Golf Ball Projectile Motion

EGR 312: Dynamics
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Abstract

This paper analyzes projectile motion of a golf ball due to air friction. MATLAB was used to run the analysis and to generate plots of the motion.

Introduction

Air friction, or drag force, has a large impact on the projectile motion of a golf ball. This paper analyzes this motion by first looking at the motion with no air friction, followed by the motion with constant air friction, and finally with air friction as a function of velocity.

Background

The drag force, D, can be approximated by equation 1 where C_D is the drag coefficient, ρ is the density of the air, A is the area of the ball normal to the air flow, and V is the speed of the ball.

$$D = \frac{1}{2} C_D \rho A V^2 \tag{1}$$

The drag coefficient is a function of air viscosity (μ), air density, ball diameter, and ball speed. These four parameters are included in one parameter known as Reynolds number (Re) shown in equation 2. This makes the drag coefficient a function of Re.

$$\mathrm{Re} = \frac{\rho V d}{\mu} \tag{2}$$

τ is a time constant and is defined in equation 3 where m is the mass of the golf ball.

$$\tau = \frac{m}{3\pi d \mu} \tag{3}$$

For the purpose of this analysis, the following will be assumed: Initial velocity of the golf ball will be 120 ft/s at an angle of $30°$ with the horizontal, the weight of the golf ball is

1.5 oz with a diameter of 1.75 in., air viscosity is 0.375×10^{-6} (lb-s)/ft and air density is 0.002378 slug/ft^3. Also, for the last part of the analysis, the drag coefficient will be assumed constant for the following conditions:

$$C_D = 0.4 \qquad for \quad \mathrm{Re} \le 9\times10^4 \tag{4}$$

$$C_D = 0.1 \qquad for \quad \mathrm{Re} > 9\times10^4 \tag{5}$$

Calculations and Results

The motion of a golf ball with no air friction must be analyzed first. x will be the horizontal distance the golf ball has traveled, y will be the height, u will be the velocity in the x direction, and v will be the velocity in the y direction. Equations 6 and 7 show the general equations for projectile motion.

$$x(t) = x_0 + u_0 t + \frac{1}{2}a_x t^2 \tag{6}$$

$$y(t) = y_0 + v_0 t + \frac{1}{2}a_y t^2 \tag{7}$$

The initial distance and height is zero, the initial velocity is given, and the only force on the golf ball is gravity, g. Equations 8 and 9 show the final position equations.

$$x(t) = u_0 t = 103.9t \quad \text{ft} \tag{8}$$

$$y(t) = v_0 t - \frac{1}{2}gt^2 = 60t - 16.1t^2 \quad \text{ft} \tag{9}$$

From equations 8 and 9, the velocities in the x and y directions were found by taking the derivatives with respect to time. Equations 10 and 11 show these velocity equations.

$$u(t) = \frac{dx}{dt} = u_0 = 103.9 \quad \text{ft/s} \tag{10}$$

$$v(t) = \frac{dy}{dt} = v_0 - gt = 60 - 32.2t \quad \text{ft/s} \tag{11}$$

With these equations, MATLAB was used to plot the path of the golf ball. Appendix A shows the MATLAB code used for this analysis. Figure 1 shows the path of the golf ball with no air friction.

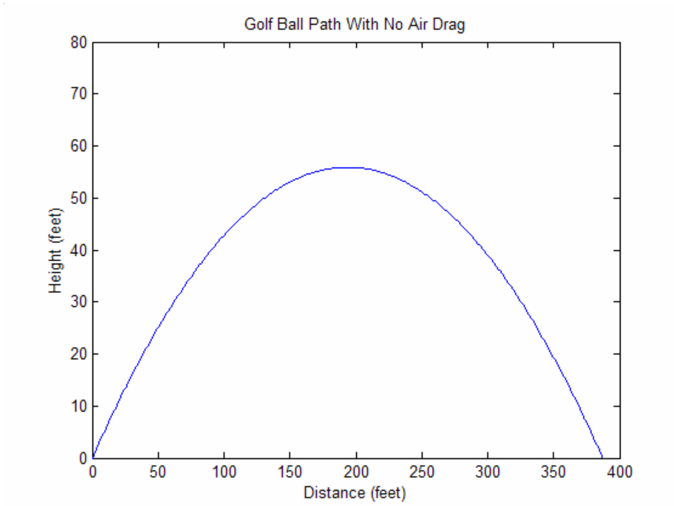

Figure 1: Projectile Motion without Air Friction

The range of the golf ball was 387.6 ft and the max height was 55.9 ft.

For the second part of the analysis, the air friction is assumed to be constant with Re = 9×10^4 and $C_D = 0.25$. Using equations 1 through 3, expressions for the motion can be derived. Appendix B contains the derivation. Equations 12 through 15 show the derived equations of motion.

$$u = \frac{dx}{dt} \tag{12}$$

$$v = \frac{dy}{dt} \tag{13}$$

$$\frac{du}{dt} = -\frac{C_D \operatorname{Re}}{24\tau} u \tag{14}$$

$$\frac{dv}{dt} = -\frac{C_D \operatorname{Re}}{24\tau} v - g \tag{15}$$

Since $C_D \text{Re}$ is constant, equations 14 and 15 are simply first order differential equations. Appendix C contains the derivation to solve these four equations for the position of the golf ball. Equations 16 and 17 show the position of the golf ball as a function of time with air friction constant.

$$x(t) = 624 - 624e^{-.1665t} \quad \text{ft} \tag{16}$$

$$y(t) = 1522 - 1522e^{-.1665t} - 193.4t \quad \text{ft} \tag{17}$$

With equations 16 and 17, MATLAB was used to plot the path of the golf ball. Appendix D contains the MATLAB code used for this analysis. Figure 2 shows the path of the golf ball with constant air friction.

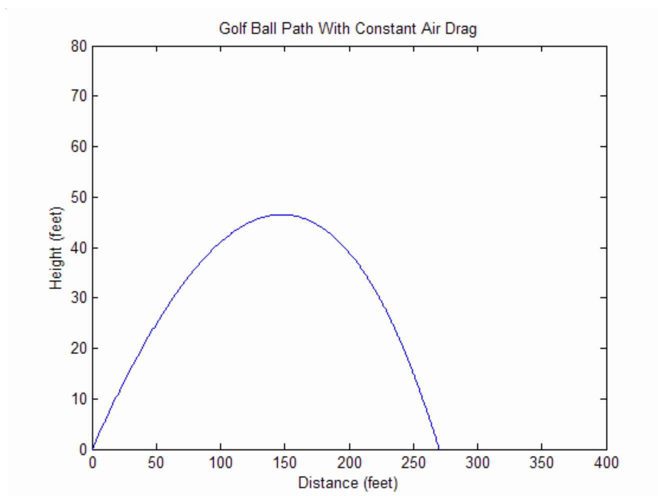

Figure 2: Projectile Motion with Constant Air Friction

The range of the golf ball was 270.3 ft and the max height was 46.5 ft.

The third part of the analysis is to assume air friction is defined by the conditions in equations 4 and 5. Equations 12 – 15 were used to derive expressions for position and velocity. This was done by substituting the derivatives with limit equations where t goes to zero. The following equations were derived from equations 12 – 15:

$$x(t) = x(t-\Delta t) + \frac{\Delta t}{2}[u(t-\Delta t) + u(t)] \tag{18}$$

$$y(t) = y(t-\Delta t) + \frac{\Delta t}{2}[v(t-\Delta t) + v(t)] \tag{19}$$

$$u(t) = u(t-\Delta t) - \frac{C_D \operatorname{Re}}{24\tau}[\frac{u(t-\Delta t) + u(t)}{2}]\Delta t \tag{20}$$

$$v(t) = v(t-\Delta t) - \frac{C_D \operatorname{Re}}{24\tau}[\frac{v(t-\Delta t) + v(t)}{2}]\Delta t - g\Delta t \tag{21}$$

Solving first for u(t), all u(t) terms were brought to the left side of equation 20:

$$u(t) + \frac{C_D \operatorname{Re} \Delta t}{48\tau} u(t) = u(t-\Delta t) - \frac{C_D \operatorname{Re} \Delta t}{48\tau} u(t-\Delta t)$$

Therefore,

$$u(t)\left[1 + \frac{C_D \operatorname{Re} \Delta t}{48\tau}\right] = u(t-\Delta t) - \frac{C_D \operatorname{Re} \Delta t}{48\tau} u(t-\Delta t)$$

Isolating u(t):

$$u(t) = \frac{u(t-\Delta t) - \frac{C_D \operatorname{Re} \Delta t}{48\tau} u(t-\Delta t)}{\left[1 + \frac{C_D \operatorname{Re} \Delta t}{48\tau}\right]}$$

Simplifying the above equation:

$$u(t) = \frac{48\tau[u(t-\Delta t)] - C_D \operatorname{Re} \Delta t[u(t-\Delta t)]}{48\tau + C_D \operatorname{Re} \Delta t} \tag{22}$$

The v(t) equation was derived by first bringing all of the v(t) terms to the left side of equation 21:

$$v(t) + \frac{C_D \operatorname{Re} \Delta t}{48\tau} v(t) = v(t-\Delta t) - \frac{C_D \operatorname{Re} \Delta t}{48\tau} v(t-\Delta t) - g\Delta t$$

Therefore,

$$v(t)\left[1+\frac{C_D \operatorname{Re}\Delta t}{48\tau}\right] = v(t-\Delta t) - \frac{C_D \operatorname{Re}\Delta t}{48\tau} v(t-\Delta t) - g\Delta t$$

Isolating v(t):

$$v(t) = \frac{v(t-\Delta t) - \dfrac{C_D \operatorname{Re}\Delta t}{48\tau} v(t-\Delta t) - g\Delta t}{\left[1+\dfrac{C_D \operatorname{Re}\Delta t}{48\tau}\right]}$$

Simplifying the above equation:

$$v(t) = \frac{48\tau[v(t-\Delta t)] - C_D \operatorname{Re}\Delta t[v(t-\Delta t)] - 48\tau g\Delta t}{48\tau + C_D \operatorname{Re}\Delta t} \tag{23}$$

From the velocity equations that were derived from equations 22 and 23, the Central Differences and Runge Kutta Methods were used in order to further, and more accurately, analyze the projectile motion of the golf ball. Appendix E and F contain the MATLAB code used for these analyses. Both methods gave the same result, which is shown in figure 3.

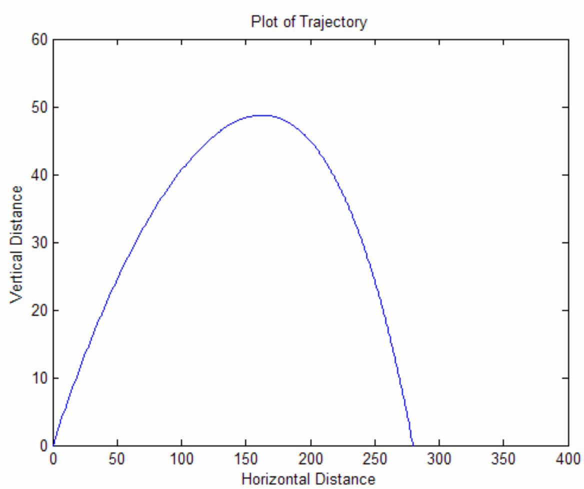

Figure 3: Central Difference and Runge Kutta

The range of the golf ball was 280.1 ft and the max height was 48.8 ft. Figure 4 shows all three analyses on one plot.

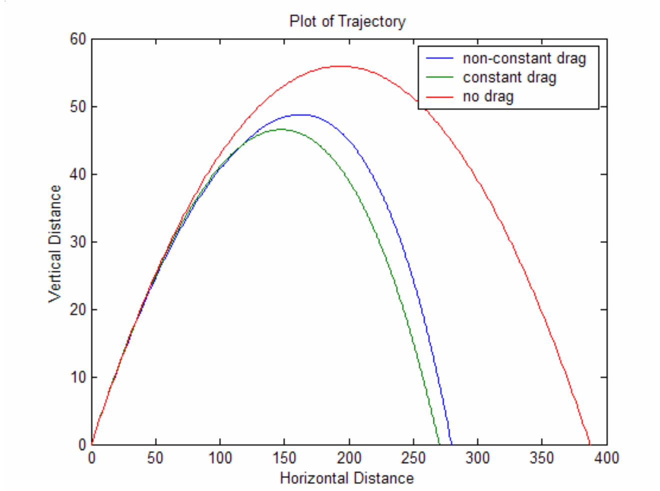

Figure 4: Combination Plot

Discussion

Since the first analysis considered no air friction, it is logical that the largest range was obtained. The second analysis considered a constant air friction, and does not regenerate the Reynolds number for the changing velocity. Since the drag force is constant and not reliant on velocity, it makes sense that the impact on the range of the golf ball was the most. In the third analysis, drag was a function of the Reynolds number and velocity. The Reynolds number also varies with velocity, so it has an exponential effect. The faster the golf ball travels, the greater the aerodynamic drag force. Shown below in table 1 is a comparison of the MATLAB outputs for the maximum distance and maximum height of the golf ball.

Table 1: Maximum Distance and Height Values

	x-max [ft]	y-max [ft]
No Drag	387.63	55.9
Drag	270.32	46.53
Central Differences	280.1	48.76
Runge Kutta	280.1	48.76

The Runge Kutta Method is a more precise numerical method than Central Differences Method, but both yield the same results. The time step in the Central Differences program was very small, which can explain the matching distances and heights.

Conclusion

Theoretically, projectile motion follows a perfectly parabolic path. However, when drag forces are considered, the path of the golf ball varies from this parabolic behavior. The higher the velocity of the golf ball, the greater the effects of aerodynamic drag on the path and range of the golf ball.

Appendix A: MATLAB code for no air friction

```
% Engineering 312 - Dynamics
% Golf Ball Projectile Motion
% Matt Brower, Nate Bassett, Matt Shinew, Steve Michel
% Projectile Motion assuming no air friction
% ****Initialize Vectors and Constants****

i=1;
t=[];
t(i)=0;
x=[];
y=[];
x(i)=0;
y(i)=0;
dt=.01;
g=32.2;     % ft/s^2
theta=pi/6; % 30 degrees
Vo=120;     % ft/s
Vx=Vo*cos(theta);
Vy=Vo*sin(theta);

% Test whether ball has hit the ground
% ****x and y position****

while ((y(i)>0)|(i==1))
    i=i+1;
    t(i)=t(i-1)+dt;
    x(i)=Vx*t(i);
    y(i)=Vy*t(i) - .5*g*(t(i))^2;
end

% Plot Path of Ball

plot(x,y)
title('Golf Ball Path With No Air Drag')
axis([0,400,0,80])
xlabel('Distance (feet)')
ylabel('Height (feet)')

% Display the max height and distance

fprintf('Max Distance = %-5.1f ft\n',max(x));
fprintf('Max Height = %-5.1f ft\n',max(y));
break
```

Appendix B: Derivation of differential equations of motion

Equations 1-3 are given.

$$D = \frac{1}{2} C_D \rho A V^2 \quad (1)$$

$$\mathrm{Re} = \frac{\rho V d}{\mu} \quad (2)$$

$$\tau = \frac{m}{3\pi d \mu} \quad (3)$$

First the differential equation for the x direction is found. Equation 4 shows Newton's second law for the x direction.

$$F_x = m a_x \quad (4)$$

Since the only force in the x direction is the air resistance, which is opposite to the direction of the velocity, equation 1 can be substituted into equation 4. This gives equation 5.

$$a_x = -\frac{C_D \rho A u^2}{2m} \quad (5)$$

The projected area of the ball normal to the air flow can be written in terms of the diameter of the golf ball. Multiplying by $\frac{\mu}{\mu}$ will give the terms of Re. Equation 6 shows this simplification.

$$a_x = -\frac{C_D \pi d \mu}{8m} u \frac{\rho u d}{\mu} = -\frac{C_D \, \mathrm{Re}}{8} \frac{\pi d \mu}{m} u \quad (6)$$

Equation 3 can be substituted into equation 6 and a_x substituted as $\frac{du}{dt}$. Equation 7 shows the final differential equation for the motion in the x direction.

$$\frac{du}{dt} = -\frac{C_D \operatorname{Re}}{24\tau} u \tag{7}$$

Next the differential equation for the y direction is found. Equation 8 shows Newton's second law for the y direction.

$$F_y = ma_y \tag{8}$$

Both gravity and air friction are forces that act on the motion in the y direction. Equation 9 shows these two forces substituted into equation 8.

$$a_y = \frac{-C_D \rho A v^2 - mg}{m} = -\frac{C_D \rho A v^2}{m} - g \tag{9}$$

The same steps that were done for the x direction can be done for equation 9. Equation 10 shows the final differential equation for the motion in the y direction.

$$\frac{dv}{dt} = -\frac{C_D \operatorname{Re}}{24\tau} v - g \tag{10}$$

Since u and v are the velocities in the x and y direction, respectively, they can be written as the derivatives of the position with respect to time. Equations 11 and 12 show this relationship.

$$u = \frac{dx}{dt} \tag{11}$$

$$v = \frac{dy}{dt} \tag{12}$$

Appendix C: Solving the differential equations with constant air friction

Equations 1 and 2 are the differential equations to be solved.

$$\frac{du}{dt} = -\frac{C_D \operatorname{Re}}{24\tau} u \tag{1}$$

$$\frac{dv}{dt} = -\frac{C_D \operatorname{Re}}{24\tau} v - g \tag{2}$$

Since $-\frac{C_D \operatorname{Re}}{24\tau}$ is a constant, equations 1 and 2 can be written as equations 3 and 4 where

$k = -\frac{C_D \operatorname{Re}}{24\tau} = -.1665$.

$$\frac{du}{dt} - ku = 0 \tag{3}$$

$$\frac{dv}{dt} - kv = -g \tag{4}$$

Equation 5 shows equation 3 solved for u.

$$u = u_0 e^{kt} = 103.9 e^{-.1665t} \quad \text{ft/s} \tag{5}$$

$u = \frac{dx}{dt}$ can be written as $\int dx = \int u dt$. This is used to solve for the position in the x

direction shown in equation 6.

$$x = -\frac{u_0}{k} + \frac{u_0}{k} e^{kt} = 624 - 624 e^{-.1665t} \quad \text{ft} \tag{6}$$

Equation 7 shows equation 4 solved for v.

$$v = \left(v_0 - \frac{g}{k}\right) e^{kt} + \frac{g}{k} = 253.4 e^{-.1665t} - 193.4 \quad \text{ft/s} \tag{7}$$

$v = \frac{dy}{dt}$ can be written as $\int dy = \int v dt$. This is used to solve for the position in the y

direction shown in equation 8.

$$y = \frac{1}{k}\left(v_0 - \frac{g}{k}\right) e^{kt} + \left(\frac{g}{k}\right) t - \frac{1}{k}\left(v_0 - \frac{g}{k}\right) = -1522 e^{-.1665t} - 193.4t + 1522 \tag{8}$$

Appendix D: MATLAB code for constant air friction

```
% Engineering 312 - Dynamics
% Golf Ball Projectile Motion
% Matt Brower, Nate Bassett, Matt Shinew, Steve Michel
% Projectile Motion assuming constant air drag
% ****Initialize Vectors and Constants****

i=1;
t=[];
t(i)=0;
x=[];
y=[];
x(i)=0;
y(i)=0;
dt=.01;
g=32.2;     % ft/s^2
k=-.1665;
theta=pi/6; % 30 degrees
Vo=120;     % ft/s
Vx=Vo*cos(theta);
Vy=Vo*sin(theta);

% Test whether ball has hit the ground
% ****x and y position****

while ((y(i)>0)|(i==1))
    i=i+1;
    t(i)=t(i-1)+dt;
    x(i)=624-624*exp(k*t(i));
    y(i)=-1522*exp(k*t(i))+(g/k)*t(i)+1522;
end

% Plot Path of Ball

plot(x,y)
title('Golf Ball Path With Constant Air Drag')
axis([0,400,0,80])
xlabel('Distance (feet)')
ylabel('Height (feet)')

% Display the max height and distance

fprintf('Max Distance = %-5.1f ft\n',max(x));
fprintf('Max Height = %-5.1f ft\n',max(y));
break
```

Appendix E: MATLAB code for air friction as a function of speed

```
% Engineering 312 - Dynamics
% Golf Ball Projectile Motion - Part 5
% Matt Brower, Nate Bassett, Matt Shinew, Steve Michel
% Program to calculate the friction on a golf ball

function y = friction(t,y) %Function definition for drag
force

%Define constants and variables
g = 32.2; %Gravity constant (ft/s^2)
w = 1.5/16; %Golf ball weight (lb)
d = 1.75/12; %Golf ball diameter (ft)
mu = 0.375*10^(-6); %Viscosity of air (lb-sec/ft)
m = w/g; %Golf ball mass (slugs)
tau = m/(3*mu*pi*d); %Time constant (sec)
p = 0.002378; %Air density (slugs/ft^3)
ymax = 0; %Maximum height
xmax = 0; %Maximum distance
Reu = (p*d*y(3))/(mu);
Rev = (p*d*y(4))/(mu);

%Calculate drag as a function of Reynolds number
%which is a fucntion of velocity
if Reu>90000 %When Re is > 90,000, the drag constant is 0.1
Cdu=0.1;
else %When Re is <= 90,000, the drag constant is 0.4
Cdu=0.4;
end
if Rev>90000 %When Re is > 90,000, the drag constant is 0.1
Cdv=0.1;
else %When Re is <= 90,000, the drag constant is 0.4
Cdv=0.4;
end
%State matrix of [x,y,u,v]
y = [y(3);y(4);((-Cdu*Reu*y(3))/(24*tau));(((-
Cdv*Rev*y(4))/(24*tau))-g)];
```

```
% Engineering 312 - Dynamics
% Matt Brower, Nate Bassett, Matt, Steve
% Description: This program plots the curve of a golf ball
% with drag force using the method of central differences.

%Define constants and variables
i = 1; %Index
Cdh = 0.4; %Drag Coefficient for Reynolds Number <= 90000
Cdl = 0.1; %Drag Coefficient for Reynolds Number > 90000
g = 32.2; %Definition of gravity (ft/s^2)
dt = 0.01; %Differential time step definition (s)
w = 1.5/16; %Golf ball weight (lb)
d = 1.75/12; %Golf ball diameter (ft)
p = 0.002378; %Air density (slugs/ft^3)
mu = 0.375*10^(-6); %Viscosity of air (lb-sec/ft)
m = w/g; %Golf ball mass (slugs)
tau = m/(3*mu*pi*d); %Time constant (sec)
y1max = 0; %Maximum height
x1max = 0; %Maximum distance

%Define arrays
x1 = []; %Distance array
y1 = []; %Height array
u1 = []; %Horizontal velocity component array
v1 = []; %Vertical velocity component array
t = []; %Time array (sec)

%Initialize constants and variables
x1(i) = eps; %Distance initialization (ft)
y1(i) = eps; %Height initialization (ft)
u1(i) = 103.9230; %Horizontal velocity component
initialization (ft)
v1(i) = 60; %Vertical velocity component initialization
(ft)
Reu = eps; %Horizontal Reynolds Number
Rev = eps; %Vertical Reynolds Number
t(i) = eps; %Time initialization (sec)

% calculations
while y1(i)>=eps
i = i+1; %Increment the counter
Reu = (p*d*u1(i-1))/(mu); %Define Re in the x-direction
Rev = (p*d*v1(i-1))/(mu); %Define Re in the y-direction

%The calculation of velocity
if Reu > (90000) %When Re is > 90,000, the drag constant is
0.1
```

```
u1(i)=((u1(i-1))*((48*tau)-
(Cdl*Reu*dt)))/((48*tau)+(Cdl*Reu*dt));
else %When Re is <= 90,000, the drag constant is 0.4
u1(i)=((u1(i-1))*((48*tau)-
(Cdh*Reu*dt)))/((48*tau)+(Cdh*Reu*dt));
end

if Rev > (90000) %When Re is > 90,000, the drag constant is
0.1
v1(i)=((v1(i-1)*(48*tau-Cdl*Rev*dt))-
(48*tau*g*dt))/(48*tau+Cdl*Rev*dt);
else %When Re is <= 90,000, the drag constant is 0.4
v1(i)=((v1(i-1)*(48*tau-Cdh*Rev*dt))-
(48*tau*g*dt))/(48*tau+Cdh*Rev*dt);
end

%The calcualtion of position
x1(i) = x1(i-1)+(dt/2)*(u1(i-1)+u1(i)); %Defines x-position
y1(i) = y1(i-1)+(dt/2)*(v1(i-1)+v1(i)); %Defines y-position
t(i) =(i-1)*dt; %Increments the time

%Find the maximum distance and height
if y1(i)>y1max
y1max = y1(i);
end

if x1(i)>x1max
x1max = x1(i);
end

end
%Print the results
fprintf('Max Distance with non-constant drag = %-5.2f
ft\n', x1max);
fprintf('Max Height with non-constant drag = %-5.2f ft\n',
y1max);

plot(x1,y1);
Title('Plot of Trajectory');
axis([0,400,0,60]);
xlabel('Horizontal Distance');
ylabel('Vertical Distance');
```

Appendix F: MATLAB code for air friction as a function of speed using Fourth Order Runge_Kutta Method

```
% Engineering 312 - Dynamics
% Golf Ball Projectile Motion
% Matt Brower, Nate Bassett, Matt Shinew, Steve Michel
% Description: This program plots the curve of a golf ball
% with drag force using the fourth-order Runge-Kutta
  method.

%The time span is between 0 and 4 seconds.
%there is a time step of 0.01 seconds
for j=1:400
tspan(j)=0+(j-1)*0.01;
end
%Initial conditions for state matrix
y0=[0;0;103.923;60];
%Call on ode45.m and fric.m to perform the integration.
[t,y]=ode45(@friction,tspan,y0);
% Test whether or not the golf ball has hit the ground.

ymax = eps;
xmax = eps;

for i=1:400
s=y(i,2);
%Defines the maximum distance and height values
if y(2)>ymax
ymax = y(2);
end
if y(1)>xmax
xmax = y(1);
end

if s > eps
%Plot the trajectory of the golf ball
plot(y(:,1), y(:,2))
Title('Plot of Golf Ball Trajectory Using Runge-Kutta')
axis([0,400,0,60])
xlabel('Horizontal Position (ft)')
ylabel('Vertical Position (ft)')
break
end
end
%End program
```