}}{\tau_{flow}} \right]$$

What is solution of the  $\frac{d}{dt}(T_{soot} - T_{gas})$  equation?

$$\frac{1 + fr}{\tau_{cool}} + \frac{1}{\tau_{flow}}$$

Get  $T_{soot} - T_{gas}$  terms on LHS of equation and Multiply left hand side by integrating factor  $\exp\left[t \cdot \left(\frac{1 + fr}{\tau_{cool}} + \frac{1}{\tau_{flow}}\right)\right]$

$$\exp\left[t \cdot \left(\frac{1 + fr}{\tau_{cool}} + \frac{1}{\tau_{flow}}\right)\right] \cdot \left[ \frac{d}{dt}(T_{soot} - T_{gas}) + (T_{soot} - T_{gas}) \cdot \left(\frac{1 + fr}{\tau_{cool}} + \frac{1}{\tau_{flow}}\right) \right] = \frac{d}{dt}\left[\exp\left[t \cdot \left(\frac{1 + fr}{\tau_{cool}} + \frac{1}{\tau_{flow}}\right)\right] \cdot (T_{soot} - T_{gas})\right] = \exp\left[t \cdot \left(\frac{1 + fr}{\tau_{cool}} + \frac{1}{\tau_{flow}}\right)\right] \cdot [\text{Lasheat} \cdot (\sin(\omega \cdot t) + 1)]$$

Integrating this expression in time t we have

$$\exp\left[t \cdot \left(\frac{1 + fr}{\tau_{cool}} + \frac{1}{\tau_{flow}}\right)\right] \cdot (T_{soot} - T_{gas}) = \int \exp\left[t \cdot \left(\frac{1 + fr}{\tau_{cool}} + \frac{1}{\tau_{flow}}\right)\right] \cdot [\text{Lasheat} \cdot (\sin(\omega \cdot t) + 1)] dt$$

$fr := fr$

Mathcad derived integral

$$\int \exp\left[t \cdot \left(\frac{1 + fr}{\tau_{cool}} + \frac{1}{\tau_{flow}}\right)\right] \cdot [\text{Lasheat} \cdot (\sin(\omega \cdot t) + 1)] dt \rightarrow \frac{\text{Lasheat} \cdot \tau_{flow} \cdot \tau_{cool} \cdot e^{\frac{t}{\tau_{flow}} + \frac{t}{\tau_{cool}} + \frac{fr \cdot t}{\tau_{cool}}} \cdot \left(\tau_{flow}^2 \cdot \sin(\omega \cdot t) + \tau_{cool}^2 \cdot \sin(\omega \cdot t) + 2 \cdot fr \cdot \tau_{flow}^2 + \tau_{flow}^2 + \tau_{cool}^2 + fr^2 \cdot \tau_{flow}^2 + 2 \cdot \tau_{flow} \cdot \tau_{cool}^2\right)}{(\tau_{flow} + \tau_{cool})^2}$$

Original form of integral

$$T_{soot} - T_{gas} = \frac{\text{Lasheat} \cdot \tau_{flow} \cdot \tau_{cool} \cdot \left( \tau_{flow}^2 \cdot \sin(\omega \cdot t) + \tau_{cool}^2 \cdot \sin(\omega \cdot t) + 2 \cdot fr \cdot \tau_{flow}^2 + \tau_{flow}^2 + \tau_{cool}^2 + fr^2 \cdot \tau_{flow}^2 + 2 \cdot \tau_{flow} \cdot \tau_{cool} + 2 \cdot \tau_{flow} \cdot \tau_{cool} \cdot \sin(\omega \cdot t) + 2 \cdot fr \cdot \tau_{flow} \cdot \tau_{cool} + 2 \cdot fr \cdot \tau_{flow}^2 \cdot \sin(\omega \cdot t) \right)}{\left( \tau_{flow} + \tau_{cool} + fr \cdot \tau_{flow} \right) \cdot \left( \omega^2 \cdot \tau_{flow}^2 \cdot \tau_{cool}^2 + fr^2 \cdot \tau_{flow}^2 + 2 \cdot fr \cdot \tau_{flow}^2 \cdot \sin(\omega \cdot t) \right)}$$

This is very complex try simplifying

$$\text{Simplify by defining } Cn = \frac{1 + fr}{\tau_{cool}} + \frac{1}{\tau_{flow}}$$

$$\int \exp(t \cdot Cn) \cdot [\text{Lasheat} \cdot (\sin(\omega \cdot t) + 1)] dt \rightarrow \frac{\text{Lasheat} \cdot e^{Cn \cdot t} \cdot \left( Cn^2 + \omega^2 + Cn^2 \cdot \sin(\omega \cdot t) - Cn \cdot \omega \cdot \cos(\omega \cdot t) \right)}{Cn^3 + Cn \cdot \omega^2}$$

$$T_{soot} - T_{gas} = \frac{\text{Lasheat} \cdot \left( Cn^2 + \omega^2 + Cn^2 \cdot \sin(\omega \cdot t) - Cn \cdot \omega \cdot \cos(\omega \cdot t) \right)}{Cn^3 + Cn \cdot \omega^2}$$

Define phase angle

$$\frac{\omega}{\left( \frac{1 + fr}{\tau_{cool}} + \frac{1}{\tau_{flow}} \right)} = \tan(\phi) = \frac{\omega}{Cn}$$

Note

$$\tan(\phi)^2 + 1 = \frac{1}{\cos(\phi)^2} \quad Cn = \frac{\omega}{\tan(\phi)}$$

$$\frac{\omega \cdot \tau_{cool}}{1 + fr(T, svf)} = \tan(\phi)$$

$$T_{soot} - T_{gas} = \frac{\text{Lasheat} \cdot \left( Cn^2 + \omega^2 + Cn^2 \cdot \sin(\omega \cdot t) - Cn \cdot \omega \cdot \cos(\omega \cdot t) \right)}{Cn^3 + Cn \cdot \omega^2}$$

$$\frac{\text{Lasheat} \cdot \left( Cn^2 + \omega^2 + Cn^2 \cdot \sin(\omega \cdot t) - Cn \cdot \omega \cdot \cos(\omega \cdot t) \right)}{Cn^3 + Cn \cdot \omega^2} \text{ substitute, } Cn = \frac{\omega}{\tan(\phi)} \rightarrow \frac{\text{Lasheat} \cdot \tan(\phi) \cdot \left( \tan(\phi)^2 - \cos(\omega \cdot t) \cdot \tan(\phi) + \sin(\omega \cdot t) + 1 \right)}{\omega \cdot \tan(\phi)^2 + \omega}$$

$$\frac{\text{Lasheat} \cdot \tan(\phi) \cdot \left( \tan(\phi)^2 - \cos(\omega \cdot t) \cdot \tan(\phi) + \sin(\omega \cdot t) + 1 \right)}{\omega \cdot \tan(\phi)^2 + \omega} = \frac{\text{Lasheat} \cdot \tan(\phi)}{\omega} \left[ 1 - \cos(\phi)^2 \cdot (\cos(\omega \cdot t) \cdot \tan(\phi) + \sin(\omega \cdot t)) \right]$$

should it be a subtraction rather than addition of the sin omega t?

$$\frac{\text{Lasheat} \cdot \tan(\phi)}{\omega} \left[ 1 - \cos(\phi)^2 \cdot (\cos(\omega \cdot t) \cdot \tan(\phi) + \sin(\omega \cdot t)) \right] = \frac{\text{Lasheat} \cdot \tan(\phi)}{\omega} [1 + \cos(\phi) \cdot (-\sin(\phi) \cdot \cos(\omega \cdot t) + \cos(\phi) \cdot \sin(\omega \cdot t))]$$

where is the close bracket for the large open bracket?  
note that subtraction error of the line above fixed here.

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$$

$$\frac{\text{Lasheat} \cdot \tan(\phi)}{\omega} [1 + \cos(\phi) \cdot (-\sin(\phi) \cdot \cos(\omega \cdot t) + \cos(\phi) \cdot \sin(\omega \cdot t))] = \frac{\text{Lasheat} \cdot \tan(\phi)}{\omega} [1 + \cos(\phi) \cdot (\sin(\omega \cdot t - \phi))]$$

Simplified expression for integral

$$T_{soot} - T_{gas} = \frac{\text{Lasheat} \cdot \tan(\phi)}{\omega} [1 + \cos(\phi) \cdot (\sin(\omega \cdot t - \phi))]$$

plus an integration constant IC?

$$fr(T, svf) := \frac{HC_{soot}(T, svf)}{HC_{air}(T)}$$

$$\phi(\omega, \tau_{flow}, \tau_{cool}, svf, T_{0gas}) := \text{atan} \left[ \frac{\omega}{\left( \frac{1 + fr(T_{0gas}, svf)}{\tau_{cool}} + \frac{1}{\tau_{flow}} \right)} \right]$$

$$fr(T_{0gas}, svf)$$

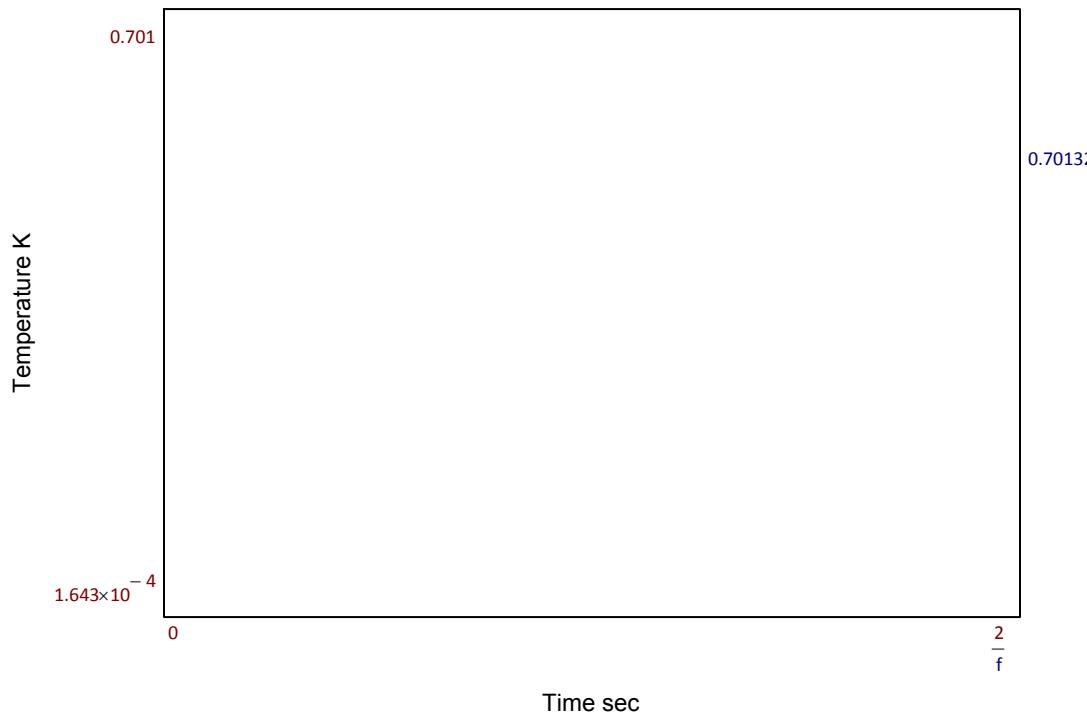
$$Eq_{original}(\omega, \tau_{flow}, \tau_{cool}, T_{0gas}, svf, t) := \frac{\text{Lasheat} \cdot \tau_{flow} \cdot \tau_{cool} \cdot \left( \tau_{flow}^2 \cdot \sin(\omega \cdot t) + \tau_{cool}^2 \cdot \sin(\omega \cdot t) + 2 \cdot fr(T_{0gas}, svf) \cdot \tau_{flow}^2 + \tau_{flow}^2 + \tau_{cool}^2 + fr(T_{0gas}, svf)^2 \cdot \tau_{flow}^2 + 2 \cdot \tau_{flow} \cdot \tau_{cool} + 2 \cdot \tau_{flow} \cdot \tau_{cool} \cdot \sin(\omega \cdot t) \right)}{(\tau_{flow} + \tau_{cool} + fr(T_{0gas}, svf) \cdot \tau_{flow})}$$

$$\text{Lasheat} = 1.824556 \times 10^5 \frac{\text{K}}{\text{s}}$$

$$Eq_{simplified}(\omega, \tau_{flow}, \tau_{cool}, T_{0gas}, svf, t) := \frac{\text{Lasheat} \cdot \tan(\phi(\omega, \tau_{flow}, \tau_{cool}, svf, T_{0gas}))}{\omega} [1 + \cos(\phi(\omega, \tau_{flow}, \tau_{cool}, svf, T_{0gas})) \cdot (\sin(\omega \cdot t - \phi(\omega, \tau_{flow}, \tau_{cool}, svf, T_{0gas})))]$$

$$f := 2500$$

### Comparison of original solution and simplified expression fo Tsoot-Tgas



### Consideration of IC

$$\text{When } t=0 \quad T_{\text{soot}} = T_{\text{gas}}$$

$$IC = \frac{-Lasheat \cdot \tan(\phi)}{\omega} (1 + \cos(\phi) \cdot \sin(-\phi))$$

$$T_{\text{soot}} - T_{\text{gas}} = \frac{Lasheat \cdot \tan(\phi)}{\omega} \cdot [\cos(\phi) \cdot \sin(\phi) + \cos(\phi) \cdot (\sin(\omega \cdot t - \phi))]$$

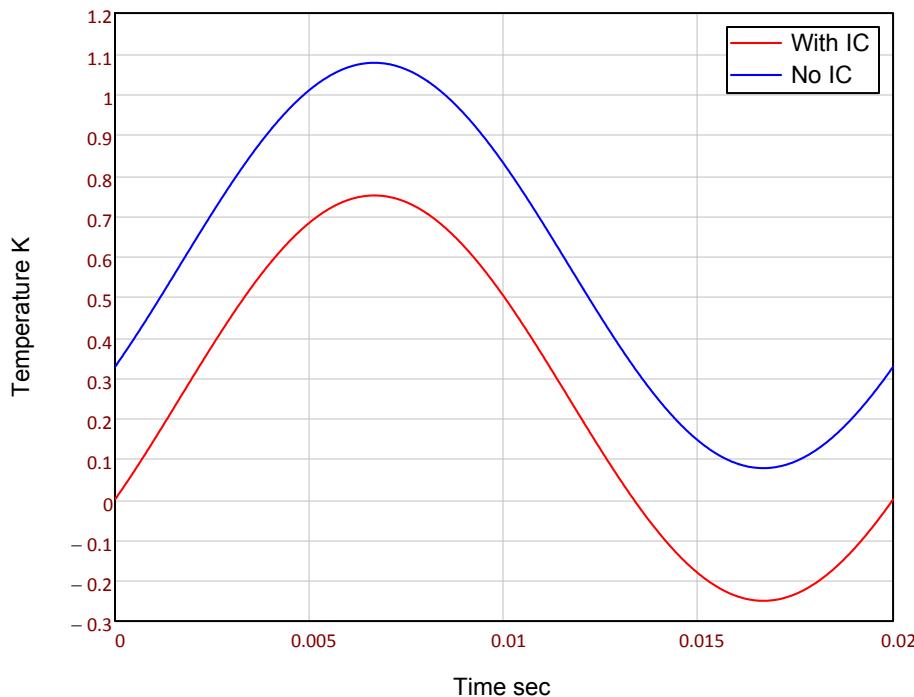
$$T_{\text{soot}} - T_{\text{gas}} = \frac{Lasheat \cdot \sin(\phi)}{\omega} \cdot (\sin(\phi) + \sin(\omega \cdot t - \phi)) \quad (\sin(\phi) + \sin(\omega \cdot t - \phi)) = 2 \cdot \sin\left(\frac{\omega \cdot t}{2}\right) \cdot \cos\left(\frac{\omega \cdot t}{2} + \phi\right)$$

We have ignored any integrating factor, IC, but when  $t=0$   $(T_{\text{soot}} - T_{\text{gas}}) = 0$  therefore  $IC=0$

$$sf2(\phi, \omega, t) := \tan(\phi)[1 + \cos(\phi) \cdot (\sin(\omega \cdot t - \phi))]$$

$$sf1(\phi, \omega, t) := \sin(\phi) \cdot (\sin(\phi) + \sin(\omega \cdot t - \phi))$$

### Tsoot-Tgas K



$$T_{soot} - T_{gas} = 0$$

The introduction of the integrating constant based on  $T_{soot} - T_{gas} = 0$  at  $t=0$  causes the expression for the difference between the soot and the gas temperature to go negative which is not a physically meaningful solution. The application of an integrating constant thus seems inappropriate here. In fact setting the temperature difference to zero at time zero is equivalent to forcing the modulated temperature to be zero at  $t=0$ . However, the solution is still not physically meaningful because the temperature difference cannot be negative. In what follows I will ignore the integration constant, but cannot rigorously mathematically justify that procedure.

Solve for  $T_{gas}$

Substitute  $f_r = f_r(T, svf)$

$$\text{Now } \frac{d}{dt} T_{gas} = f_r \cdot \frac{(T_{soot} - T_{gas})}{\tau_{cool}} - \frac{(T_{gas} - T_{0gas})}{\tau_{flow}}$$

$$\frac{d}{dt} T_{gas} = f_r \cdot \left[ \frac{\frac{Lasheat \cdot \tan(\phi)}{\omega} [1 + \cos(\phi) \cdot (\sin(\omega \cdot t - \phi))]}{\tau_{cool}} - \frac{(T_{gas} - T_{0gas})}{\tau_{flow}} \right]$$

$$\frac{d}{dt} T_{gas} + \frac{T_{gas}}{\tau_{flow}} = \left[ \frac{\frac{f_r \cdot Lasheat \cdot \tan(\phi)}{\omega} [1 + \cos(\phi) \cdot (\sin(\omega \cdot t - \phi))]}{\tau_{cool}} \right] + \frac{T_{0gas}}{\tau_{flow}}$$

Multiplying both sides by the integrating factor  $\exp\left(\left(\frac{t}{\tau_{\text{flow}}}\right)\right)$  we get

$$\left[ \left( \frac{d}{dt} T_{\text{gas}} + \frac{T_{\text{gas}}}{\tau_{\text{flow}}} \right) \cdot \exp\left(\frac{t}{\tau_{\text{flow}}}\right) \right] = \frac{d}{dt} \left[ T_{\text{gas}} \cdot \exp\left(\frac{t}{\tau_{\text{flow}}}\right) \right] = \exp\left(\frac{t}{\tau_{\text{flow}}}\right) \cdot \left[ \frac{\left[ \frac{fr \cdot Lasheat \cdot \tan(\phi)}{\omega} [1 + \cos(\phi) \cdot (\sin(\omega \cdot t - \phi))] \right]}{\tau_{\text{cool}}} + \frac{T_{0\text{gas}}}{\tau_{\text{flow}}} \right]$$

$$\phi := \phi \quad T_{0\text{gas}} := T_{0\text{gas}} \quad \tau_{\text{flow}} := \tau_{\text{flow}} \quad \tau_{\text{cool}} := \tau_{\text{cool}} \quad \omega := \omega \quad Lasheat := Lasheat \quad fr := fr$$

$$\int \exp\left(\frac{t}{\tau_{\text{flow}}}\right) \cdot \left[ \frac{\left[ \frac{fr \cdot Lasheat \cdot \sin(\phi)}{\omega} (\sin(\phi) + \sin(\omega \cdot t - \phi)) \right]}{\tau_{\text{cool}}} + \frac{T_{0\text{gas}}}{\tau_{\text{flow}}} \right] dt \rightarrow \frac{t}{e^{\frac{t}{\tau_{\text{flow}}}}} \cdot \left( Lasheat \cdot fr \cdot \tau_{\text{flow}} + 2 \cdot T_{0\text{gas}} \cdot \omega \cdot \tau_{\text{cool}} - Lasheat \cdot fr \cdot \tau_{\text{flow}} \cdot \cos(\omega \cdot t) + Lasheat \cdot \omega^2 \cdot fr \cdot \tau_{\text{flow}}^3 + 2 \cdot T_{0\text{gas}} \cdot \omega^3 \cdot \tau_{\text{flow}}^2 \cdot \tau_{\text{cool}} + Lasheat \cdot fr \cdot \tau_{\text{flow}} \cdot \cos(2 \cdot \phi - \omega \cdot t) - Lasheat \cdot fr \cdot \tau_{\text{flow}} \cdot \cos(2 \cdot \phi) - Lasheat \cdot fr \cdot \tau_{\text{flow}} \cdot \sin(2 \cdot \phi - \omega \cdot t) - Lasheat \cdot fr \cdot \tau_{\text{flow}} \cdot \sin(2 \cdot \phi) \right)$$

$$T_{\text{gas}} = \frac{\left( Lasheat \cdot fr \cdot \tau_{\text{flow}} + 2 \cdot T_{0\text{gas}} \cdot \omega \cdot \tau_{\text{cool}} - Lasheat \cdot fr \cdot \tau_{\text{flow}} \cdot \cos(\omega \cdot t) + Lasheat \cdot \omega^2 \cdot fr \cdot \tau_{\text{flow}}^3 + 2 \cdot T_{0\text{gas}} \cdot \omega^3 \cdot \tau_{\text{flow}}^2 \cdot \tau_{\text{cool}} + Lasheat \cdot fr \cdot \tau_{\text{flow}} \cdot \cos(2 \cdot \phi - \omega \cdot t) - Lasheat \cdot fr \cdot \tau_{\text{flow}} \cdot \cos(2 \cdot \phi) - Lasheat \cdot fr \cdot \tau_{\text{flow}} \cdot \sin(2 \cdot \phi - \omega \cdot t) - Lasheat \cdot fr \cdot \tau_{\text{flow}} \cdot \sin(2 \cdot \phi) \right)}{2 \cdot \tau_{\text{cool}} \cdot \omega^3 \cdot \tau_{\text{flow}}^2 + 2 \cdot \tau_{\text{cool}} \cdot \omega}$$

$$T_{\text{gas}} = \frac{\left( Lasheat \cdot fr \cdot \tau_{\text{flow}} + 2 \cdot T_{0\text{gas}} \cdot \omega \cdot \tau_{\text{cool}} - Lasheat \cdot fr \cdot \tau_{\text{flow}} \cdot \cos(\omega \cdot t) + Lasheat \cdot \omega^2 \cdot fr \cdot \tau_{\text{flow}}^3 + 2 \cdot T_{0\text{gas}} \cdot \omega^3 \cdot \tau_{\text{flow}}^2 \cdot \tau_{\text{cool}} + Lasheat \cdot fr \cdot \tau_{\text{flow}} \cdot \cos(2 \cdot \phi - \omega \cdot t) - Lasheat \cdot fr \cdot \tau_{\text{flow}} \cdot \cos(2 \cdot \phi) - Lasheat \cdot fr \cdot \tau_{\text{flow}} \cdot \sin(2 \cdot \phi - \omega \cdot t) - Lasheat \cdot fr \cdot \tau_{\text{flow}} \cdot \sin(2 \cdot \phi) \right)}{2 \cdot \tau_{\text{cool}} \cdot \omega^3 \cdot \tau_{\text{flow}}^2 + 2 \cdot \tau_{\text{cool}} \cdot \omega}$$

$$T_{\text{gas}} = \frac{\left( -2 \cdot fr \cdot \tau_{\text{flow}} \cdot \sin\left(\phi - \frac{\omega \cdot t}{2}\right)^2 - 2 \cdot fr \cdot \tau_{\text{flow}} \cdot \sin\left(\frac{\omega \cdot t}{2}\right)^2 - 2 \cdot fr \cdot \tau_{\text{flow}} \cdot \sin(\phi)^2 + \omega \cdot fr \cdot \tau_{\text{flow}}^2 \cdot \sin(2 \cdot \phi - \omega \cdot t) + \omega \cdot fr \cdot \tau_{\text{flow}}^2 \cdot \sin(\omega \cdot t) - 2 \cdot \omega^2 \cdot fr \cdot \tau_{\text{flow}}^3 \cdot \sin(\phi)^2 \right)}{2 \cdot \tau_{\text{cool}} \cdot \omega^3 \cdot \tau_{\text{flow}}^2 + 2 \cdot \tau_{\text{cool}} \cdot \omega} \cdot Lasheat + T_{0\text{gas}}$$

$$T_{\text{gas}} = \frac{\left( -2 \cdot \sin\left(\phi - \frac{\omega \cdot t}{2}\right)^2 - 2 \cdot \sin\left(\frac{\omega \cdot t}{2}\right)^2 - 2 \cdot \sin(\phi)^2 + \omega \cdot \tau_{\text{flow}} \cdot \sin(2 \cdot \phi - \omega \cdot t) + \omega \cdot \tau_{\text{flow}} \cdot \sin(\omega \cdot t) - 2 \cdot \omega^2 \cdot \tau_{\text{flow}}^2 \cdot \sin(\phi)^2 \right)}{2 \cdot \tau_{\text{cool}} \cdot \omega^2 \cdot \tau_{\text{flow}} + 2 \cdot \frac{\tau_{\text{cool}}}{\tau_{\text{flow}}}} \cdot \frac{fr \cdot Lasheat}{\omega} + T_{0\text{gas}}$$

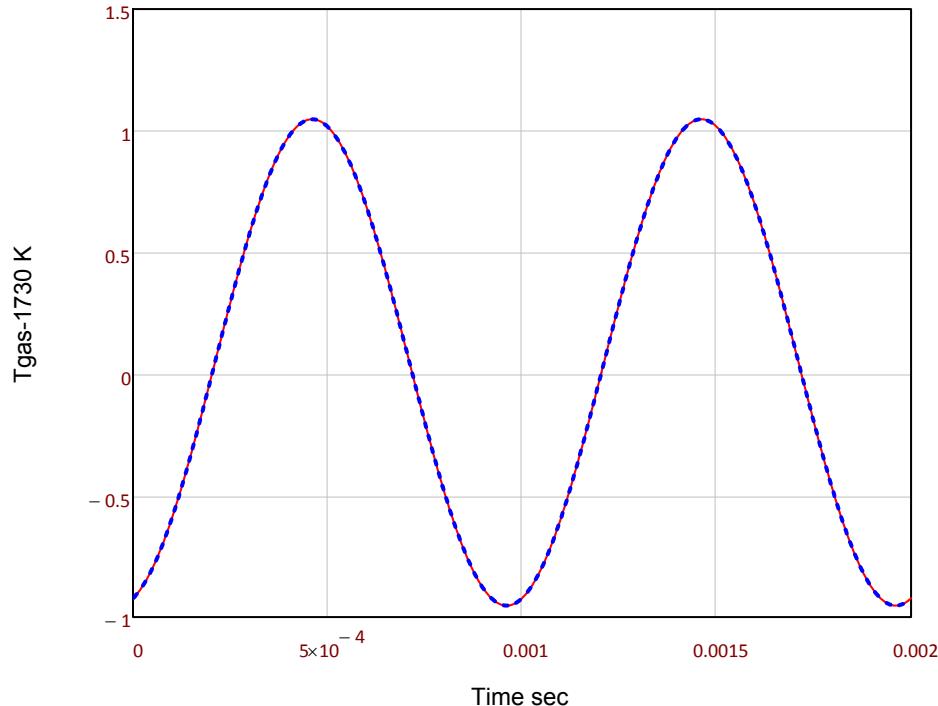
Check expressions numerically

$$\phi(\omega, fr, \tau_{flow}, \tau_{cool}) := \text{atan} \left[ \frac{\omega}{\left( \frac{1+fr}{\tau_{cool}} + \frac{1}{\tau_{flow}} \right)} \right]$$

$$\text{resoriginal}(\omega, fr, T_0\text{gas}, \tau_{flow}, \tau_{cool}, t) := \frac{\text{Lasheat} \cdot fr \cdot \tau_{flow} + 2 \cdot T_0\text{gas} \cdot \omega \cdot \tau_{cool} - \text{Lasheat} \cdot fr \cdot \tau_{flow} \cdot \cos(\omega \cdot t) + \text{Lasheat} \cdot \omega^2 \cdot fr \cdot \tau_{flow}^3 + 2 \cdot T_0\text{gas} \cdot \omega^3 \cdot \tau_{flow}^2 \cdot \tau_{cool} + \text{Lasheat} \cdot fr \cdot \tau_{flow} \cdot \cos(2 \cdot \phi(\omega, fr, \tau_{flow}, \tau_{cool}, t))}{\text{Lasheat} \cdot fr \cdot \tau_{flow} + 2 \cdot T_0\text{gas} \cdot \omega \cdot \tau_{cool} - \text{Lasheat} \cdot fr \cdot \tau_{flow} \cdot \cos(\omega \cdot t) + \text{Lasheat} \cdot \omega^2 \cdot fr \cdot \tau_{flow}^3 + 2 \cdot T_0\text{gas} \cdot \omega^3 \cdot \tau_{flow}^2 \cdot \tau_{cool} + \text{Lasheat} \cdot fr \cdot \tau_{flow} \cdot \cos(2 \cdot \phi(\omega, fr, \tau_{flow}, \tau_{cool}, t))}$$

$$\text{ressimplified}(\omega, fr, T_0\text{gas}, \tau_{flow}, \tau_{cool}, t) := \frac{\left[ -\frac{2 \cdot \sin\left(\phi(\omega, fr, \tau_{flow}, \tau_{cool}) - \frac{\omega \cdot t}{2}\right)^2 - 2 \cdot \sin\left(\frac{\omega \cdot t}{2}\right)^2 - 2 \cdot \sin\left(\phi(\omega, fr, \tau_{flow}, \tau_{cool})\right)^2 + \omega \cdot \tau_{flow} \cdot \sin(2 \cdot \phi(\omega, fr, \tau_{flow}, \tau_{cool}) - \omega \cdot t) + \omega \cdot \tau_{flow} \cdot \sin(\omega \cdot t) - 2 \cdot \omega^2 \cdot \tau_{cool} \cdot \tau_{flow}^2 \cdot \tau_{cool} + 2 \cdot \tau_{cool} \cdot \omega^2 \cdot \tau_{flow}^2 + 2 \cdot \frac{\tau_{cool}}{\tau_{flow}} \right]}{2 \cdot \tau_{cool} \cdot \omega^2 \cdot \tau_{flow}^2 + 2 \cdot \frac{\tau_{cool}}{\tau_{flow}}} \right] \cdot \tau_{cool}}$$

Comparison of original and simplified expression for Tgas



I will use simplified expression

$$T_{\text{gas}}(\omega, fr, T_0\text{gas}, \tau_{flow}, \tau_{cool}, t) := \frac{\left[ -\frac{2 \cdot \sin\left(\phi(\omega, fr, \tau_{flow}, \tau_{cool}) - \frac{\omega \cdot t}{2}\right)^2 - 2 \cdot \sin\left(\frac{\omega \cdot t}{2}\right)^2 - 2 \cdot \sin\left(\phi(\omega, fr, \tau_{flow}, \tau_{cool})\right)^2 + \omega \cdot \tau_{flow} \cdot \sin(2 \cdot \phi(\omega, fr, \tau_{flow}, \tau_{cool}) - \omega \cdot t) + \omega \cdot \tau_{flow} \cdot \sin(\omega \cdot t) - 2 \cdot \omega^2 \cdot \tau_{flow}^2 \cdot \tau_{cool} + 2 \cdot \tau_{cool} \cdot \omega^2 \cdot \tau_{flow}^2 + 2 \cdot \frac{\tau_{cool}}{\tau_{flow}}} {2 \cdot \tau_{cool} \cdot \omega^2 \cdot \tau_{flow}^2 + 2 \cdot \frac{\tau_{cool}}{\tau_{flow}}} \right] \cdot \tau_{cool}}{2 \cdot \tau_{cool} \cdot \omega^2 \cdot \tau_{flow}^2 + 2 \cdot \frac{\tau_{cool}}{\tau_{flow}}}$$

## Solve for Tsoot

$$T_{\text{gas}} = \left( -\frac{2 \cdot \sin\left(\phi - \frac{\omega \cdot t}{2}\right)^2 - 2 \cdot \sin\left(\frac{\omega \cdot t}{2}\right)^2 - 2 \cdot \sin(\phi)^2 + \omega \cdot \tau_{\text{flow}} \cdot \sin(2 \cdot \phi - \omega \cdot t) + \omega \cdot \tau_{\text{flow}} \cdot \sin(\omega \cdot t) - 2 \cdot \omega^2 \cdot \tau_{\text{flow}}^2 \cdot \sin(\phi)^2}{2 \cdot \tau_{\text{cool}} \cdot \omega^2 \cdot \tau_{\text{flow}} + 2 \cdot \frac{\tau_{\text{cool}}}{\tau_{\text{flow}}}} \right) \cdot \frac{fr \cdot Lasheat}{\omega} + T_{0\text{gas}}$$

$$Tsoot - T_{\text{gas}} = \frac{Lasheat \cdot \tan(\phi)}{\omega} [1 + \cos(\phi) \cdot (\sin(\omega \cdot t - \phi))]$$

$$Tsoot = \left( -\frac{2 \cdot \sin\left(\phi - \frac{\omega \cdot t}{2}\right)^2 - 2 \cdot \sin\left(\frac{\omega \cdot t}{2}\right)^2 - 2 \cdot \sin(\phi)^2 + \omega \cdot \tau_{\text{flow}} \cdot \sin(2 \cdot \phi - \omega \cdot t) + \omega \cdot \tau_{\text{flow}} \cdot \sin(\omega \cdot t) - 2 \cdot \omega^2 \cdot \tau_{\text{flow}}^2 \cdot \sin(\phi)^2}{2 \cdot \tau_{\text{cool}} \cdot \omega^2 \cdot \tau_{\text{flow}} + 2 \cdot \frac{\tau_{\text{cool}}}{\tau_{\text{flow}}}} \right) \cdot \frac{fr \cdot Lasheat}{\omega} + T_{0\text{gas}} + \frac{Lasheat \cdot \tan(\phi)}{\omega} [1 + \cos(\phi) \cdot (\sin(\omega \cdot t - \phi))]$$

$$\omega := \omega \quad t := t \quad \phi := \phi \quad \tau_{\text{flow}} := \tau_{\text{flow}} \quad \tau_{\text{cool}} := \tau_{\text{cool}} \quad Lasheat := Lasheat \quad T_{0\text{gas}} := T_{0\text{gas}} \quad Tsoot := Tsoot$$

$$\left( -\frac{2 \cdot \sin\left(\phi - \frac{\omega \cdot t}{2}\right)^2 - 2 \cdot \sin\left(\frac{\omega \cdot t}{2}\right)^2 - 2 \cdot \sin(\phi)^2 + \omega \cdot \tau_{\text{flow}} \cdot \sin(2 \cdot \phi - \omega \cdot t) + \omega \cdot \tau_{\text{flow}} \cdot \sin(\omega \cdot t) - 2 \cdot \omega^2 \cdot \tau_{\text{flow}}^2 \cdot \sin(\phi)^2}{2 \cdot \tau_{\text{cool}} \cdot \omega^2 \cdot \tau_{\text{flow}} + 2 \cdot \frac{\tau_{\text{cool}}}{\tau_{\text{flow}}}} \right) \cdot \frac{fr \cdot Lasheat}{\omega} + T_{0\text{gas}} + \frac{Lasheat \cdot \tan(\phi)}{\omega} [1 + \cos(\phi) \cdot (\sin(\omega \cdot t - \phi))] \text{ collect, Lasheat}$$

$$Tsoot = \left[ \frac{\tan(\phi) \cdot (\sin(\omega \cdot t - \phi) \cdot \cos(\phi) + 1)}{\omega} - \frac{fr \cdot \left( 2 \cdot \sin\left(\phi - \frac{\omega \cdot t}{2}\right)^2 - 2 \cdot \sin\left(\frac{\omega \cdot t}{2}\right)^2 - 2 \cdot \sin(\phi)^2 + \omega \cdot \tau_{\text{flow}} \cdot \sin(\omega \cdot t) + \omega \cdot \tau_{\text{flow}} \cdot \sin(2 \cdot \phi - \omega \cdot t) - 2 \cdot \omega^2 \cdot \tau_{\text{flow}}^2 \cdot \sin(\phi)^2 \right)}{\omega \cdot \left( \frac{2 \cdot \tau_{\text{cool}}}{\tau_{\text{flow}}} + 2 \cdot \omega^2 \cdot \tau_{\text{flow}} \cdot \tau_{\text{cool}} \right)} \right] \cdot Lasheat + T_{0\text{gas}}$$

I Have not found a way to simplify this symbolic expression other than the obvious of collecting Lasheat terms I have tried expressing time constants as phase terms e.g.  $\omega \cdot \tau_{\text{col}} = \tan(\phi_{\text{col}})$  but it produced little useful simplification

$$\phi := \phi \quad T_{0\text{gas}} := T_{0\text{gas}} \quad \tau_{\text{flow}} := \tau_{\text{flow}} \quad \tau_{\text{cool}} := \tau_{\text{cool}} \quad \omega := \omega \quad Lasheat := Lasheat \quad fr := fr$$

$$\frac{d}{dt} \left[ \frac{\tan(\phi) \cdot (\sin(\omega \cdot t - \phi) \cdot \cos(\phi) + 1)}{\omega} - \frac{fr \cdot \left( 2 \cdot \sin\left(\phi - \frac{\omega \cdot t}{2}\right)^2 - 2 \cdot \sin\left(\frac{\omega \cdot t}{2}\right)^2 - 2 \cdot \sin(\phi)^2 + \omega \cdot \tau_{flow} \cdot \sin(\omega \cdot t) + \omega \cdot \tau_{flow} \cdot \sin(2 \cdot \phi - \omega \cdot t) - 2 \cdot \omega^2 \cdot \tau_{flow}^2 \cdot \sin(\phi)^2 \right)}{\omega \cdot \left( \frac{2 \cdot \tau_{cool}}{\tau_{flow}} + 2 \cdot \omega^2 \cdot \tau_{flow} \cdot \tau_{cool} \right)} \right] \bullet \text{Lasheat} + T_0 \text{gas} \right] \text{collect, Lasheat}$$

$$Dif1 = \left[ \cos(\omega \cdot t - \phi) \cdot \cos(\phi) \cdot \tan(\phi) + \frac{fr \cdot \left( 2 \cdot \omega \cdot \cos\left(\phi - \frac{\omega \cdot t}{2}\right) \cdot \sin\left(\phi - \frac{\omega \cdot t}{2}\right) - \omega^2 \cdot \tau_{flow} \cdot \cos(\omega \cdot t) + \omega^2 \cdot \tau_{flow} \cdot \cos(2 \cdot \phi - \omega \cdot t) + 2 \cdot \omega \cdot \cos\left(\frac{\omega \cdot t}{2}\right) \cdot \sin\left(\frac{\omega \cdot t}{2}\right) \right)}{\omega \cdot \left( \frac{2 \cdot \tau_{cool}}{\tau_{flow}} + 2 \cdot \omega^2 \cdot \tau_{flow} \cdot \tau_{cool} \right)} \right] \bullet \text{Lasheat}$$

$$\left[ \cos(\omega \cdot t - \phi) \cdot \cos(\phi) \cdot \tan(\phi) + \frac{fr \cdot \left( 2 \cdot \omega \cdot \cos\left(\phi - \frac{\omega \cdot t}{2}\right) \cdot \sin\left(\phi - \frac{\omega \cdot t}{2}\right) - \omega^2 \cdot \tau_{flow} \cdot \cos(\omega \cdot t) + \omega^2 \cdot \tau_{flow} \cdot \cos(2 \cdot \phi - \omega \cdot t) + 2 \cdot \omega \cdot \cos\left(\frac{\omega \cdot t}{2}\right) \cdot \sin\left(\frac{\omega \cdot t}{2}\right) \right)}{\omega \cdot \left( \frac{2 \cdot \tau_{cool}}{\tau_{flow}} + 2 \cdot \omega^2 \cdot \tau_{flow} \cdot \tau_{cool} \right)} \right] \bullet \text{Lasheat} = 0 \text{ solve, } t \rightarrow$$

I Tried above to find the value of t that would make the first differential zero. That would've given me an expression which identified the first turning point (maximum or minimum).

Calculate numerical results

$$fr(T, svf) := \frac{HC_{soot}(T, svf)}{HC_{air}(T)} \quad \phi_c(\omega, \tau_{cool}) := \text{atan}(\omega \cdot \tau_{cool}) \quad \phi_f(\omega, \tau_{flow}) := \text{atan}(\omega \cdot \tau_{flow}) \quad \phi(\omega, \tau_{flow}, \tau_{cool}, svf, T_0 \text{gas}) := \text{atan} \left[ \frac{\omega}{\left( \frac{1 + fr(T_0 \text{gas}, svf)}{\tau_{cool}} + \frac{1}{\tau_{flow}} \right)} \right]$$

$$T_{soot}(\omega, T_0 \text{gas}, \tau_{flow}, \tau_{cool}, svf, t) := \left[ \frac{\tan(\phi(\omega, \tau_{flow}, \tau_{cool}, svf, T_0 \text{gas})) \cdot (\sin(\omega \cdot t - \phi(\omega, \tau_{flow}, \tau_{cool}, svf, T_0 \text{gas})) \cdot \cos(\phi(\omega, \tau_{flow}, \tau_{cool}, svf, T_0 \text{gas})) + 1)}{\omega} - \frac{fr(T_0 \text{gas}, svf) \cdot \left( 2 \cdot \sin\left(\phi(\omega, \tau_{flow}, \tau_{cool}, svf, T_0 \text{gas}) - \frac{\omega \cdot t}{2}\right)^2 \right.} \right]$$

$$T_{gas}(\omega, T_0 \text{gas}, \tau_{flow}, \tau_{cool}, svf, t) := \left[ \frac{-2 \cdot \sin\left(\phi(\omega, \tau_{flow}, \tau_{cool}, svf, T_0 \text{gas}) - \frac{\omega \cdot t}{2}\right)^2 - 2 \cdot \sin\left(\frac{\omega \cdot t}{2}\right)^2 - 2 \cdot \sin(\phi(\omega, \tau_{flow}, \tau_{cool}, svf, T_0 \text{gas}))^2 + \omega \cdot \tau_{flow} \cdot \sin(2 \cdot \phi(\omega, \tau_{flow}, \tau_{cool}, svf, T_0 \text{gas}) - \omega \cdot t) + \omega \cdot \tau_{flow} \cdot \sin(2 \cdot \phi(\omega, \tau_{flow}, \tau_{cool}, svf, T_0 \text{gas}) + \omega \cdot t)}{2 \cdot \tau_{cool} \cdot \omega^2 \cdot \tau_{flow} + 2 \cdot \frac{\tau_{cool}}{\tau_{flow}}} \right]$$

$$\text{Lasheat}(P_d, T) = \frac{6 \cdot P_d \cdot \pi \cdot E_m}{\lambda_{laser} \cdot fC_p(T) \cdot RHOS}$$

$$Tdif(T_0 \text{gas}, \text{Lasheat}, \omega, fr, \tau_{flow}, \tau_{cool}, t) := \frac{\text{Lasheat}(P_d, T_0 \text{gas}) \cdot \tan(\phi(\omega, \tau_{flow}, \tau_{cool}, svf, T_0 \text{gas}))}{\omega} \cdot (1 + \cos(\phi(\omega, \tau_{flow}, \tau_{cool}, svf, T_0 \text{gas})) \cdot \sin(\omega \cdot t - \phi(\omega, \tau_{flow}, \tau_{cool}, svf, T_0 \text{gas})))$$

f := 2500

f kHz simulation  $\tau_{\text{flow}}=0.0006$  sec and  $\tau_{\text{cooling}}=2 \times 10^{-6}$  sec, 2.5 ppm soot

Temperature K



Calculate numeric first and second derivatives of Tsoot equation

$$\phi := \phi \quad T_{0\text{gas}} := T_{0\text{gas}} \quad \tau_{\text{flow}} := \tau_{\text{flow}} \quad \tau_{\text{cool}} := \tau_{\text{cool}} \quad \omega := \omega \quad \text{Lasheat} := \text{Lasheat} \quad fr := fr$$

$$\text{Dif1} = \left[ \cos(\omega \cdot t - \phi) \cdot \cos(\phi) \cdot \tan(\phi) + \frac{fr \cdot \left( 2 \cdot \omega \cdot \cos\left(\phi - \frac{\omega \cdot t}{2}\right) \cdot \sin\left(\phi - \frac{\omega \cdot t}{2}\right) - \omega^2 \cdot \tau_{\text{flow}} \cdot \cos(\omega \cdot t) + \omega^2 \cdot \tau_{\text{flow}} \cdot \cos(2 \cdot \phi - \omega \cdot t) + 2 \cdot \omega \cdot \cos\left(\frac{\omega \cdot t}{2}\right) \cdot \sin\left(\frac{\omega \cdot t}{2}\right) \right)}{\omega \cdot \left( \frac{2 \cdot \tau_{\text{cool}}}{\tau_{\text{flow}}} + 2 \cdot \omega^2 \cdot \tau_{\text{flow}} \cdot \tau_{\text{cool}} \right)} \right] \cdot \text{Lasheat}$$

$$\text{Lasheat}(P_d, T) := \frac{6 \cdot P_d \cdot \pi \cdot E_m}{\lambda_{\text{laser}} \cdot fC_p \left( \frac{T}{K} \right) \cdot RHOS}$$

$$fr(T, svf) := \frac{HC_{\text{soot}}(T, svf)}{HC_{\text{air}}(T)}$$

$$\phi(\omega, \tau_{\text{flow}}, \tau_{\text{cool}}, T_{0\text{gas}}, svf) := \text{atan} \left[ \frac{\omega}{\left( \frac{1 + fr(T_{0\text{gas}}, svf)}{\tau_{\text{cool}}} + \frac{1}{\tau_{\text{flow}}} \right)} \right]$$

$$\text{dif1}(\omega, \tau_{\text{flow}}, \tau_{\text{cool}}, T_0_{\text{gas}}, \text{svf}, t) := \left[ \cos(\omega \cdot t - \phi(\omega, \tau_{\text{flow}}, \tau_{\text{cool}}, T_0_{\text{gas}}, \text{svf})) \cdot \cos(\phi(\omega, \tau_{\text{flow}}, \tau_{\text{cool}}, T_0_{\text{gas}}, \text{svf})) \cdot \tan(\phi(\omega, \tau_{\text{flow}}, \tau_{\text{cool}}, T_0_{\text{gas}}, \text{svf})) + \frac{\text{fr}(T_0_{\text{gas}}, \text{svf}) \cdot \left( 2 \cdot \omega \cdot \cos(\phi(\omega, \tau_{\text{flow}}, \tau_{\text{cool}}, T_0_{\text{gas}}, \text{svf})) - \frac{\omega \cdot t}{2} \right) \cdot \sin(\phi(\omega, \tau_{\text{flow}}, \tau_{\text{cool}}, T_0_{\text{gas}}, \text{svf}))}{2} \right]$$

$$\text{dif1}\left(\frac{1000 \cdot 2 \cdot \pi}{\text{sec}}, 0.00045 \cdot \text{sec}, 1 \cdot 10^{-6} \cdot \text{sec}, 1730 \cdot K, 2.5 \cdot 10^{-6}, .0002474 \cdot \text{sec}\right) = \blacksquare$$

$$\phi := \phi \quad T_0_{\text{gas}} := T_0_{\text{gas}} \quad \tau_{\text{flow}} := \tau_{\text{flow}} \quad \tau_{\text{cool}} := \tau_{\text{cool}} \quad \omega := \omega \quad \text{Lasheat} := \text{Lasheat} \quad t := t \quad \phi_f := \phi_f \quad \text{fr} := \text{fr}$$

$$\frac{d}{dt} \left[ \left[ \cos(\omega \cdot t - \phi) \cdot \cos(\phi) \cdot \tan(\phi) + \frac{\text{fr} \cdot \left( 2 \cdot \omega \cdot \cos\left(\phi - \frac{\omega \cdot t}{2}\right) \cdot \sin\left(\phi - \frac{\omega \cdot t}{2}\right) - \omega^2 \cdot \tau_{\text{flow}} \cdot \cos(\omega \cdot t) + \omega^2 \cdot \tau_{\text{flow}} \cdot \cos(2 \cdot \phi - \omega \cdot t) + 2 \cdot \omega \cdot \cos\left(\frac{\omega \cdot t}{2}\right) \cdot \sin\left(\frac{\omega \cdot t}{2}\right) \right)}{\omega \cdot \left( \frac{2 \cdot \tau_{\text{cool}}}{\tau_{\text{flow}}} + 2 \cdot \omega^2 \cdot \tau_{\text{flow}} \cdot \tau_{\text{cool}} \right)} \right] \cdot \text{Lasheat} \right] \rightarrow \text{Lasheat} \cdot \left[ \frac{\text{fr} \cdot \left( \omega^2 \cdot \sin\left(\phi - \frac{\omega \cdot t}{2}\right) - \omega^2 \cdot \cos\left(\phi - \frac{\omega \cdot t}{2}\right)^2 + \omega^2 \cdot \cos\left(\frac{\omega \cdot t}{2}\right)^2 - \omega^2 \cdot \sin\left(\frac{\omega \cdot t}{2}\right)^2 + \omega^3 \cdot \tau_{\text{flow}} \cdot \sin(\omega \cdot t) + \omega^3 \cdot \tau_{\text{flow}} \cdot \sin(2 \cdot \phi - \omega \cdot t) \right)}{\omega \cdot \left( \frac{2 \cdot \tau_{\text{cool}}}{\tau_{\text{flow}}} + 2 \cdot \omega^2 \cdot \tau_{\text{flow}} \cdot \tau_{\text{cool}} \right)} - \omega \cdot \sin(\omega \cdot t - \phi) \cdot \cos(\phi) \cdot \tan(\phi) \right]$$

$$\text{Dif2} = \text{Lasheat} \cdot \left[ \frac{\text{fr} \cdot \left( \omega^2 \cdot \sin\left(\phi - \frac{\omega \cdot t}{2}\right)^2 - \omega^2 \cdot \cos\left(\phi - \frac{\omega \cdot t}{2}\right)^2 + \omega^2 \cdot \cos\left(\frac{\omega \cdot t}{2}\right)^2 - \omega^2 \cdot \sin\left(\frac{\omega \cdot t}{2}\right)^2 + \omega^3 \cdot \tau_{\text{flow}} \cdot \sin(\omega \cdot t) + \omega^3 \cdot \tau_{\text{flow}} \cdot \sin(2 \cdot \phi - \omega \cdot t) \right)}{\omega \cdot \left( \frac{2 \cdot \tau_{\text{cool}}}{\tau_{\text{flow}}} + 2 \cdot \omega^2 \cdot \tau_{\text{flow}} \cdot \tau_{\text{cool}} \right)} - \omega \cdot \sin(\omega \cdot t - \phi) \cdot \cos(\phi) \cdot \tan(\phi) \right]$$

$$\text{dif2}(\omega, \tau_{\text{flow}}, \tau_{\text{cool}}, T_0_{\text{gas}}, \text{svf}, t) := \text{Lasheat} \cdot \left[ \frac{\text{fr}(T_0_{\text{gas}}, \text{svf}) \cdot \left( \omega^2 \cdot \sin\left(\phi(\omega, \tau_{\text{flow}}, \tau_{\text{cool}}, T_0_{\text{gas}}, \text{svf}) - \frac{\omega \cdot t}{2}\right)^2 - \omega^2 \cdot \cos\left(\phi(\omega, \tau_{\text{flow}}, \tau_{\text{cool}}, T_0_{\text{gas}}, \text{svf}) - \frac{\omega \cdot t}{2}\right)^2 + \omega^2 \cdot \cos\left(\frac{\omega \cdot t}{2}\right)^2 - \omega^2 \cdot \sin\left(\frac{\omega \cdot t}{2}\right)^2 + \omega^3 \cdot \tau_{\text{flow}} \cdot \sin(\omega \cdot t) \right)}{\omega \cdot \left( \frac{2 \cdot \tau_{\text{cool}}}{\tau_{\text{flow}}} + 2 \cdot \omega^2 \cdot \tau_{\text{flow}} \cdot \tau_{\text{cool}} \right)} \right]$$

$$\text{dif2}\left(\frac{1000 \cdot 2 \cdot \pi}{\text{sec}}, 0.00045 \cdot \text{sec}, 1 \cdot 10^{-6} \cdot \text{sec}, 1730 \cdot K, 2.5 \cdot 10^{-6}, .0002474 \cdot \text{sec}\right) = \blacksquare$$

$$\text{freq} := \text{stack}(25, 100, 500, 1000, 2500, 5000, 7500, 10000, 20000, 30000, 40000, 100000)$$

$$\text{din} := 1 .. \text{rows(freq)}$$

$$\tau_{\text{flow}} = 0.0005 \quad \tau_{\text{cool}}$$

$$\text{CTOL} = 0.001 \quad \text{TOL} = 0.001$$

$$\text{Guess} \quad x := -.25$$

Given

$$\text{dif1}\left(\frac{\text{freq}_{\text{din}} \cdot 2 \cdot \pi}{\text{sec}}, \tau_{\text{flow}} \cdot \text{sec}, \tau_{\text{cool}} \cdot \text{sec}, 1730 \cdot K, \text{svf}, \frac{x}{\text{freq}_{\text{din}}} \cdot \text{sec}\right) = 0 \quad x < 0 \quad x > \frac{-\pi}{2} \quad \frac{1}{\text{freq}}$$

$$\text{Res}(\text{din}) := \text{Find}(x)$$

The solve block finds the value of X for which the function dif1 is 0. dif1 is  $\frac{d}{dt} T_{\text{soot}}$  and we are finding the time, x, in units of  $\frac{1}{\text{freq}}$  or the period of the laser driving

function. There are multiple such minima and maxima and the solve block is limited to the interval x of  $-\pi/2$  to 0

$$\frac{1}{\text{freq}_{\text{din}}} = \begin{pmatrix} 0.04 \\ 0.01 \\ 0.002 \\ 0.001 \\ 0.0004 \\ 0.0002 \\ 0.000133 \\ 0.0001 \\ 5 \times 10^{-5} \\ 3.333333 \times 10^{-5} \\ 2.5 \times 10^{-5} \\ 1 \times 10^{-5} \end{pmatrix}$$

Res(din) = ■

$$\frac{\text{atan}\left(\text{Res}(\text{din}) \cdot 2 \cdot \pi + \frac{\pi}{2}\right)}{\text{deg}} = ■$$

$$\text{phasetheory}_{\text{din}} := \frac{\text{atan}\left(\text{Res}(\text{din}) \cdot 2 \cdot \pi + \frac{\pi}{2}\right)}{\text{deg}}$$

What is dif1 for the solve  
blocksolutions

$$\text{dif1}\left(\frac{\text{freq}_{\text{din}} \cdot 2 \cdot \pi}{\text{sec}}, \tau_{\text{flow}} \cdot \text{sec}, \tau_{\text{cool}} \cdot \text{sec}, 1730 \cdot K, \text{svf}, \frac{\text{Res}(\text{din})}{\text{freq}_{\text{din}}} \cdot \text{sec}\right) = ■$$

Guess     $x := -.08$

$$\text{Given} \quad \text{dif1}\left(\frac{\text{freq}_5 \cdot 2 \cdot \pi}{\text{sec}}, \tau_{\text{flow}} \cdot \text{sec}, \tau_{\text{cool}} \cdot \text{sec}, 1730 \cdot K, 2.5 \cdot 10^{-6}, \frac{x}{\text{freq}_5} \cdot \text{sec}\right) = 0$$

Res2 := Find(x) = ■

$$\text{dif1}\left(\frac{\text{freq}_5 \cdot 2 \cdot \pi}{\text{sec}}, \tau_{\text{flow}} \cdot \text{sec}, \tau_{\text{cool}} \cdot \text{sec}, 1730 \cdot K, \text{svf}, \frac{\text{Res2}}{\text{freq}_5} \cdot \text{sec}\right) = ■$$

$$\frac{\text{Res2}}{\text{Res}(5)} - 1 = ■$$

This individual fit was an attempt to see if a more accurate solution would be found by starting the fit with a guess value very close to the solution obtained in the first solve block. Indeed

it did find a more accurate solution as evidenced by the fact that the value of dif1 at this new solution was approximately 1/3 the original value. The ratio of the two solutions

$$\frac{\text{Res2}}{\text{Res}(5)} - 1 = \blacksquare \quad \text{indicates that the improvement was in the 15th decimal place – not significant}$$

$$\text{dif2} \left( \frac{\text{freq}_{\text{din}} \cdot 2 \cdot \pi}{\text{sec}}, \tau_{\text{flow}} \cdot \text{sec}, \tau_{\text{cool}} \cdot \text{sec}, 1730 \cdot K, \text{svf}, \frac{\text{Res}(\text{din})}{\text{freq}_{\text{din}}} \cdot \text{sec} \right) = \blacksquare$$

The value of the second differential for each of the solutions is positive indicating that we have found a minimum turning point rather than a maximum.

$$\begin{matrix} \text{orig} := \\ \begin{pmatrix} 5.15 \\ 31.58 \\ 45.15 \\ 49.16 \\ 47.59 \\ 30.38 \\ 22.1 \\ 20.99 \end{pmatrix} \end{matrix}$$

## Experimental data

phase := stack(0, 29, 53, 65, 57, 29, 25, 32)

$$\text{phase} := \begin{pmatrix} -0.6 & 0.3 \\ 0.4 & 2.7 \\ 1.2 & -0.4 \\ -12.2 & -11.6 \\ -46.8 & -46.4 \\ -59.8 & -59.5 \\ -56.0 & -56.8 \\ -40.8 & -40.9 \\ -32.0 & -33.3 \\ -27.9 & -28.7 \\ -26.2 & -25.7 \\ -27.9 & -27.3 \\ -28.9 & -31.9 \\ -31.9 & -30.6 \end{pmatrix}$$

$$\text{phaseexp} := -\frac{\text{phase}^{\langle 1 \rangle} + \text{phase}^{\langle 2 \rangle}}{2}$$

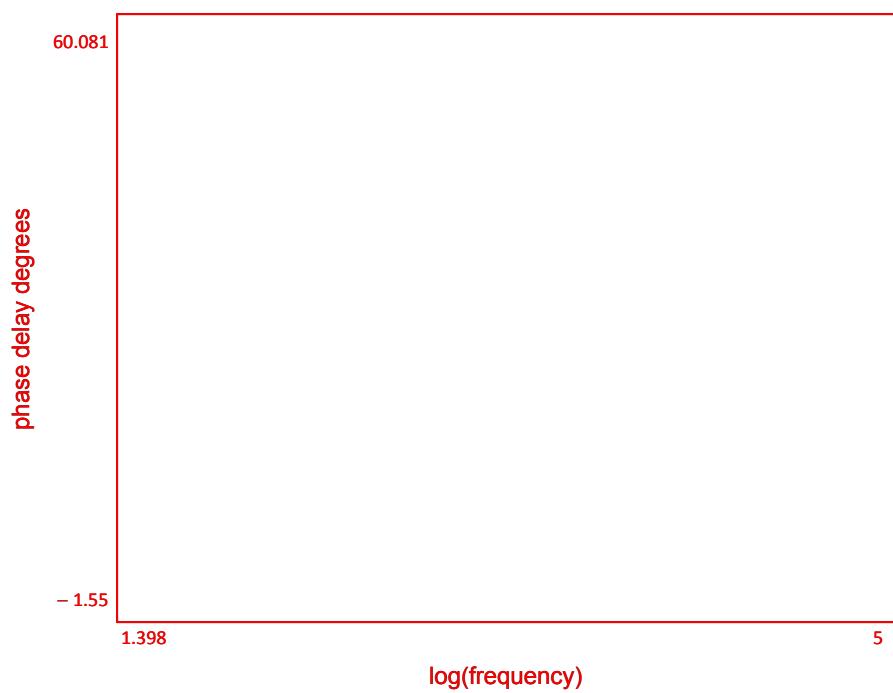
$$\text{freqexp} := \begin{pmatrix} 25 \\ 25 \\ 25 \\ 100 \\ 500 \\ 1000 \\ 2500 \\ 5000 \\ 7500 \\ 10000 \\ 20000 \\ 30000 \\ 40000 \\ 40000 \end{pmatrix}$$

$$\text{phaseexp} = \begin{pmatrix} 0.15 \\ -1.55 \\ -0.4 \\ 11.9 \\ 46.6 \\ 59.65 \\ 56.4 \\ 40.85 \\ 32.65 \\ 28.3 \\ 25.95 \\ 27.6 \\ 30.4 \\ 31.25 \end{pmatrix}$$

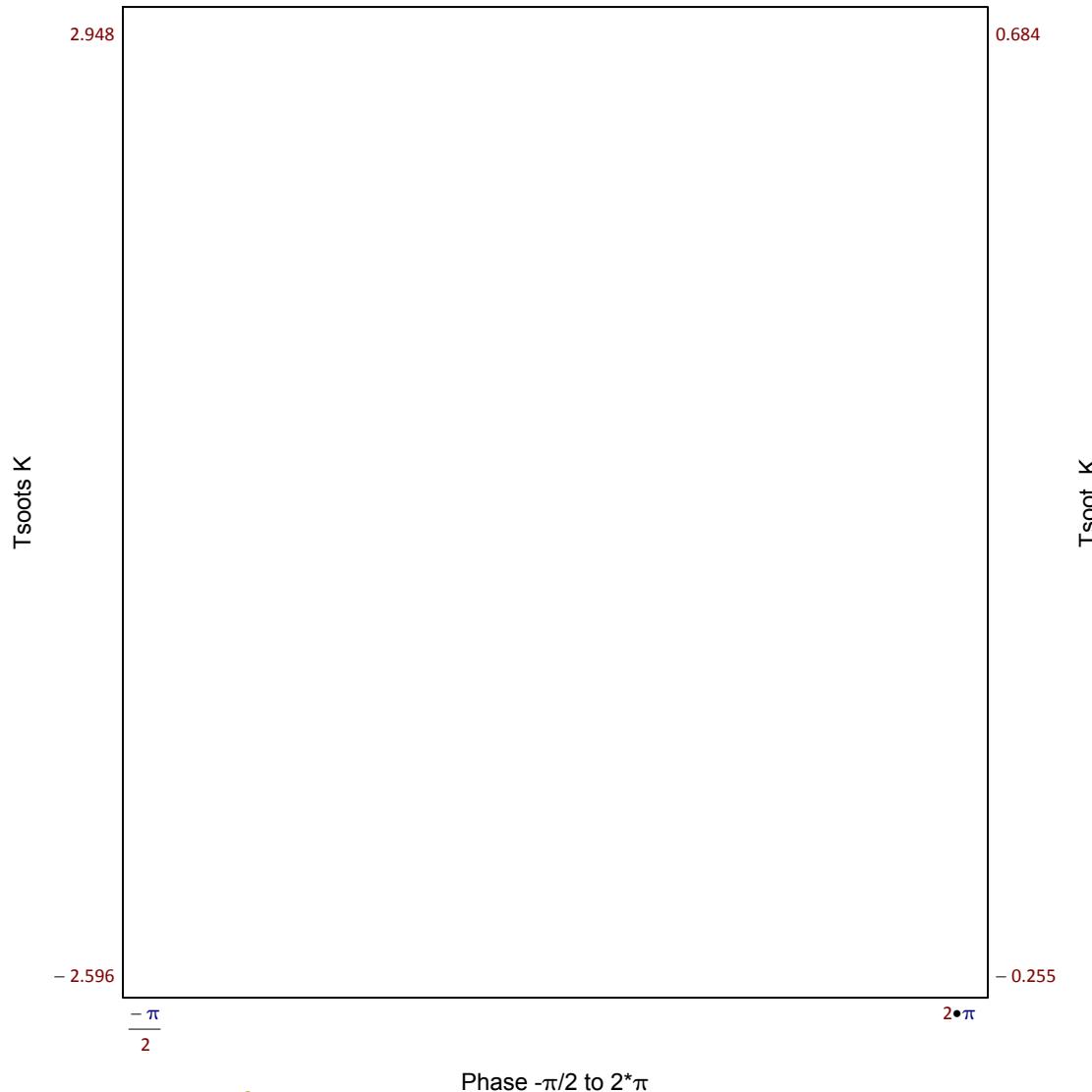
`vs_phase := loess(log(freqexp), phaseexp, 0.5)`

`fit_phase(z) := interp(vs_phase, log(freqexp), phaseexp, z)`

Phase delay



f kHz simulation  $\tau_{\text{flow}}=0.0004$  sec and  $\tau_{\text{cooling}}=1*10^{-6}$  sec, 2.5 ppm soot



Calculate amplitude of signal

`ind := 1 .. rows(freq)`

$$\text{Amp}_{\text{ind}} := \text{Tsoot} \left( \frac{2 \cdot \pi \cdot \text{freq}_{\text{ind}}}{\text{sec}}, 1730 \cdot K, \tau_{\text{flow}} \cdot \text{sec}, \tau_{\text{cool}} \cdot \text{sec}, 2.5 \cdot 10^{-6}, \frac{\text{Res(ind)} + .5}{\text{freq}_{\text{ind}}} \cdot \text{sec} \right) - \text{Tsoot} \left( \frac{2 \cdot \pi \cdot \text{freq}_{\text{ind}}}{\text{sec}}, 1730 \cdot K, \tau_{\text{flow}} \cdot \text{sec}, \tau_{\text{cool}} \cdot \text{sec}, 2.5 \cdot 10^{-6}, \frac{\text{Res(ind)}}{\text{freq}_{\text{ind}}} \cdot \text{sec} \right)$$

$$\text{Amp}_{\text{ind}} := \frac{\text{Amp}_{\text{ind}}}{\text{scale}} = \blacksquare$$

$$\text{scale} := \frac{\text{Amp}_1}{K} = \blacksquare$$

$$\text{up}_{\text{ind}} := \text{Tsoot} \left( \frac{2 \cdot \pi \cdot \text{freq}_{\text{ind}}}{\text{sec}}, 1730 \cdot K, .0004 \cdot \text{sec}, 1 \cdot 10^{-6} \cdot \text{sec}, 2.5 \cdot 10^{-6}, \frac{\text{Res(ind)} + .5}{\text{freq}_{\text{ind}}} \cdot \text{sec} \right) - 1730 \cdot K$$

$$\text{lo}_{\text{ind}} := \text{Tsoot} \left( \frac{2 \cdot \pi \cdot \text{freq}_{\text{ind}}}{\text{sec}}, 1730 \cdot K, .0004 \cdot \text{sec}, 1 \cdot 10^{-6} \cdot \text{sec}, 2.5 \cdot 10^{-6}, \frac{\text{Res(ind)}}{\text{freq}_{\text{ind}}} \cdot \text{sec} \right) - 1730 \cdot K$$

$$\text{Amp}_1 = \blacksquare$$

$$\text{Amp} = \blacksquare$$

$$\text{up} = \blacksquare$$

$$\text{lo} = \blacksquare$$

$$\text{Amp} = \blacksquare$$

$$\text{Amp}_1 = \blacksquare$$

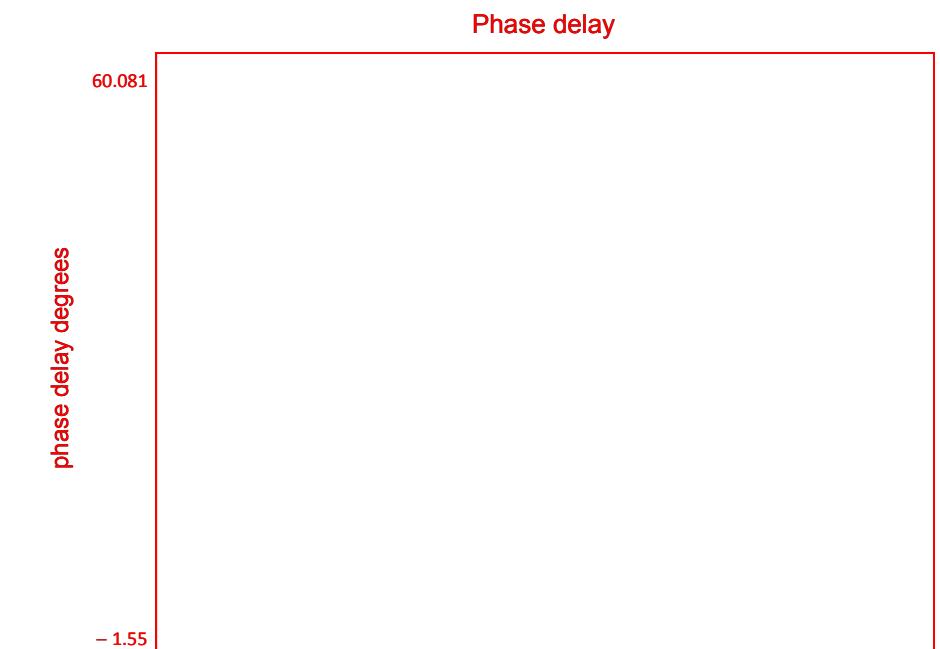
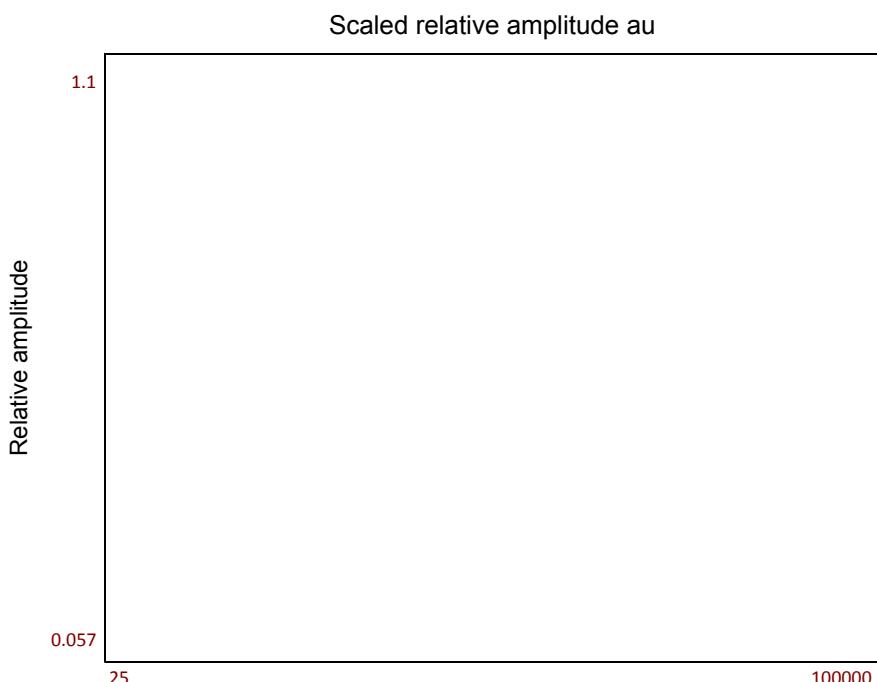
Are the negatives a function of the IC (or not having solved for it)?

$$\text{vs_amp} := \text{loess} \left( \ln(\text{freq}), \frac{\text{Amp}}{K}, .45 \right)$$

$$\text{fit_amp}(z) := \text{interp} \left( \text{vs_amp}, \ln(\text{freq}), \frac{\text{Amp}}{K}, \ln(z) \right)$$

<pre> Ampexp := {35907.3 1683472.3            17059.5 830642.6            9655.9 439698.0            32184.7 1599872.3            19599.7 1007944.8            11324.3 589196.2            6072.4 262902.2            3347.0 179169.1            3052.8 160848.2            2816.7 146999.5            2685.5 138556.5            2545.3 137386.2            2641.4 138765.5            2654.6 141106.1} </pre>	<pre> freqexp := {25              25              25              100              500              1000              2500              5000              7500              10000              20000              30000              40000              40000} </pre>	<pre> laserint := stack(1, .5, .3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1) </pre>	$\text{Ampexp}^{(1)} := \frac{\text{Ampexp}}{\text{laserint}}$ $\text{Ampexp}^{(2)} := \frac{\text{Ampexp}}{\text{laserint}}$ $\text{Ampexp}^{(1)} := \frac{\text{Ampexp}}{\text{Ampexp}_{1, 1}}$ $\text{Ampexp}^{(2)} := \frac{\text{Ampexp}}{\text{Ampexp}_{1, 2}}$	<pre> Ampexp = {35907.3 1.683472 × 10<sup>6</sup>            34119 1.661285 × 10<sup>6</sup>            32186.333333 1.46566 × 10<sup>6</sup>            32184.7 1.599872 × 10<sup>6</sup>            19599.7 1.007945 × 10<sup>6</sup>            11324.3 5.891962 × 10<sup>5</sup>            6072.4 2.629022 × 10<sup>5</sup>            3347 1.791691 × 10<sup>5</sup>            3052.8 1.608482 × 10<sup>5</sup>            2816.7 1.469995 × 10<sup>5</sup>            2685.5 1.385565 × 10<sup>5</sup>            2545.3 1.373862 × 10<sup>5</sup>            2641.4 1.387655 × 10<sup>5</sup>            2654.6 1.411061 × 10<sup>5</sup>} </pre>
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Is this a comparison of temperature and signal intensity? Is that an appropriate comparison?



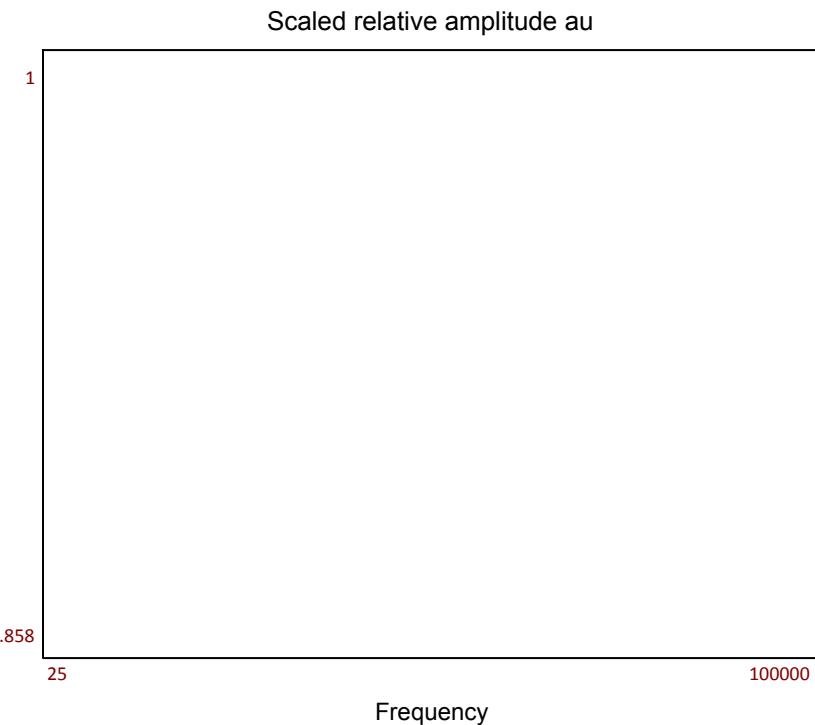
Frequency

1.398

5

log(frequency)

$$\tau_{\text{flow}} \equiv .0005 \quad \tau_{\text{cool}} \equiv 1.6 \cdot 10^{-6} \quad svf \equiv 2.5 \cdot 10^{-6}$$



$$\begin{aligned}\tau_{\text{flow}} &\equiv .0006 & \tau_{\text{cool}} &\equiv 2 \cdot 10^{-6} & svf &\equiv 2.5 \cdot 10^{-6} \\ \tau_{\text{flow}} &\equiv .0005 & \tau_{\text{cool}} &\equiv 1.6 \cdot 10^{-6} & svf &\equiv 2.5 \cdot 10^{-6} \\ \tau_{\text{flow}} &\equiv .0004 & \tau_{\text{cool}} &\equiv 1.2 \cdot 10^{-6} & svf &\equiv 2.5 \cdot 10^{-6}\end{aligned}$$

## Cooling rate

$$T_0 \text{gas} = 1730 \text{K}$$

$$\frac{dT}{dt} = - \frac{12 \cdot KAD \cdot \alpha \cdot (T - T_{\text{gas}})}{[Rg + (G(T_{\text{gas}})) \cdot MFP] \cdot (fCp(T) \cdot RHOS \cdot dp)}$$

$$\frac{T - T_{\text{gas}}}{T_{\text{initial}} - T_{\text{gas}}} = \exp \left[ - \frac{12 \cdot KAD \cdot \alpha \cdot t}{Rg + (G(T_{\text{gas}})) \cdot MFP) \cdot (Cp_{\text{fix}} \cdot RHOS \cdot dp)} \right] = \exp \left( \frac{t}{\tau} \right)$$

Since  $T_{\text{gas}}$  is essentially constant - if we ignore gas heating - then the soot temperature decays exponentially with a time constant  $\tau$  given below where we have ignored the mean aggregate size  $Rg$  as being negligible in the free molecular regime of a flame

$$\tau = \frac{[(G(T_{\text{gas}})) \cdot MFP) \cdot (Cp \cdot RHOS \cdot dp)]}{12 \cdot KAD \cdot \alpha}$$

Create Eucken factor equation:

$$vs := \text{loess}(T_{\text{Eucken}}, \text{Eucken}, 0.8)$$

$$vs := \text{lspline}(T_{\text{Eucken}}, \text{Eucken})$$

$$f(\text{temp}) := \text{interp}(vs, T_{\text{Eucken}}, \text{Eucken}, \text{temp})$$

$$KAD := .001 \cdot \left( fTC \left( \frac{T_{\text{gas}}}{K} \right) \right) \cdot \frac{\text{watt}}{\text{m} \cdot \text{K}}$$

$$MFP := \frac{.001 \cdot \left( fTC \left( \frac{T_{\text{gas}}}{K} \right) \right) \cdot \frac{\text{watt}}{\text{m} \cdot \text{K}}}{1 \cdot \text{atm} \cdot Cv_{\text{air}} \left( \frac{T_{\text{gas}}}{K} \right) \cdot f \left( \frac{T_{\text{gas}}}{K} \right)} \cdot \left[ \frac{\pi \cdot R \cdot T_{\text{gas}}}{2 \cdot \left( 28.96 \cdot \frac{\text{gm}}{\text{mole}} \right)} \right]^{\frac{1}{2}}$$

$$MFP = \blacksquare$$

$$Cp := fCp \left( \frac{T_{\text{gas}}}{K} \right)$$

Assume

$$\alpha := 0.2$$

$$\tau := \frac{[(G \left( \frac{T_{\text{gas}}}{K} \right)) \cdot MFP) \cdot (Cp \cdot RHOS \cdot 30 \cdot \text{nm})]}{12 \cdot KAD \cdot \alpha}$$

$$\tau = \blacksquare$$

## Stop point

## Calculate modulated radiation

$$\frac{1}{40} = 0.025$$

Theoretical Radiation power of a Soot Particle:

Using Wein approximation

$$1807.2 - 1731 = 76.2$$

$$\text{Rad\_Par} = \frac{8 \cdot \pi^3 \cdot h \cdot c^2 \cdot dp^3}{\lambda^6 \cdot \left( \exp\left(\frac{h \cdot c}{k_b \cdot \lambda \cdot TP}\right) \right)} \cdot Em$$

where Em is the scattering function

phase<sub>theory</sub> = ■

The total (over 4π steradians) spectral power radiated at wavelength λ by a single particle of diameter dp,

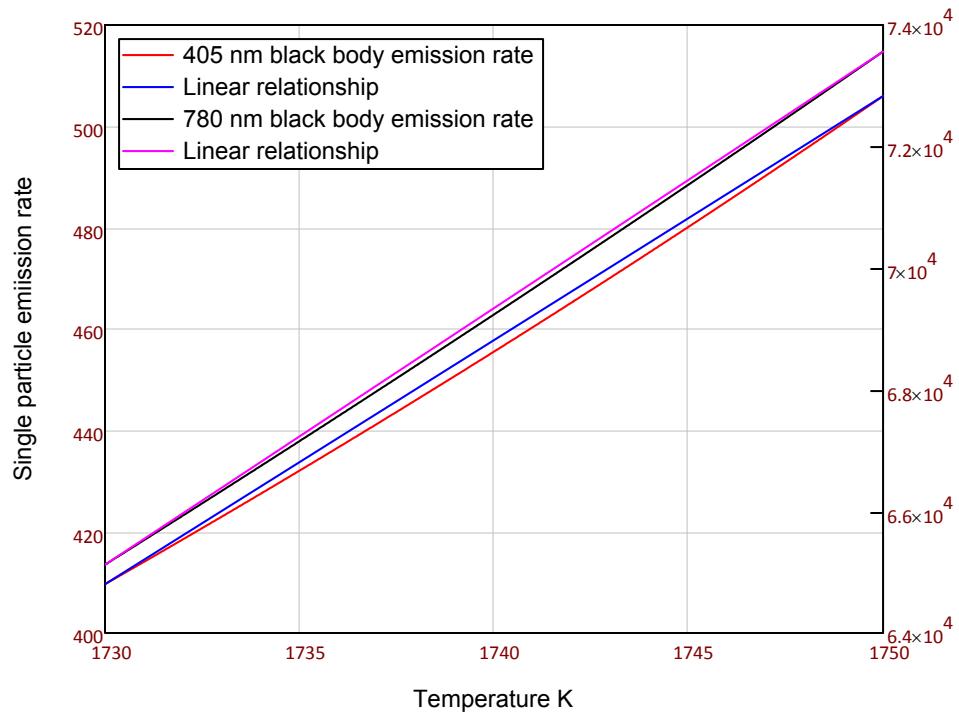
If we want the signal in photons/sec rather than watts we must divide by  $\frac{h \cdot c}{\lambda}$  the energy of a single photon

$$\text{Pho\_Par} = \frac{8 \cdot \pi^3 \cdot c \cdot dp^3}{\lambda^5 \cdot \left( \exp\left(\frac{h \cdot c}{k_b \cdot \lambda \cdot TP}\right) \right)} \cdot Em$$

$$\text{Pho\_Par}(TP, \lambda, dp) := \frac{8 \cdot \pi^3 \cdot c \cdot dp^3}{\lambda^5 \cdot \left( \exp\left(\frac{h \cdot c}{k_b \cdot \lambda \cdot TP}\right) \right)} \cdot Em$$

$$\text{Pho\_Par}(1730 \cdot K, 450 \cdot nm, 30 \cdot nm) = 409.635819 \frac{1}{s} \cdot nm^{-1}$$

This is the number of photons per nm wavelength interval into  $4\pi$  steradians



$$c := c \quad h := h \quad k_b := k_b$$

$$\frac{d}{dT_P} \left[ \frac{8 \cdot \pi^3 \cdot c \cdot d_p^3}{\lambda^5 \cdot \left( \exp\left(\frac{h \cdot c}{k_b \cdot \lambda \cdot T_P}\right) \right)} \cdot E_m \right] \rightarrow \frac{\frac{c \cdot h}{T_P \cdot \lambda \cdot k_b}}{T_P^2 \cdot \lambda^6 \cdot k_b}$$

To a good approximation we can calculate modulated radiation intensity from the product of this differential and temperature change using an average temperature

$$\frac{\tau_{cool} \cdot (1 + \cos(\phi(\tau_{cool}, \omega, T, svf)) \cdot \sin(\omega \cdot t - \phi(\tau_{cool}, \omega, T, svf))))}{\left( \frac{HC_{soot}(T, svf)}{HC_{air}(T)} + 1 \right)^2} + \left[ \frac{\left( \frac{\cos(\omega \cdot t - \omega \cdot \tau_{flow}) - \cos(\omega \cdot t)}{\omega} \right)}{\left( 1 + \frac{HC_{air}(T)}{HC_{soot}(T, svf)} \right)} + T_0_{gas} \right]$$

$$sf1(\textcolor{red}{t}) := \frac{2 \cdot \pi \cdot \text{cnst} \cdot \cos(\omega \cdot \tau_{flow}) - 2 \cdot \pi \cdot \sin(\omega \cdot \tau_{flow})}{\omega}$$

$$svf := svf$$

$$\frac{\tau_{cool} \cdot (1 + \cos(\phi(\tau_{cool}, \omega, T, svf)) \cdot \sin(\omega \cdot t - \phi(\tau_{cool}, \omega, T, svf))))}{cNST^2} + \left[ \left( \tau_{flow} + \frac{\cos(\omega \cdot t - \omega \cdot \tau_{flow}) - \cos(\omega \cdot t)}{\omega} \right) \right] \text{fourier} \rightarrow$$

$$sf2(\textcolor{red}{t}) := \frac{\pi \cdot \omega \cdot \tau_{cool} \cdot \cos(2 \cdot \phi(\tau_{cool}, \omega, T, svf))}{2}$$

Here I get the error: "This value must be a scalar" and it points to variable svf which is a scalar and has been reset with  $svf := svf$

The symbolic Fourier transform is evaluated in a separate Mathcad sheet see below

$$\frac{\tau_{cool} \cdot (1 + \cos(\phi(\tau_{cool}, \omega, T, svf)) \cdot \sin(\omega \cdot t - \phi(\tau_{cool}, \omega, T, svf))))}{cNST^2} + \left[ \left( \tau_{flow} + \frac{\cos(\omega \cdot t - \omega \cdot \tau_{flow}) - \cos(\omega \cdot t)}{\omega} \right) \right] \text{fourier} \rightarrow \frac{2 \cdot \pi \cdot cNST \cdot \Delta(\omega + \omega_0) + 2 \cdot \pi \cdot cNST \cdot \Delta(\omega\omega - \omega) - 2 \cdot \pi \cdot cNST \cdot \Delta(\omega + \omega_0)}{2 \cdot \pi \cdot cNST^2}$$

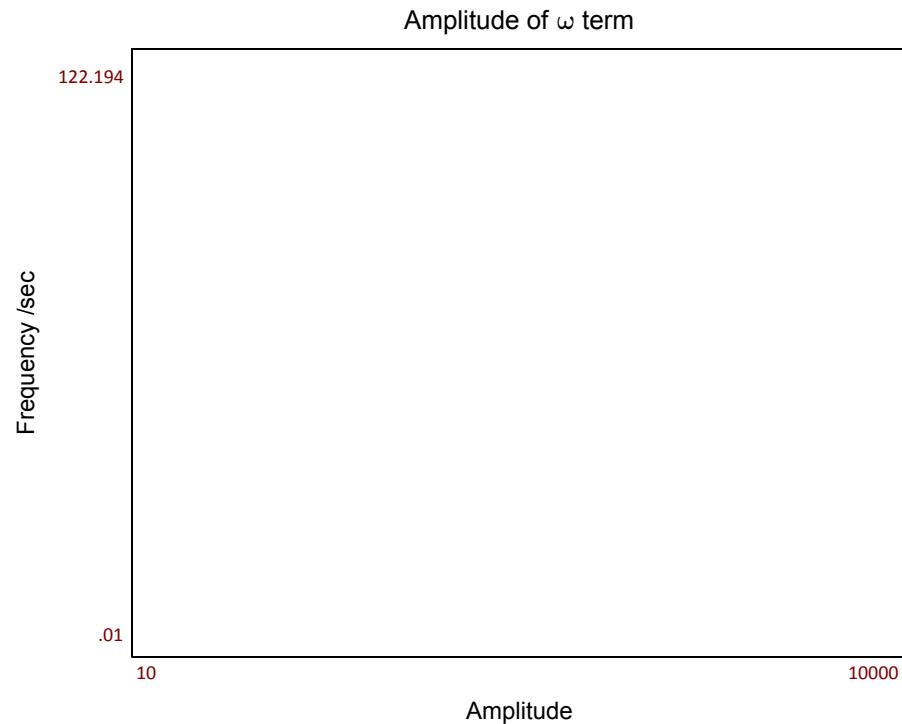
FT term at fundamental frequency

$$\left( \frac{2 \cdot \pi \cdot cNST \cdot \cos(\omega \cdot \tau_{flow}) - 2 \cdot \pi \cdot cNST - \pi \cdot \omega \cdot \tau_{cool} \cdot \sin(2 \cdot \phi(\tau_{cool}, \omega, T, svf)) + \pi \cdot \omega \cdot \tau_{cool} \cdot \cos(2 \cdot \phi(\tau_{cool}, \omega, T, svf)) \cdot 1i + 2i \cdot \pi \cdot cNST \cdot \sin(\omega \cdot \tau_{flow}) + \pi \cdot \omega \cdot \tau_{cool} \cdot 1i}{2 \cdot \omega \cdot cNST^2} \right) \cdot \Delta(\omega\omega)$$

$$cNST(T, svf) := 1 + \frac{HC_{air}(T)}{HC_{soot}(T, svf)}$$

$$sf3(\omega, \tau_{cool}, \tau_{flow}, svf, T) := \text{Lasheat}(P_d, T) \cdot -\frac{2 \cdot \pi \cdot cNST(T, svf) \cdot \cos(\omega \cdot \tau_{flow}) - 2 \cdot \pi \cdot cNST(T, svf) - \pi \cdot \omega \cdot \tau_{cool} \cdot \sin(2 \cdot \phi(\tau_{cool}, \omega, T, svf)) + \pi \cdot \omega \cdot \tau_{cool} \cdot \cos(2 \cdot \phi(\tau_{cool}, \omega, T, svf)) \cdot i + 2i \cdot \pi \cdot cNST(T, svf) \cdot \sin(\omega \cdot \tau_{flow})}{s \cdot 2 \cdot \omega \cdot cNST(T, svf)^2}$$

$$-\frac{2 \cdot \pi \cdot cNST \cdot \cos(\omega \cdot \tau_{flow}) - 2 \cdot \pi \cdot cNST - \pi \cdot \omega \cdot \tau_{cool} \cdot \sin(2 \cdot \phi(\tau_{cool}, \omega, T, svf)) + \pi \cdot \omega \cdot \tau_{cool} \cdot \cos(2 \cdot \phi(\tau_{cool}, \omega, T, svf)) \cdot i + 2i \cdot \pi \cdot cNST \cdot \sin(\omega \cdot \tau_{flow}) + \pi \cdot \omega \cdot \tau_{cool} \cdot 1i}{2 \cdot \omega \cdot cNST^2}$$



$$sf3\left(\frac{2 \cdot \pi \cdot z}{\text{sec}}, 2 \cdot 10^{-6}\right)$$

$$1807.2 - 1731 = 76.2$$

$$2 \cdot \frac{120.08}{76.2} = 3.151706$$

$\frac{1}{40}$

$\frac{0.0}{4.4}$

$$\frac{0.0003256}{1.0685e-005} = 30.472625$$

$$\frac{0.00052593}{1.4731e-005} = 35.702261$$

$\frac{0.00016}{5.6057e-}$

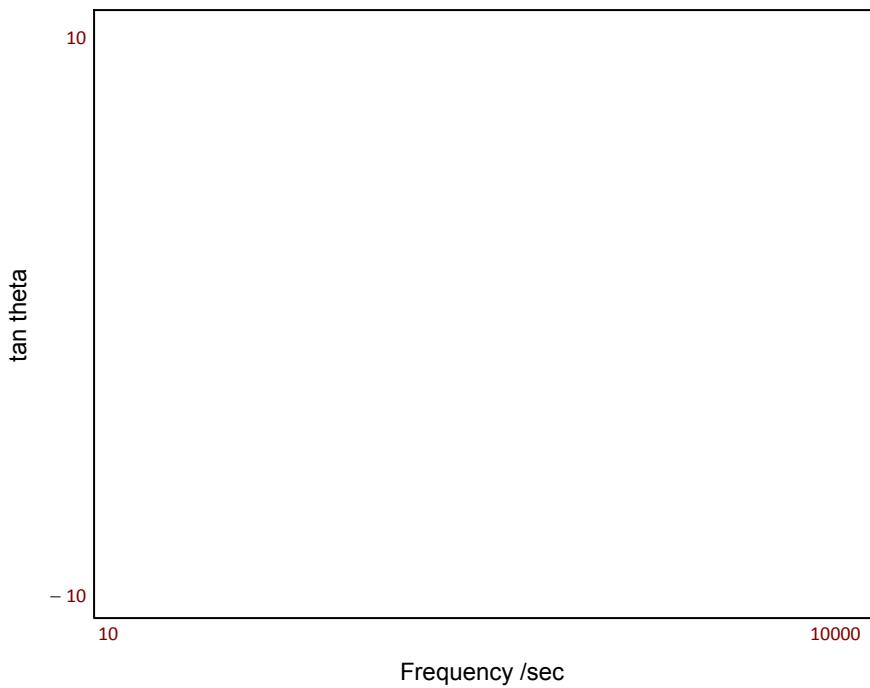
$$sf3\left(\frac{2 \cdot \pi \cdot 20000}{\text{sec}}, 2 \cdot 10^{-6} \cdot \text{sec}, .006 \cdot \text{sec}, 2.5 \cdot 10^{-6}, 1730 \cdot K\right) = ■$$

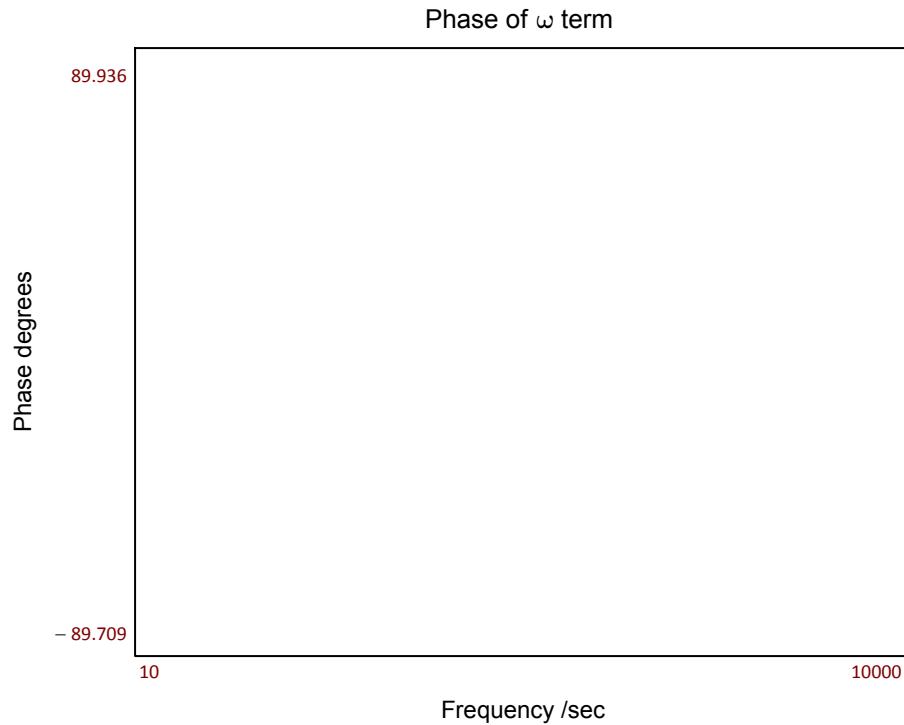
$$\frac{0.00064579}{1.6891e-005} = 38.232787$$

$$sf3\left(\frac{2 \cdot \pi \cdot 20}{\text{sec}}, 2 \cdot 10^{-6} \cdot \text{sec}, .006 \cdot \text{sec}, 2.5 \cdot 10^{-6}, 1730 \cdot K\right) = ■$$

Re

### Phase of $\omega$ term





$$\frac{\operatorname{Im}\left(\operatorname{sf3}\left(\frac{2 \cdot \pi \cdot 10}{\text{sec}}, 2 \cdot 10^{-6} \cdot \text{sec}, .005 \cdot \text{sec}, 1. \cdot 10^{-6}, 1730 \cdot K\right)\right)}{\operatorname{Re}\left(\operatorname{sf3}\left(\frac{2 \cdot \pi \cdot 10}{\text{sec}}, 2 \cdot 10^{-6} \cdot \text{sec}, .005 \cdot \text{sec}, 1. \cdot 10^{-6}, 1730 \cdot K\right)\right)} = \blacksquare$$

$$\frac{\operatorname{atan}(5)}{\text{deg}} = 78.690068$$

$$\frac{\operatorname{atan}(-5)}{\text{deg}} = -78.690068$$

$$\frac{\operatorname{atan}(-10)}{\text{deg}} = -84.289407$$

Constant term

$$-\left(\frac{4 \cdot \pi \cdot \omega \cdot \tau_{\text{cool}} + 4 \cdot \pi \cdot \omega \cdot \text{cnst} \cdot \tau_{\text{flow}}}{2 \cdot \omega \cdot \text{cnst}^2}\right) \cdot \Delta(\omega\omega)$$

$$\frac{4 \cdot \pi \cdot \omega \cdot \tau_{\text{cool}} + 4 \cdot \pi \cdot \omega \cdot \text{cnst} \cdot \tau_{\text{flow}}}{2 \cdot \omega \cdot \text{cnst}^2}$$

$$sf4(\omega, \tau_{cool}, \tau_{flow}, svf, T) := \left( \frac{4 \cdot \pi \cdot \omega \cdot \tau_{cool} + 4 \cdot \pi \cdot \omega \cdot cnst(T, svf) \cdot \tau_{flow}}{2 \cdot \omega \cdot cnst(T, svf)^2} \right) \cdot Lasheat(P_d, T)$$

0.00022502

